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ON THE TRAPPED SURFACE CHARACTERIZATION OF BLACK HOLE REGIONS IN THE SPHERICALLY AND AXIALLY SYMMETRIC SPACETIMES

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DECLARATION

I hereby declare that this submission is my own work towards the Doctor of Philosophy (PhD) in Mathematical Physics and that, to the best of my knowledge, it contains no material previously published by another person nor material which has been accepted for the award of any other degree of the University, except where due acknowledgment has been made in the text.

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DEDICATION

This research work is dedicated to my Supervisor, Prof. F. T Oduro, Department of Mathematics, Kwame Nkrumah University of Science and Technology (K.N.U.S.T.)

ABSTRACT

The characterization of black holes by means of classical event horizon is a global concept since one need to have the knowledge of the whole spacetime in order to locate a black hole region and the event horizon. To surmount these issues, we investigate alternative approach based on the concept of trapped surfaces in a variety of spacetimes. Specifically, to compute the expansions of the appropriate null vectors in both spherically and axiallysymmetric spacetimes and thus explicitly determine the existence of trapped and marginally trapped surfaces in their respective black hole regions.

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NOTATION

The following are notations used in this thesis

ADM	Arnowitt, Deser Misner
OS	Oppenheimer-Snyder
MTS	marginally trapped surfaces
MOTS	marginally outer trapped surfaces
FOTH	Future outer trapping horizon
DH	dynamic horizon
MTT	marginally trapped tube
IH	isolated horizon
PG	Painleve-Gullstrand coordinates
EF	Eddington-Finkelstein coordinates systems
KS	Kruskal-Szekeres coordinates
RN	Reissner-Nordstrom solution
M	Mass
Q	Charge

Chapter 1

INTRODUCTION

1.1 Background to the Study

One of the most striking results of General Relativity is its prediction of black holes which are spacetime regions from which no signal can be seen by an observer located sufficiently far from the matter sources (Frolov and Zelnikov (2011); Page1). Black holes in the universe form as the final state of gravitational collapse of sufficiently massive objects, such as massive stars demonstrated by the work of Chandrasekhar (1983). General relativity shows that black holes are remarkably simple objects characterized by just a few numbers. As Chandrasekhar put it ”the black holes of nature are the most perfect macroscopic objects there are in the universe”: the only elements in their construction are our concepts of space and time (Hartle (2003)). The very possibility of the existence of black holes was first discussed by Michell and Laplace within the framework of the Newtonian theory at the end of the 18th century, (Frolov and Zelnikov (2011); Page 8). They viewed as a star which has strong gravitational field such that the Newtonian escape velocity $\sqrt{2GM/R}$ (where M and R are respectively the mass and radius of the star) is greater than the speed of light. The inequality $R \leq 2GM/C^2$ for escape velocity also holds in general relativity (Penrose (2004); Page 707; Krishnan (2013); Page 1).

The history of black holes started just after general relativity was discovered. Schwarzschild in 1916 discovered the first exact (spherically symmetric) solution of the Einstein equations in vacuum and it was named after him was in fact, a black hole. This solution describes the gravitational field of spherically compact objects. Apart from the singularity at the center of symmetry (at $r = 0$), this solution had another singularity on the gravitational radius surface (at $r = 2M$). It was soon understood that the latter singularity is quite different from singularity at the region. The nature of this Schwarzschild singularity was a mystery for many years. However its properties were fully appreciated after four decades. Many scientists contributed to the solution of this problem (Frolov and Zelnikov (2011); Page 8).

Kerr discovered a solution of the Einstein equation, which describes the gravitational field of a stationary rotating black hole in 1963 (Jakobsson (2017)). This solution has a gravitational radius which describes the position of the event horizon (Frolov and Zelnikov (2011); Page 8). Carter in (1966) explained its global properties. The charged spinning black holes which represent the Kerr-Newman solution was discovered in1965. The term black hole was introduced in 1967 by John Wheeler (Frolov and Zelnikov (2011); Page 10); Krishnan (2012); Page 1). There were seminal developments at that very time to understand the general properties of black holes. These include the study of the global properties of black hole spacetimes, the event horizon definition, the singularity theorems of Penrose and

Hawking and the introduction of trapped surfaces by Penrose (Krishnan (2012); Page 1). In the 1980s, Robinson and Carter established the uniqueness of the Kerr metric for the description of the black holes of nature. This theorem states that; stationary axiallysymmetric solutions of Einstein's equation for the vacuum, which have a smooth convex event horizon, are asymptotically flat and are non-singular outside of the horizon are uniquely specified by the two parameters the Mass and the Angular Momentum and these two parameters only (Chandrasekhar (1983); Page 298).These theorems assert that, given a matter model (for example vacuum), a static or a stationary black hole spacetime belongs to a specific class of spacetimes (in the vacuum case, they are Schwarzschild in the static regime and Kerr for the stationary case) which are characterized by a few parameters that describe the fundamental properties of the black hole (for vacuum these parameters are the mass and the angular momentum of the black hole). The Kerr solution represents the unique solution which the general theory of relativity provides for the description of all black holes that can occur in the astronomical universe by the gravitational collapse of stellar masses; and it is the only physical theory that provides an exact description of a macroscopic object (Chandrasekhar (1983); Page 273). The study of black holes has for many years depended on event horizons as the boundary of the region of the black hole from where one can send signals to infinity (Senovilla (2011); Page 1). However, the study of black holes based on the concept of classical event horizon has the following drawbacks: to determine the event horizon requires the knowledge of the entire future null infinity, the definition has no direct relation with the notion of strong gravitational field as shown by (Ashtekar and Krishnan (2004); Page 9) and (Krishnan (2013); Page 16) on an example based on the Vaidya metric, event horizon can form in a flat region of spacetime. Another non-local feature of event horizons is their teleological nature (Gourgoulhon and Jaramillo (2008); Page 2). The classical black hole boundary, i.e. the event horizon, responds in advance to what will happen in the future. This is shown by (Booth (2005); Page 4), on the explicit example of a black hole formed by the collapse of two successive matter shells: after the first shell has collapsed to form the event horizon, the latter remains stationary for a while and then starts to grow before the second collapsing shell reaches it (Gourgoulhon and Jaramillo (2008); Page 2). If we consider black holes as ”ordinary” physical objects, for instance in quantum gravity or numerical relativity, the above mentioned non-local behaviour of the event horizon would be problematic (Gourgoulhon and Jaramillo (2008); Page 2). To circumvent these problems, the seminal notion of a trapped surface could play a crucial role, capturing as it does the idea that all light rays emitted from the surface locally converge. Through the Hawking and Penrose’s singularity theorems and weak cosmic censorship, the existence of a black hole region is indicated. In fact, in the so-called strongly predictable spacetimes which satisfies proper energy conditions, these trapped surfaces are guaranteed to lie inside the black hole region. Moreover; to locate these trapped surfaces does not involve a whole future spacetime development (Jaramillo (2011); Page 2).

1.2 Problem Statement

Characterizing black holes by means of classical event horizon is a global concept which has the following drawbacks: It is a teleological concept, i.e. we need to know the whole spacetime in order to locate event horizon and black hole region. The event horizon can enter into flat spacetime regions. This has motivated the need to use local approach as a complementary means of characterizing black holes. Specifically, to compute using appropriate null vectors the covariant divergences and the fluxes in spherically symmetric spacetime (Schwarzschild and Vaidya spacetimes) and axially symmetric spacetime (Kerr and Kerr Vaidya spacetimes) which Text books and Journals have not used.

1.3 Research Objective

The objective of this study is as follows:

To demonstrate the existence of black hole region by showing the existence of trapped and marginally trapped surfaces in spherically symmetric spacetime (Schwarzschild metric), the Vaidya spacetime, axially symmetric spacetime (Kerr metric) and the Kerr Vaidya spacetime.

1.4 Methodology

Using covariant divergence and Gauss divergence theorems in spherically symmetric spacetime and axiallysymmetric spacetime to exploit the definitions of trapped

surfaces.

1.5 Structure of the Thesis

This thesis is organized as follows; the Background of the study, the Problem Statement and research objectives. This is followed by Methodology, structure of the thesis, Notations. Review of literature is done in chapter two. Chapter three discusses several properties of spacetime including the classification of various components of the curvature tensor and Killing vectors. These Killing vectors play important role in spacetimes. For example a spacetime is static if it possesses a timelike Killing vector field and spherically symmetric if it possesses a full set of three rotational (hence spacelike) Killing vectors.

In chapter four, we discuss a hypersurface which is the main ideas related to null vectors and curves. We will define a non-affinely parameterized geodesic in terms of parallel transport and properties of non-affinely parameterized geodesics will also be discussed. This chapter also discusses the celebrated Raychaudhuri equation for timelike geodesic congruences and affine null geodesic congruences. By Raychaudhuri equation, we obtain the expansion of null vectors which play a very important role in the singularity theorems of general relativity (where the trapped surfaces are characterized by negative expansions for both ingoing and outgoing null vectors), and in the study of (black holes and the laws governing the evolution of the surface of the black hole). Another important property that is discussed is the surface gravity. This property does not change over the horizon except for a bifurcation two-sphere where the Killing vector vanishes and can change in sign. As an example we show that the surface gravity of the Schwarzschild black hole is non-degenerate.

In this same chapter we discuss how various spacetimes can be classified in terms of their curvature and introduce several quasi-local definitions of mass in curved spacetimes that are used to assign a mass to a black hole. Various definitions of black hole horizons that have been proposed in the literature are also discussed in this chapter. Some properties of trapped surfaces under deformations and time evolution are also discussed in chapter four, and this leads naturally to the notions of marginally trapped tubes, trapping and dynamical horizons.

In chapter five, we discuss the spherically symmetric Schwarzschild spacetime which happens to be the simplest black hole, and investigate the properties of its black hole region. This will eventually lead to the notions of event horizons and trapped surfaces, and the boundary of the trapped region. We will see that, the different definitions of the black hole horizon are the same in Schwarzschild black hole but will be different in more general situations. Perhaps the simplest example we shall discuss is the dynamic black hole (Vaidya spacetime) in the same chapter. We shall see in this simple spherically symmetric black hole that, the locations of trapped surfaces are different.

In chapter six, we investigate axiallysymmetric spacetimes in different coordinate systems to locate trapped surfaces for Kerr in advanced Eddington-Finkelstein coordinates, Doran coordinates, Kerr Vaidya coordinates and non extreme Kerr black hole. We shall see that, for the extreme Kerr black hole i.e. when $a = M$, there are no trapped surfaces. We use metric signature (- + ++) throughout and geometric units $C = G = 1$.

Finally, chapter seven provides a summary and some open issues.

Chapter 2

LITERATURE REVIEW

The traditional way of approaching black holes involves global spacetime concepts, in particular to have a good control of the notion of infinity (Jaramillo (2011); Page 2). Given a strongly asymptotically predictable spacetime M, the black hole region B is defined as $B = M - J^{-}(p^{+})$ where $J^{-}(p^{+})$ is the causal past of future null infinity p^{+}. That is, B is the spacetime region that cannot communicate with p^{+} (Jaramillo (2011); Page 2; Wald (2001); Page 6; Hawking and Ellis (1973); Page 312). In terms of spacetime the boundary of a black hole is its event horizon
(Bengtsson et al. (2013); Page 1; Hayward (2000); Page 2; Booth (2005); Page

2). In this global context, the event horizon E, defined by the boundary of B, is $E = \partial J^{-}(p^{+}) \cap M$ (Bengtsson et al. (2013); Page 1), or the boundary of the region from where one can send signals to a far away asymptotic external region (Senovilla (2011); Page 1). The most interesting geometric and physical properties of the event horizon are: E is a null hypersurface in M; it satisfies an *Area Theorem* (Jaramillo (2011); Page 2; Hayward (2000); Page 2); Booth (2005); Page 2) so that the area of spatial sections S of E does not decrease in the evolution and, beyond that, a set of black hole mechanics laws are fulfilled (Jaramillo (2011); Page 2; Bardeen et al. (1973); Page 167).

It is very unfortunate that, the event horizon is essentially a global object because it depends on the whole future evolution of the spacetime (Gourgoulhon and Jaramillo (2008); Page 2): it is a *teleological* concept (Hawking and Hartle (1972);) given in the work of (Gourgoulhon and Jaramillo (2008); Page 2) i.e. the knowledge of the full future spacetime is needed in order to locate the event horizon and the black hole region. Also the event horizon can enter into flat spacetime regions (Jaramillo (2011); Page 2). Moreover, this black hole definition does not have direct relation with the notion of strong gravitational field: as shown by (Ashtekar and Krishnan (2004); Page 9) on an example based on the Vaidya metric, an event horizon can form in a flat region of spacetime, where by flat it means a vanishing Riemann tensor, i.e. no gravitational field at all. As noticed also by Demianski and Lasota (1973) long time ago (Gourgoulhon and Jaramillo (2008); Page 2), this definition is also not applicable to cosmology, for usually a cosmological spacetime (M,g) is not asymptotically flat.

This global approach requires controlling structures that are not accessible during the evolution (Jaramillo (2011); Page 2). One possible alternative is to use the notion of trapped surfaces introduced by (Penrose (1965a), Helou (2015), page 3). *These trapped Surfaces are closed spacelike 2-surfaces (usually topological spheres)* *which are such that their area decreases locally along any possible future direction,* (Senovilla (2011); Page 2, Senovilla and Garfinkle (2015)). Trapped surfaces according to the singularity theorems and weak cosmic censorship, are local characterization of black holes (Bengtsson et al. (2013); Page 1). Moreover, their location does not involve a whole future spacetime development (Jaramillo (2011); Page 2). Since trapped surfaces are closed spacelike surfaces, they provide a quasi-local alternative which an observer could in principle locate in order to detect the presence of a black hole. Trapped surfaces therefore lead to quasi-local horizons such as isolated, dynamical and trapping horizons (Krishnan (2013); Page 2).

However, not all quasi-local horizon definitions are the same. Most of them depend on a choice of foliation for the spacetime. This dependence on the foliation is a crucial aspect of their definition (Senovilla (2011); Page 2). For instance, Wald and Iyer (1991) showed that certain foliations of the Schwarzschild spacetime do not contain apparent horizon. (Eardley (1997); Page 4), showed that, under certain conditions, marginally outer trapped surfaces can be perturbed outwards by choosing a new foliation and it was conjectured that the outer boundary of all marginally trapped surfaces was actually the event horizon. This conjecture was supported numerically by (Schnetter et al. (2006); Page 5), in the context of marginal surfaces and analytically was shown to be true for Vaidya spacetime by (Ben-Dov (2007); Page 6). In situations without spherical symmetry, it may not be clear what a foliation should be and in many situations the areas of the various quasi-locally defined horizons do not coincide (Ben-Dov (2007); Page 6). Chapter four of this thesis investigates the various definitions of horizons and their properties.

The concept of trapped surfaces was originally formulated in terms of the signs or the vanishing of the null expansions and has remained as such for many years (Senovilla (2011); Page 2). In a general spacetime $(M, g_{\mu,\nu})$ with the metric $g_{\mu,\nu}$ having signature $(- + ++)$, one can define two future directed null vectors l^{μ} and n^{μ} whose expansion scalars θ_l and θ_n are given by $\theta_l = q^{\mu\nu}\nabla_\mu l_\nu$ and $\theta_n = q^{\mu\nu}\nabla_\mu n_\nu$ where $q_{\mu\nu} = g_{\mu\nu} + l_\mu n_\nu + n_\mu l_\nu$ is the metric induced by $g_{\mu\nu}$ on the two dimensional spacelike surface formed by spatial foliation of the null hypersurface generated by l^μ and n^μ. Then a two dimensional spacelike surface S is said to be trapped if both $\theta_l < 0$ and $\theta_n < 0$, and S is marginally trapped surface if one of the two null expansions vanishes i.e. $\theta_l = 0$ or $\theta_n = 0$ (Pradhan and Majumdar (2011); Page 17). The set of all points in M contained in at least one trapped surface is called the trapped region.

In flat space, the expansion of S along l is always positive: $\theta_l > 0$, whereas that along n is negative: θ_n (Gourgoulhon and Jaramillo (2008); Page 3). A trapped surface may also be characterized by $\theta_l\theta_n > 0$ (Jaramillo (2011); Page 3). In the black hole context, in which the singularity occurs in the future, we refer to S as a future *trapped surface* (TS) if $\theta_l < 0$, $\theta_n < 0$ and as future *marginally trapped surface* (MTS) if one of the expansions, say θ_l, vanishes: $\theta_l = 0$, $\theta_n \leq 0$. If a notion of *naturally expanding* direction for the light rays exists (e.g. in isolated systems, the *outer* null direction l^μ pointing to infinity), a related notion of outer trapped surface is given by $\theta_l < 0$ (Hawking and Ellis (1973); Page 319). A particular case of weakly outer trapped surfaces, the so-called *marginally outer trapped surfaces* (MOTS) (defined as compact surfaces without boundary with vanishing outer null expansion i.e. $\theta_l = 0$), are widely considered as the best quasi-local replacements for the event horizon (Coley et al. (2017); page 1). It is worth noticing that in the work by Demianski and Lasota (1973), the ”local event horizon” defines nothing but a marginally trapped surface. A classical result by Hawking and Ellis (1973); (Page 320) states that the apparent horizon is a marginally outer trapped surface;

(see also the recent study by Demianski and Lasota (1973), Andersson and Metzger (2008)). A theorem by (Hawking and Ellis (1973); Page 320) states that, provided that the cosmic censorship conjecture holds, if the spacetime contains a trapped surface S, then it necessarily contains a black hole.

According to Penrose (1965a) in his singularity theorem (Hawking and Penrose (2010); Page 28) the notion of trapped surfaces, captures the idea that in a sufficiently strong gravitational field, as in gravitational collapse, even outgoing light rays are bent inwards. Indeed, in numerical relativity, the signal for a black hole is the presence of outer trapped surfaces on a given spatial slice (Baumgarte and Shapiro, 2010). In a dynamical situation these lie inside the event horizon, but by considering all possible slicings outer trapped surfaces can probably be found passing through every point inside the event horizon (Eardley (1997); Page 4) while trapped surfaces cannot (Ben-Dov (2007); Page 27). The distinction between trapped and outer trapped surfaces arise because the latter are required to be weakly trapped, that is, to have negative future null expansions both outwards and inwards to obviate the need for using a spatial hypersurface to provide the meaning of outer. It is remarkably difficult to determine the boundary of the region which contains trapped surfaces (Bengtsson et al. (2013); Page 2). This is also true for the simplest possible models of matter collapsing to form black holes, the Oppenheimer-Snyder (OS) and Vaidya solutions. Both of them are spherically symmetric and can be constructed by matching regions with collapsing matter to vacuum regions. In both cases they have a central world line surrounded by a tube of round marginally trapped surfaces (MTS) (Bengtsson et al. (2013); Page 2). In the case of the Vaidya model, this tube is spacelike and composed of outermost stable MTS and also lies outside the causal past of the central world line. But for the OS model, the tube is timelike, it is composed of unstable MTS, and is visible from the central world line (Bengtsson et al. (2013); Page 2).

In physically reasonable black hole spacetimes, trapped surfaces, marginally trapped surfaces and marginally trapped tubes (MTTs) always lie inside the black hole region, but MTTs act as a boundary between the regular space and the trapped region (Williams (2007); Page 3). Some of the MTTs have special names: a dynamical horizon (DH) is a MTT which is simply spacelike, while an isolated horizon (IH) is a MTT which is null (Ashtekar and Galloway (2005); Page 3). Dynamical and isolated horizons are likely to be good models of the surfaces of dynamical and equilibrium black holes respectively (Ashtekar and Galloway (2005); Page 3; Williams (2007): Page 3). In fact in numerical simulations of black holes, many people use DHs and IHs and some of those developing loop quantum gravity have found them to be well-suited for quantum considerations (Ashtekar and Krishnan (2004); Page 11; Schnetter et al. (2006); Page 3; Williams (2007): Page 3).

In this thesis, we investigate how black holes can be described in the frame work of quasi-local horizons, which takes trapped and marginally trapped surfaces as its starting point, provides a unified approach for studying various aspects of black hole physics. An important theme in this discussion is not to base our understanding of black holes on stationary spacetimes alone. We look at dynamical situations which have essential different features (Coley et al. (2017); page 1). We illustrate this by the Schwarzschild spacetime and Vaidya spacetime. We also investigate local horizons in axiallysymmetric spacetimes by considering the Kerr and KerrVaidya spacetimes. The eventual goal of these studies (from a physics viewpoint) is to explicitly demonstrate the existence of trapped surfaces and marginally trapped surfaces in black hole regions by computing the covariant divergences and the fluxes of null vectors in spherically and axisymmetric spacetimes. We discussed the inadequacy of event horizons for this purpose due to its teleological properties and it is desirable to find a suitable replacement. Penrose's trapped surfaces and the boundary of the trapped region seem ideally suited for this task and lead naturally to the various definitions of quasi-local horizons.

Chapter 3

PRELIMINARY NOTIONS IN DIFFERENTIAL GEOMETRY

The mathematical representation of spacetime in theoretical physics would presumably be with some set M whose elements representing 'spacetime points'. In practice, M should be a topological space whose open sets represent certain privileged regions in the spacetime. However, the topological spaces that have actually been used historically to represent spacetime (or space and time separately) are of a very special type. Namely, they have the property that it is possible to uniquely label any particular spacetime point by specifying the values of a finite set of real numbers, the number of which is identified as the 'dimension' of the space. Thus, in Newtonian physics, three-dimensional physical space is represented mathematically by the Euclidean space $\mathbf{R}^3$, and one-dimensional time is represented by $\mathbf{R}^1$; in special relativity, the combined notion of 'spacetime' is represented by the Euclidean space $\mathbf{R}^4$.

The use of such familiar mathematical models has an important implication that differentiation can be defined, thus opening up the very fruitful idea that the dynamical evolution of a physical system can be modeled by differential equations defined on the spacetime. Also, there is an explicit underlying topology on such spaces: namely, the metric-space topology induced by the usual metric function.

However, one of Einstein's major contributions to physics was his realization that, it is possible to generalize the mathematical model of spacetime whilst keeping the basic ideas of (i) being able to locate a spacetime point via the values of a set of real numbers; and (ii) being able to describe the dynamical evolution of a system using differential equations. Specifically, in general relativity a spacetime is modeled by a 'differentiable manifold', of which the Euclidean space $\mathbf{R}^4$ of special relativity is just a special example. This remains one of the major motivations for studying differential geometry. However, differential geometry enters into many other areas of modern theoretical physics, and it has become an indispensable tool for many scientists who work in these fields. We can now proceed to give the formal definition of a differentiable manifold. For convenience, and unless stated otherwise, it will be assumed that M is a connected Hausdorff topological space.

3.1 The Elements of Differential Geometry

Differential geometry deals with manifolds. A manifold, generally speaking is a topological space which is locally Euclidean. An Euclidean space of *n*-dimension, $\mathbf{R}^n$, is the set of all *n*-topples, $(x^1 \ldots x^n)(-\infty < x^i < +\infty)$, with open and close sets or neighbourhoods defined in the usual way (Chandrasekhar (1983); Page 3); Hawking and Ellis (1973); Page 11). A manifold *M*, is locally identical to Euclidean space because *M* is covered i.e. a union of neighbourhoods, u_α and that associated with each u_α there is a one-one map φ_α, which images each point $p \in u_\alpha$ to a point in an open neighbourhood of $\mathbf{R}^n$ onto which u_α is imaged by φ_α with coordinates $x^i,\ (i = 1, \ldots, n)$. Moreover, if two neighbourhoods, u_α and v_α of *M*, intersect and have points in common i.e. $u_\alpha \cap v_\alpha$ 6= {}, and if φ_α and ψ_α are the associated maps onto neighbourhoods in $\mathbf{R}^n$, then the map $\phi_\alpha \cap \psi_\alpha^{-1}$ images a point $\psi_\alpha(p)$, $p \in u_\alpha \cap v_\alpha$ with the coordinates $x_i^{-1},\ (i = 1, \ldots, n)$ then as a part of the definition of *M*, it is required that $x^i(i = 1,\ldots,n)$ are smooth functions of x^{-1} $(i = 1,\ldots,n)$) (Chandrasekhar (1983); Page 3). Smooth functions are functions which have continuous partial derivatives of all orders. Cartesian product $M \times N$ of two manifolds *M* and *N* is the ordered pair of points, (p,q), where $p \in M$ and $q \in N$; moreover if u_α and v_α are neighbourhoods in *M* and in *N* especially, φ and ψ are the associated maps, and $\phi_\alpha(p) = x^i, (i = 1, \ldots, n)$ and $\phi_\beta(q) = y^i, (i = 1, \ldots, m)$ where *m* is not necessarily equal to n, then the map

$$\varphi_\alpha \times \psi_\beta(p,q) = (x^1,\ldots,x^n)(y^1,\ldots,y^m) \tag{3.1}$$

Suffices to complete the definition of $M \times N$ as a manifold of $(m + n)$-dimensions. The study of differential manifolds involves topology, since differentiability implies continuity. The set $\mathbf{R}^n$ of n-tuples of real numbers is not only a vector space, but also a topological space and the vector operations are continuous with respect to the topology (Schutz (1980); Page 3).

Definition 3.1.1 *A topological space X is locally Euclidean of dimension n if for each $x \in X$, there exists an open set $U \subset X$ and a map $\varphi : U \to \mathbf{R}^n$ such that $\varphi : U \to \varphi(U)$ is a homeomorphism (in particular, $\varphi(U)$ is an open subset of $\mathbf{R}^n$)*

Definition 3.1.2 *A topological space X is Hausdorff if for any two points $p,q \in X$ such that p 6= q, there exist open neighborhoods U,V such that $p \in U$, $q \in V$ and $U \cap V = \emptyset$.*

A Hausdorff space M is called a topological if each point of M has a neighborhood homeomorphic to an open set in $\mathbf{R}^n$ (Schutz (1980); Page 3). An n-manifold is locally $\mathbf{R}^n$.

Examples: $\mathbf{R}^n$ and $n-$sphere S^n are $n-$manifolds. A 2−dimensional manifold is called a surface.

Definition 3.1.3 *A topological space X is second countable if it has a countable basis for the topology, i.e. there exists a countable collection of open sets $\{U_\alpha\}_{\alpha\in N}$ such that for any open set $U \subset X$ containing a point x, there exists $\beta \in N$ such that $x \in U_\beta \subseteq U$ (Schutz (1980); Page 3)*

Definition 3.1.4 (Topological Manifold) *A topological space M is a manifold of dimension n if*

(i) M is Hausdorff

(ii) M is second countable

(iii) M is locally Euclidean of dimension n

Definition 3.1.5 (Charts) *A chart on M is a pair* (U,φ) *where U is an open set in M and* $\varphi : U \to \varphi(U) \subseteq \mathbf{R}^n$ *is homeomorphism onto it image. The set U is*

called a coordinate domain or coordinate neighborhood or coordinate patch. If $\varphi(U)$ *is a ball in* $\mathbf{R}^n$*, U is called coordinate ball. A coordinate is* (U,φ) *is centered at P if* $\varphi(P) = 0$*. Given two charts* (U_1,φ_1) *and* (U_2,φ_2) *then we get overlap or transition maps*

$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2) \tag{3.2}$$
$$\phi_1 \circ \phi_2^{-1} : \phi_2(U_1 \cap U_2) \to \phi_1(U_1 \cap U_2) \tag{3.3}$$

Definition 3.1.6 (Atlas) *An atlas for M is a collection of coordinate charts* $\{U_\alpha, \varphi_\alpha\}, \alpha \in I$ *such that*

(i) M is covered by the $\{U_\alpha\}, \alpha \in I$

(ii) For each $\alpha,\beta \in I, \varphi_\alpha(U_\alpha, U_\beta)$ *is open in* $\mathbf{R}^n$

(iii) The map $\varphi_\beta : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ *is* C^∞ *with* C^∞ *inverse.*

Since a chart is a pair consisting of a neighborhood and its overlap, it is easy to see that these open neighborhoods must have overlaps if all points of *M* are to be included in at least one. It is these overlaps which enable us to give further characteristics of the manifold (Schutz (1980); Page 24).

Definition 3.1.7 (Differentiable Manifold) *If the partial derivatives of order k or less of all these functions* $\{y^i\}$ *with respect to all the* $\{x^i\}$ *exist and are continuous, then the maps f and g are strictly, the charts* (U,f) *and* (V,g) *are said to be* C^k*-related. If it is possible to construct a whole system of charts called appropriately enough, an atlas in* *such a way that every point of M is in at least one neighborhood and every chart is C^k-related to every other one it overlaps with, then the manifold M is said to be a C^k-manifold. The differentiability of a manifold endows it with an enormous amount of structure: the possibility of defining tensors differential forms and Lie derivatives (Schutz (1980); Page 26).*

Definition 3.1.8 *A curve or path in a manifold or affine space M is a (smooth unless otherwise stated) map $c : J \to M$, where J is an interval on the real line. The interval may be open or closed, finite or infinite at either end (O'Neill (1995); page 4).*
If for all choices of x,y ∈ M, there is a curve from x to y, M is path connected. ($\mathbf{R}\backslash\{0\}$ for instance is not path-connected as by the intermediate value theorem there is no path from -1 *to* $+1$*).*

Definition 3.1.9 (Tangent Vectors) *Tangent vectors are equivalent classes of curves that are tangent at a given point. That is, they have the same time derivative in a chart at that point.*

Definition 3.1.10 (Tangent Space) *If M is an n-manifold, and p ∈ M, then the tangent space at p is the set T_p of all tangent vectors at p or the tangent space T_pM is the space of all tangent vectors at point p (Schutz (1980); Page 34).*

3.1.1 Basis Vectors and Basis Vector Fields

At any point P, the space T_p is a vector space with the same dimension n as the manifold. Any collection of n linearly independent vectors in T_p is a basis for T_p. If we have a coordinate system $\{x^i\}$ in a neighborhood U of P, then the coordinates define the coordinate basis $\left\{\frac{\partial}{\partial x^i}\right\}$ at all points in U. But one need not use the coordinate basis: we could refer vectors to some arbitrary basis $\{\bar{e}_i\}$ where the subscript i is used as a label to distinguish one basis vector from another. At a point P, an arbitrary vector $\bar{V}$ can be written as

$$\bar{V} = \sum V^i \frac{\partial}{\partial x^i} = \sum_j V^j \bar{e}_j \tag{3.4}$$

(i) The numbers $\{V^i\}$ are the components of $\bar{V}$ on $\{\frac{\partial}{\partial x^i}\}$ and related to V^i by the usual vector transformation laws.

(ii) If $\bar{V}$ and the basis $\{\frac{\partial}{\partial x^i}\}$ and $\bar{e}$ are regarded as vector fields then the components $\{V^i\}$ and $\{V^j\}$ of the field V are functions on M.

(iii) A vector field is said to be differentiable if these functions are differentiable (Schutz (1980); Page 3).

Definition 3.1.11 (Tangent Bundles) *A particular manifold is formed by combining a manifold M with all its tangent spaces T_p. The simplest case: a onemanifold M (a curve) and its tangent spaces (lines tangent to it at each point) (Schutz (1980); Page 35)*

Definition 3.1.12 (One forms) *Let T_p be the space of all tangent vectors at P. we define a one-form as a linear real-valued function of the vectors. This means that a one form at P associates with a vector $\bar{V}$ at P a real number which we call $\tilde{\omega}$(Schutz (1980); Page 55).*

3.2 Lie Derivatives

In the concept of dragging the definition of a derivative along the congruence is permitted. Covariant differentiation introduces a rule that transports a tensor from one point to another at which the derivative is evaluated. This rule then introduces connection as a new structure on the manifold. In this section, we define Lie derivative as another type of derivative which does not introduce any additional structure (Schutz (1980); page 73).

Definition 3.2.1 (Lie Derivatives) *is the derivative along the congruence of a* *vector field.*

When we compare vectors at points λ and $\lambda + 4\lambda$ on a certain curve, we can Lie drag the vector at λ+4λ back to the point λ. This defines a new vector at λ, which can be subtracted from the old one to define the deference between them. Notice that this is a unique difference and hence a unique derivative given the congruence.

To derive the analytic expression for this, first consider a scalar function say *f*. Evaluate the scalar at the point $\lambda_0 + 4\lambda$, drag it back to λ_0, subtract the value of the scalar at λ_0, divide by 4λ and take the limit $4\lambda \to 0$. Its value at $\lambda_0 + 4\lambda$ is $f(\lambda_0 + 4\lambda)$. By dragging, we define a new scalar field f^*, whose value is defined by the rule $\frac{df^*}{d\lambda} = 0$. Therefore its value at λ_0 is the same as at $\lambda_0 + 4\lambda$: $f^*(\lambda_0) = f(\lambda_0 + 4\lambda)$. The derivative so defined is

$$\lim_{\triangle\lambda \to 0} \frac{f^*(\lambda_0) - f(\lambda_0)}{\triangle\lambda} = \lim_{\triangle\lambda \to 0} \frac{f(\lambda_0 + \triangle\lambda) - f(\lambda_0)}{\triangle\lambda} = \left[\frac{df}{d\lambda}\right]_{\lambda_0} \qquad (3.5)$$

There is a special notation for the Lie derivative operator $£_{\bar{V}}$ where $\bar{V}$ is the vector field generating the mapping $\left(\frac{d}{d\lambda}\right.$ in this case . So that the analytic expression is

$$£_{\bar{V}} f = \bar{V} f = \frac{df}{d\lambda} \qquad (3.6)$$

In the same way, for a vector field $\bar{U}$ of tangent vector $\overline{\frac{d}{d\mu}}$ its Lie derivative is $£_{\bar{U}} f = \bar{U} f = \frac{d}{d\mu}$. Since a vector is defined by its effect on functions, we use an arbitrary function *f* in the following. At λ_0 the field $\bar{U}$ gives derivative $\left(\frac{df}{d\mu}\right)_{\lambda_0}$ while at $\lambda_0 + 4\lambda$ it gives $\left(\frac{df}{d\mu}\right)_{\lambda_0+\triangle\lambda}$. By dragging $\bar{U}(\lambda_0 + 4\lambda)$, we get a new field $\bar{U}^* = \frac{d}{d\mu^*}$ defined by $[\bar{U}, \bar{V}] = 0$ and by $\bar{U}^*(\lambda_0 + 4\lambda) = \bar{U}(\lambda_0 + 4\lambda)$. The vanishing of the commutator implies

$$\frac{df}{d\lambda}\frac{d}{d\mu^*} f = \frac{d}{d\mu^*}\frac{df}{d\lambda} \qquad (3.7)$$

everywhere. Therefore for analytic vector fields, we have by Taylor's expansion

$$\left[\frac{d}{d\mu^*}f\right]_{\lambda_0} = \left[\frac{d}{d\mu^*}f\right]_{\lambda_0+\triangle\lambda} - \triangle\lambda\left[\frac{d}{d\lambda}\left(\frac{d}{d\mu^*}f\right)\right]_{\lambda_0} + \bigcirc(\triangle\lambda^2) \tag{3.8}$$

$$= \left[\frac{d}{d\mu^*}f\right]_{\lambda_0+\triangle\lambda} - \triangle\lambda\left[\frac{d}{d\mu^*}\left(\frac{d}{d\lambda}f\right)\right]_{\lambda_0} + \bigcirc(\triangle\lambda^2) \tag{3.9}$$

$$= \left[\frac{d}{d\mu^*}f\right]_{\lambda_0} + \triangle\lambda\left[\frac{d}{d\lambda}\left(\frac{d}{d\mu}f\right)\right]_{\lambda_0} - \triangle\lambda\left[\frac{d}{d\mu^*}\left(\frac{d}{d\lambda}f\right)\right]_{\lambda_0} + \bigcirc\left(\triangle\lambda^2\right) \tag{3.10}$$

The Lie derivative $\pounds_{\bar V}\bar U$ is defined as the vector field which operates on f to give

$$\left[\pounds_{\bar V}\bar U\right]f = \lim_{\triangle\lambda\to 0}\left[\frac{\left(\frac{d}{d\mu^*}\right)_{\lambda_0} - \left(\frac{d}{d\mu}\right)_{\lambda_0}}{\triangle\lambda}\right] = \lim_{\triangle\lambda\to 0}\left(\frac{d}{d\lambda}\frac{d}{d\mu}f - \frac{d}{d\mu^*}\frac{d}{d\lambda}f\right) \tag{3.11}$$

Now the difference between μ^* and μ is clearly a term of first order in 4λ, which means we can replace μ^* by μ in the last equation above. Since this equation is true for all f, we have

$$\pounds_{\bar V}\bar U = \frac{d}{d\lambda}\bar U - \frac{d}{d\mu}\bar V = [\bar V, \bar U] \tag{3.12}$$

The Lie derivative along some vector fields is a very important concept in geometry because it tells us how that geometric object changes as it is pushed along the congruence of curves that have the tangent vector $\bar V$ (Schutz (1980); Page 77).

Theorem 3.2.1 (Properties of Lie Derivatives)

1. $[\pounds_{\bar V}, \pounds_{\bar W}] = \pounds_{[\bar V, \bar W]}$ *also* $\pounds_{\bar V} + \pounds_{\bar W} = \pounds_{\bar V + \pounds\bar W}$

2. $[[\pounds_{\bar X}, \pounds_{\bar Y}], \pounds_{\bar Z}] + [[\pounds_{\bar Y}, \pounds_{\bar Z}], \pounds_{\bar X}] + [[\pounds_{\bar Z}, \pounds_{\bar X}], \pounds_{\bar Y}] = 0$ *Jacobi's Identity*

3. $\pounds_{\bar V}(A \otimes B) = (\pounds_{\bar V}A) \otimes B + A \otimes (\pounds_{\bar V}B)$ *Leibniz rule*

4. $\pounds_{\bar V}\bar W = (V^i W^j_{i} - W^i V^j_{i})\frac{\partial}{\partial x^j}$ *Coordinate basis*

5. $\pounds_{\bar V}\bar W = (V^i \bar e_i(W^j) - W^i \bar e_i(V^j))\bar e_j + V^i W^j \pounds_{\bar e_i}\bar e_j$ *General basis*

6. $\pounds_{\frac{\partial}{\partial x^j}} \bar{W} = W_i^j \frac{\partial}{\partial x^j} = \frac{\partial W^j}{\partial x^i}\frac{\partial}{\partial x^j} = \frac{\partial \bar{W}}{\partial x^i}$

$\pounds_{\bar{V}} \frac{\partial}{\partial x^j} = -\pounds_{\frac{\partial}{\partial x^j}} \bar{V} = \frac{\partial \bar{V}}{\partial x^i} = -\frac{\partial \bar{V}^j}{\partial x^i}\frac{\partial}{\partial x^i}$ *(Chandrasekhar (1983); Page 14)*

Coordinate free partial derivative

(Chandrasekhar (1983); Page 14)

3.2.1 Lie Derivative of a one-form

Fields of one-forms and tensors of higher rank are defined in terms of vector fields and scalar functions, so we can deduce the Lie derivatives of one-forms from the Lie derivatives of vectors and scalars (Schutz (1980); Page 78).

Conceptually, the definition is the same. That is a one-form field is said to be Lie dragged if its value on many Lie dragged vector fields is constant. The derivative is found by dragging the one-form at $\lambda_0+4\lambda$ back λ_0 and taking the difference. The result is that if $\tilde{\omega}$ is a one-form, and then $\pounds_{\bar{V}}\, \tilde{\omega}$ is the one-form field which is the Lie derivative of $\tilde{\omega}$ along $\bar{V}$ defined by the product rule (the Leibniz rule for first order derivative).

$$\pounds_{\bar{V}} \left[\tilde{\omega}(\bar{W})\right] = \left(\pounds_{\bar{V}}\tilde{\omega}\right)(\bar{W}) + \tilde{\omega}\left(\pounds_{\bar{V}}\bar{W}\right) \tag{3.13}$$

for all vector fields $\bar{W}$. Since $\tilde{\omega}(\bar{W})$ is simply a function, this defines $\pounds_{\bar{V}}\, \tilde{\omega}$ in terms of known operations, the Lie derivative of functions and vectors.

3.3 Curvature

Curvature is a very important concept in general relativity, and how it can be described in tensorial terms. It is a mathematical quantity involving the second derivative of the metric which represents the essence of a curved space: the space is curved if the curvature does not vanish.

3.3.1 Covariant Derivative

A derivative operator ∇, (sometimes called a covariant derivative) on a manifold M is a map which takes each smooth (or merely differentiable) tensor filed of type (k,l) to a smooth tensor field of type (k,l+1) having the following properties

1. Linear: $\nabla(T + S) = \nabla T + \nabla S$
2. Leibniz (product) rule: $\nabla(T \otimes S) = \nabla T \otimes S + T \otimes (\nabla S)$

If the operator ∇ obeys Leibniz rule, then it can be written as the partial derivative plus some linear transformation (Carroll (2004); page 95). So, to take the covariant derivative, take the partial derivative first and then apply a correction to make the result covariant. Consider the covariant derivative of a vector V^ν. For each direction μ, the covariant derivative ∇_μ will be given by the partial derivative ∂_μ plus a correction term given by the matrix $\Gamma^\rho_{\mu\sigma}$ (a $n \times n$ matrix, where n is the dimensionality of the manifold, for each μ). We therefore have

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \tag{3.14}$$

Since equation (3.14) is the covariant derivative of a vector in terms of the partial derivative, we can determine the transformation properties of $\Gamma^\nu_{\mu\lambda}$ by demanding that the left hand side be a type (1,1) tensor. We can therefore write the transformation law as

$$\nabla_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \nabla_\mu V^\nu \tag{3.15}$$

Expanding the left hand side of equation (3.15) using equation (3.14) and doing the necessary transformation, we have

$$\nabla_{\mu'} V^{\nu'} = \partial_{\mu'} V^{\nu'} + \Gamma^{\nu'}_{\mu'\lambda'} V^{\lambda'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \partial_\mu V^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} V^\nu \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\nu} + \Gamma^{\nu'}_{\mu'\lambda'} \frac{\partial x^{\lambda'}}{\partial x^\lambda} V^\lambda \tag{3.16}$$

Expanding the right hand side of equation (3.15) we have

$$\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\nabla_{\mu}V^{\nu} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\partial_{\mu}V^{\nu} + \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\Gamma^{\nu}_{\mu\lambda}V^{\lambda} \tag{3.17}$$

Equating equations (3.16) and (3.17) the first terms being identical will cancel, we have

$$\Gamma^{\nu'}_{\mu'\lambda'}\frac{\partial x^{\lambda'}}{\partial x^{\lambda'}}V^{\lambda} + \frac{\partial x^{\mu}}{\partial x^{\mu'}}V^{\nu}\frac{\partial^2 x^{\nu'}}{\partial x^{\mu}\partial x^{\nu}} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\Gamma^{\nu}_{\mu\lambda}V^{\lambda} \tag{3.18}$$

Here, the dummy index ν has been changed to λ. Equation (3.18) should be true for all V^{λ}. The Christoffel symbols (connection coefficients) in the primed coordinates may be isolated by multiplying each side by $\frac{\partial x^{\lambda}}{\partial x^{\lambda'}}$, we have

$$\Gamma^{\nu'}_{\mu'\lambda'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\lambda}}{\partial x^{\lambda'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\Gamma^{\nu}_{\mu\lambda} - \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\lambda}}{\partial x^{\lambda'}}\frac{\partial^2 x^{\nu'}}{\partial x^{\mu}\partial x^{\nu}} \tag{3.19}$$

This can be put in the form

$$\Gamma^{\nu'}_{\mu'\lambda'} = \Lambda^{\mu}_{\mu'}\Lambda^{\lambda}_{\lambda'}\Lambda^{\nu'}_{\nu}\Gamma^{\nu}_{\mu\lambda} - \Lambda^{\mu}_{\mu'}\Lambda^{\lambda}_{\lambda'}(\nabla_{\lambda}\Lambda^{\nu}_{\mu}) \tag{3.20}$$

This is not defined as a tensor transformation law because the second term on the right spoils it. This is true because the Christoffel symbols or connection coefficients are not components of a tensor. They are constructed purposely to be non-tensorial but in a way that the combination (3.14) transforms like a tensor. The extra terms in the transformation of the partials and the Γ's exactly cancel (Carroll (2004); Page 95).

3.4 Covariant Derivative of a one-form

One-form also has covariant derivative expressed as a partial derivative plus some linear transformation. Let us write something like

$$\nabla_{\mu}\omega_{v} = \partial_{\mu}\omega_{\nu} + \tilde{\Gamma}^{\lambda}_{\mu\nu}\omega_{\lambda} \tag{3.21}$$

where $\tilde{\Gamma}^{\lambda}_{\mu\nu}$ is a new set of matrices for each μ.

In addition to properties (1) and (2) above, covariant derivative has the following properties

(2) Commutes with contraction: $\nabla_\mu(\Gamma^{\lambda}{}_{\lambda\rho}) = (\nabla\Gamma)^{\mu\lambda}{}_{\lambda\rho}$

(3) Reduces to the partial derivative on scalars: $\nabla_\mu\varphi = \partial_\mu\varphi$

Let us look at the implications of these new properties.

Given a one-form field ω_μ and vector field V^μ we can take the covariant derivative of the scalar defined by $\omega_\lambda V^\lambda$ to get

$$\nabla_\mu(\omega_\lambda V^\lambda) = (\nabla_\mu\omega_\lambda)V^\lambda + \omega_\lambda(\nabla_\nu V^\lambda) = (\partial_\mu\omega_\lambda)V^\lambda + \tilde{\Gamma}_{\sigma\mu\lambda}\omega_\sigma V^\lambda + \omega_\lambda(\partial_\mu V^\lambda) + \Gamma_{\mu\rho\lambda}V^\rho \quad (3.22)$$

Since $\omega_\lambda V^\lambda$ is a scalar, it can also be given by the partial derivative

$$\nabla_\mu(\omega_\lambda V^\lambda) = \partial_\mu(\omega_\mu V^\mu) = (\partial_\mu\omega_\lambda)V^\lambda + \omega_\lambda(\partial_\mu V^\lambda) \quad (3.23)$$

This is also true if the terms in (3.22) whose Christoffel symbols or connection coefficients cancel each other: equating (3.22) and (3.23) and rearranging dummy indices, we have

$$\tilde{\Gamma}_{\sigma\mu\lambda} = -\Gamma_{\sigma\mu\lambda} \quad (3.24)$$

where ω_σ and V^λ are completely arbitrary.

The two extra conditions we imposed have allowed us to express the covariant derivative of a one-form using the same connection coefficients as were used for the vector but with a sign now changed to minus and indices muched up differently. Hence, the covariant derivative of a one-form is

$$\nabla_\mu\omega_\nu = \partial_\mu\omega_\nu - \Gamma^{\lambda}_{\mu\nu}\omega_\lambda \quad (3.25)$$

We should not be surprise that the connection coefficients encode all the information necessary to take the covariant derivative of a tensor of arbitrary rank.

The formula is deduced as follows; for each upper index you introduce a term with a single +Γ and for each lower index a term with a single index −Γ:

$$\nabla_\sigma T^{\mu_1\mu_2\ldots\mu_k}_{\nu_1\nu_2\ldots\nu_l} = \partial_\sigma T^{\mu_1\mu_2\ldots\mu_k}_{\nu_1\nu_2\ldots\nu_l} + \Gamma^{\mu_1}_{\sigma\lambda} T^{\lambda\mu_2\ldots\mu_k}_{\nu_1\nu_2\ldots\nu_l} + \Gamma^{\mu_2}_{\sigma\lambda} T^{\mu_1\lambda\ldots\mu_k}_{\nu_1\nu_2\ldots\nu_l} + \cdots -$$

$$\Gamma^{\lambda}_{\sigma\nu_1} T^{\mu_1\mu_2\ldots\mu_k}_{\lambda\nu_2\ldots\nu_l} - \Gamma^{\lambda}_{\sigma\nu_2} T^{\mu_1\mu_2\ldots\mu_k}_{\nu_1\lambda\ldots\nu_l} - \ldots \qquad (3.26)$$

(Carroll (2004); Page 96)

3.5 The Riemann-Christoffel Curvature Tensor

The second derivatives do not in general commute on a type (p,q)-tensor unless p and q are equal. However, the commutator of covariant derivatives which act on a vector field say V^μ does not involve any derivatives of that vector. We have

$$\nabla_\alpha \nabla_\beta V^\mu = \nabla_\alpha (V^\mu_\beta) = V^\mu_{;\beta,\alpha} + \Gamma^\mu_{\alpha\alpha} V^\sigma_{;\beta} - \Gamma^\sigma_{\beta\alpha} V^\mu_{;\alpha} \qquad (3.27)$$

In a locally inertial frame $\Gamma^\mu_{\alpha\alpha}$, equation (3.27) becomes

$$\nabla_\alpha \nabla_\beta V^\mu = V^\mu_{;\beta,\alpha} = V^\mu_{,\beta,\alpha} + \Gamma^\mu_{\nu\beta,\alpha} V^\nu \qquad (3.28)$$

Interchanging α and β in (3.28)

$$\nabla_\alpha \nabla_\beta V^\mu = V^\mu_{;\alpha,\beta} = V^\mu_{,\alpha,\beta} + \Gamma^\mu_{\nu\alpha,\beta} V^\nu \qquad (3.29)$$

The commutator of the covariant derivative is

$$[\nabla_\alpha, \nabla_\beta] V^\mu = \nabla_\alpha \nabla_\beta V^\mu - \nabla_\beta \nabla_\alpha V^\mu = (\Gamma^\mu_{\nu\beta,\alpha} - \Gamma^\mu_{\nu\alpha,\beta}) V^\nu \qquad (3.30)$$

In a locally inertial frame $R^\mu_{\nu\alpha,\beta} = \Gamma^\mu_{\nu\beta,\alpha} - \Gamma^\mu_{\nu\alpha,\beta}$

$$[\nabla_\alpha, \nabla_\beta] V^\mu = R^\mu_{\nu\alpha\beta} \qquad (3.31)$$

This is a tensor equation and since it is valid in a given reference frame, it will be valid in any frame. Equation (3.31) implies that in curved spacetime covariant derivatives do not commute and therefore the order in which they appear is important. The commutator $[\nabla_\alpha,\nabla_\beta]V^\mu$ does not depend on derivatives of V. This implies we can express the commutator purely algebraically in terms of V. Since the dependence on V is clearly linear, the commutator of covariant derivatives acts as a linear transformation. By calculating the commutator explicitly, one can confirm the structure displayed in (3.30) as the Riemann-Christoffel Curvature Tensor or Riemann tensor is given by

$$R^{\mu}_{\nu\alpha\beta} = \partial_\alpha \Gamma^{\mu}_{\nu\beta} - \partial_\beta \Gamma^{\mu}_{\nu\beta} + \Gamma^{\mu}_{\alpha\rho}\Gamma^{\rho}_{\beta\nu} - \Gamma^{\mu}_{\beta\rho}\Gamma^{\rho}_{\alpha\nu} \qquad (3.32)$$

The above can be extended to an action of the commutator $[\nabla_\alpha,\nabla_\beta]$ on arbitrary tensors (Carroll (2004); Page 122).

For covectors, since we can raise and lower the indices with the metric so we have

$$[\nabla_\alpha,\nabla_\beta]V_\rho = g_{\rho\lambda}[\nabla_\alpha,\nabla_\beta]V^\lambda = g_{\rho\lambda}R_{\sigma\sigma\beta\lambda} \quad V^\sigma = R_{\rho\sigma\sigma\beta}V^\sigma = R_{\rho\sigma\beta\sigma} \quad V_\sigma \qquad (3.33)$$

Since the Riemann tensor is anti-symmetric in its first two indices, we can also write

$$[\nabla_\alpha, \nabla_\beta]V_\rho = -R^{\sigma}_{\rho\sigma\beta}V_\sigma \qquad (3.34)$$

When we extend to arbitrary tensor of type (p,q), it follows the usual pattern, with one Riemann curvature tensor, contracted as a vector, appearing for each of the p upper indices, and one Riemann curvature tensor, contracted as a convector, for each of the q lower indices. This implies, for a tensor of type (2,0) or $T^{\alpha\beta}$

$$[\nabla_\alpha,\nabla_\beta]T_{\mu\nu} = R_{\gamma\sigma\beta\sigma} \quad T_{\gamma\nu} + R_{\gamma\sigma\beta\nu} \quad T_{\alpha\gamma} \qquad (3.35)$$

And for (1,1)-tensor T^{λ}_{ρ}, we have

$$[\nabla_\alpha, \nabla_\beta]T^{\sigma}_{\rho} = R^{\lambda}_{\gamma\sigma\beta}T^{\sigma}_{\rho} + R^{\lambda}_{\rho\sigma\beta}T^{\lambda}_{\sigma} \quad (3.36)$$

3.5.1 The Algebraic Properties of Riemann Tensor

A type (0,4)-tensor $R_{\nu\lambda\alpha\beta}$ can be constructed from the Riemann Christoffel curvature tensor $R_{\nu\alpha\beta}{}^{\mu}$ by taking its inner product with the metric tensor $g_{\lambda\mu}$ i.e.

$$g_{\lambda\mu}R_{\nu\alpha\beta\mu} = R_{\nu\lambda\alpha\beta} \quad (3.37)$$

This type of (0,4)-tensor is usually called the covariant curvature tensor which is given by

$$R_{\nu\lambda\alpha\beta} = \frac{1}{2}(g_{\nu\alpha,\beta\lambda} + g_{\lambda\beta,\alpha\nu} - g_{\lambda\alpha,\beta\nu} - g_{\nu\beta,\nu\beta,\alpha\lambda}) \quad (3.38)$$

Using equation (3.38) it is very simple to read off all the symmetries in a different way (Carroll (2004); Page 126). However, we can also derive these symmetries in a different way which will also make clear why Riemann tensor has these symmetries

1. Anti-symmetry in the second pair of indices $R_{\nu\lambda\alpha\beta} = -R_{\nu\lambda\beta\alpha} = -R_{\lambda\nu\alpha\beta} = -R_{\lambda\nu\beta\alpha}$
2. Symmetry $R_{\nu\lambda\alpha\beta} = R_{\alpha\beta\nu\lambda}$
3. First Bianchi identities

$$R_{\nu[\lambda\alpha\beta]} = 0 \Leftrightarrow R_{\nu\lambda\alpha\beta} + R_{\nu\beta\lambda\alpha} + R_{\nu\alpha\beta\lambda} = 0 \quad (3.39)$$

   This Bianchi identity is due to no torsion

4. $R_{\nu\lambda\alpha\beta}$ has $\frac{1}{12}n^2(n^2 - 1)$ independent components.

3.5.2 The Covariant Divergence of a Vector Field

The covariant divergence of a vector field means the scalar defined by

$$\mathrm{div}\bar{V} = \partial_i V^i + \Gamma^i_i V^k \tag{3.40}$$

This is a scalar in all frames and reduces to the familiar form in a Cartesian system. Consider the Christoffel symbol

$$\Gamma^i_{ij} = \frac{1}{2} g^{ik} \left[\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{ij}}{\partial x^k} \right] \tag{3.41}$$

Interchange *i* and *k* in the last two terms, equation (3.41) becomes

$$\Gamma^i_{ij} = \frac{1}{2} g^{ik} \left[\frac{\partial g_{ik}}{\partial x^j} \right] \tag{3.42}$$

Relating (3.42) to the determinant of g_{ij}, we have $\frac{\partial g}{\partial x^i} = \frac{\partial g_{ik}}{\partial x^i} G^{ik}$. Where G^{ik} is the cofactor of g_{ik} but $G^{ik} = g^{ik} g$, we have

$$\frac{1}{g} \frac{\partial g}{\partial x^i} = g^{ik} \frac{\partial g_{ik}}{\partial x^j} \tag{3.43}$$

Substituting (3.43) into (3.42), we obtain

$$\Gamma^i_{ij} = \frac{1}{2g} \frac{\partial g}{\partial x^j} = \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^j} \tag{3.44}$$

Substituting (3.44) into (3.40), the covariant divergence can be written compactly as

$$\mathrm{div}\bar{V} = \nabla_i V^i = \frac{\partial V^i}{\partial x^i} + \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^j} V^i = \frac{1}{\sqrt{g}} \left(\sqrt{g} \frac{\partial V^i}{\partial x^i} + \frac{\partial \sqrt{g}}{\partial x^j} V^i \right) = \frac{1}{\sqrt{g}} \frac{\partial (\sqrt{g} V^i)}{\partial x^i} \tag{3.45}$$

From this equation, to calculate the covariant divergence of a vector field, we just have to calculate g and its derivative, not the Christoffel symbols (Carroll (2004); Page 101).

This formula is very useful and provides the quickest way to write the ordinary flat space divergence of a vector calculus on R3 for example, cylindrical or polar coordinates.

The divergence of a 3-vector $\bar{V}$ in Cartesian coordinates (x^1,x^2,x^3), is given by the expression

$$\operatorname{div}\bar{V} = \frac{\partial V^i}{\partial x^i} \qquad (3.46)$$

In spherical polar coordinates (r,θ,φ), using (3.45) and $\sqrt{g} = r^2\sin\theta$, the divergence is given

$$\operatorname{div}\bar{V} = \nabla_i V^i = \frac{1}{r^2\sin\theta}\left[\partial_r(r^2\sin\theta V^\gamma) + \partial_\theta(r^2\sin\theta V^\theta) + \partial_\phi(r^2\sin\theta V^\phi)\right] \qquad (3.47)$$

3.6 Killing Vector Fields

General relativity is one of the fields of physics in which solutions with symmetry are needed. This is due to the fact that the nonlinear nature of Einstein's equation makes it hard to find any exact solutions. In the context of curved spacetime, however, we need to be more careful about what exactly is meant by symmetry. In this section we develop some useful tools for studying symmetry. A manifold M is said to possess a symmetry if the geometry is invariant under a certain transformation that maps M to itself; that is, if the metric is the same, in some sense, from one point to another. In fact different tensor fields may possess different symmetries; symmetries of the metric are called isometries (Carroll (2004); Page 134). Sometimes the existence of isometries is obvious; for example, in the four-dimensional Minkowski space, $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$, several isometries of this space are known; these include translations ($x^\mu \rightarrow x^\mu + a^\mu$ with a^μ fixed) and Lorentz transformation ($x^\mu \rightarrow \Lambda^\mu{}_\nu x^\nu$ with $\Lambda^\mu{}_\nu$ a Lorentz matrix). The fact that the metric is invariant under transformations is made immediately apparent by the fact that the coefficients of the metric are not dependent of the individual coordinate functions x^μ (Carroll (2004); Page 134). The solution of a physical problem can be considerably simplified if it allows some symmetry. If we consider for example the equations of Newtonian gravity, it is easy to find a solution which is spherically symmetric, but it may be difficult to find the analytic solution for an arbitrary mass distribution.

In Euclidean space, symmetry is related to invariance with respect to some operation. For example plane symmetry implies invariance of the physical variables with respect to translations on a plane, spherically symmetric solutions are invariant with respect to translation on a sphere, and the equations of Newtonian gravity are symmetric with respect to time translations $t^0 \rightarrow t + \tau$. Thus, symmetry corresponds to invariance under translations along certain lines or over certain surfaces. This definition can be applied and extended to Riemannian geometry. A solution of Einstein's equations has symmetry if there exist an n-dimensional manifold, with $1 \leq n \leq 4$, such that the solution is invariant under translations which bring a point of this manifold into another point of the same manifold. For example, for spherically symmetric solutions the manifold is the 2-sphere, and $n = 2$. This is a simple example, but there exist more complicated four-dimensional symmetries. These definitions can be made more precise by introducing the notion of Killing vectors.

A Killing vector field (often just Killing field), named after Wilhelm Killing, is a vector field on a Riemannian manifold or pseudo-Riemannian manifold that preserves the metric. Killing fields are the infinitesimal generators of isometries; that is, flows generated by Killing fields are continuous isometries of the manifold. More simply, the flow generates symmetry, in the sense that moving each point on an object the same distance in the direction of the Killing vector field will not distort distances on the object.

Consider a vector field $\bar{\xi}(x^\mu)$ defined at every point x^α of a spacetime region. $\bar{\xi}$ identifies a symmetry if an infinitesimal translation along $\bar{\xi}$ leaves the line-element unchanged i.e.

$$\pounds_\xi g_{\alpha\beta} = 0 \tag{3.48}$$

Killing vectors are intimately related to the symmetries of the spacetime. The existence of a Killing vector tells us immediately about the symmetries of the metric. If, for instance, $\xi^\alpha = \partial_q$ is a Killing vector for some manifold, then it is always possible to arrange a coordinate system, with q as one of the coordinates, so that the metric components with respect to that coordinate basis do not depend on q (Schutz (1980); page 88). To see this, let us assume $\xi^\alpha = \partial_q$ and consider

$$\begin{aligned}\nabla_\alpha \xi^\beta + \nabla^\beta \xi_\alpha &= \nabla_\alpha \xi^\beta + g^{\beta\mu} g_{\alpha\nu} \nabla_\mu \xi^\nu = \Gamma^\beta_{\alpha\lambda}\xi^\lambda + g^{\beta\mu} g_{\alpha\nu}\Gamma^\nu_{\mu\lambda}\xi^\nu \\ &= \frac{1}{2} g^{\beta\gamma}(\partial_\alpha g_{\lambda\gamma} + \partial_\lambda g_{\alpha\lambda} - \partial_\gamma g_{\lambda\alpha} + \partial_\gamma g_{\alpha\lambda} + \partial_\gamma g_{\gamma\alpha} - \partial_\alpha g_{\gamma\lambda})\xi^\lambda\end{aligned}$$

Therefore, if the metric is independent of q, $\xi^\alpha = \partial_q$ (which is always possible), then the metric will be independent of q in that system. The physical meaning of the Killing's equation (3.48) is that when the metric is dragged along some congruence of curves then it is unchanged, thereby telling us that the paths to which this vector is tangent constitute symmetry of the manifold, at least locally (Schutz (1980); page 88).

3.6.1 Relationship Between Curvature Tensor and Killing Vectors

The relationship between symmetries and geometry in (pseudo-) Riemannian geometry is reflected in the relation between the curvature tensor and Killing vectors of a metric. In this section, we explore some of these relations and their consequences.

The defining relation in Riemann curvature tensor is given by (Poisson (2002); page 8),

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) V_\lambda = -R^\rho_{\lambda\mu\nu} V_\rho \tag{3.49}$$

By the cyclic permutation symmetry (or first Bianchi identity) we can deduce that for a Killing vector K^μ

$$\nabla_{(\mu} K_{\nu)} = \nabla_\mu K_\nu + \nabla_\nu K_\mu = 0 \tag{3.50}$$

By definition of the Riemann tensor, we have

$$\nabla_\mu \nabla_\nu K_\lambda - \nabla_\nu \nabla_\mu K_\lambda = R_{\mu\nu\lambda\alpha} K_\rho \tag{3.51}$$

By Killings equation (3.50), we can write equation (3.51) as

$$\nabla_\mu \nabla_\nu K_\lambda + \nabla_\nu \nabla_\lambda K_\mu = R^\alpha_{\mu\nu\rho} K_\rho \tag{3.52}$$

By writing down the same equation with cyclic permutations of the indices $(\mu\nu\lambda)$ and then add the $(\mu\nu\lambda)$ equation to the $(\nu\lambda\mu)$ equation and subtract the $(\lambda\mu\nu)$ equation, we obtain

$$2\nabla_\nu \nabla_\lambda K_\mu = \left[R^\rho_{\mu\nu\lambda} + R^\rho_{\lambda\nu\mu} - R^\rho_{\lambda\mu\nu}\right] K_\rho = -2R^\rho_{\lambda\mu\nu} K_\rho \tag{3.53}$$

Where the symmetry property of the Riemann tensor was used in obtaining (3.53). Hence, for a Killing field K^μ, we have the equation

$$\nabla_\lambda \nabla_\nu K_\rho = R^\rho_{\lambda\mu\nu} K_\rho \tag{3.54}$$

The quantity $\nabla_\nu K_\lambda$ is anti-symmetric so the total anti-symmetrization is equivalent to cyclic permutation, and we therefore have

$$\nabla_\mu \nabla_\nu K_\lambda + \nabla_\nu \nabla_\lambda K_\mu + \nabla_\lambda \nabla_\mu K_\nu = 0 \tag{3.55}$$

Applying the Killing property in the second term, we can write

$$\nabla_\lambda \nabla_\mu K_\nu - [\nabla_\mu, \nabla_\lambda] K_\lambda = R_{\lambda\mu\nu\rho} \quad K_\rho \tag{3.56}$$

This identity implies the Lie derivative of the Christoffel symbols of a metric along a Killing vector of the metric vanishes. Indeed, we can see that under a general variation of the metric, the induced variation of the Christoffel symbol can be written as follows

$$\delta\Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\rho} \left(\nabla_\nu \delta g_{\rho\nu} - \nabla_\rho \delta g_{\nu\lambda}\right) \tag{3.57}$$

In deed, this shows the fact that the metric variation of the Christoffel symbols is a tensor and moreover provides us with an explicit expression for this tensor.

Taking variation $\delta g_{\mu\nu} = L_\xi g_{\mu\nu}$ as the Lie derivative, i.e. the variation in the metric induced by an infinitesimal coordinate transformation $\delta x^\mu = \xi^\mu$, one can write this as

$$L_\xi \Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\rho} (\nabla_\nu L_\xi g_{\rho\lambda} + \nabla_\lambda L_\xi g_{\rho\nu} - \nabla_\rho L_\xi g_{\nu\lambda}) \tag{3.58}$$

By adopting this definition and using $L_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$, the right-hand side of (3.58) an be written using the definition and cyclic symmetry of the Riemann tensor

$$L_\xi \Gamma_{\mu\nu\lambda} = \nabla_\lambda \nabla_\nu \xi_\mu - R_{\nu\lambda\rho\mu}\, \xi_\rho = \nabla_\nu \nabla_\lambda \xi_\mu - R_{\lambda\rho\nu\mu}\, \xi_\rho \tag{3.59}$$

In particular, if $\xi^\mu = K^\mu$ is a Killing vector, one has

$$L_K g_{\mu\nu} = 0 \Rightarrow L_K \Gamma_{\mu\nu\lambda} = 0 \Leftrightarrow \nabla_\lambda \nabla_\nu K_\mu = R_{\mu\nu\lambda\rho} K_\rho \tag{3.60}$$

Contracting (3.54) over λ and ν we obtain a very useful and frequently used identity

$$\nabla^\nu \nabla_\mu K_\nu = K^\nu R_{\mu\nu} \tag{3.61}$$

One immediate consequence of identity (3.54) is by contracting with K^μ ”integrating by parts” and using the anti-symmetry of $\nabla_\mu K_\nu$ we have

$$R_{\mu\nu}K^\mu K^\nu = (\nabla^\mu K^\nu)(\nabla_\mu K_\nu) + \nabla_\nu(K^\mu \nabla_\mu K^\nu) \quad (3.62)$$

3.6.2 Lie Derivative of Type (0,2)-tensor

The variation of a tensor under an infinitesimal translation along the direction of a vector field $\bar{\xi}$ is the Lie-derivative ($\bar{\xi}$ must not necessarily be a Killing vector), and it is indicated as $\pounds_{\bar{\xi}}$. The Lie derivative of type (0,2)-tensor is given by

$$\pounds_{\bar{\xi}} T_{\alpha\beta} = T_{\alpha\beta.\mu}\xi^\mu + T_{\delta\beta}\xi^\delta_{.\alpha} + T_{\alpha\delta}\xi^\delta_{.\beta} \quad (3.63)$$

The Lie derivative of a metric $g_{\mu\nu}$ is given by

$$L_\xi g_{\mu\nu} = g_{\lambda\nu}\nabla_\mu \xi_\lambda + g_{\mu\nu}\nabla_\nu \xi_\lambda \quad (3.64)$$

By lowering the index of ξ with the metric, we can write this more compactly as

$$L_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \quad (3.65)$$

The vector field $\bar{\xi}$ which satisfies this equation is called a Killing vector Since they are associated with symmetries of space time, and since symmetries are always of fundamental importance in physics, Killing vectors will play an important role in conserved quantities in geodesic motion, Killing vectors and the choice of coordinate systems, Generalization of the notion of Killing vector fields etc.

3.6.3 Killing Vectors and the Choice of Coordinate Systems

The existence of Killing vectors remarkably simplifies the problem of choosing a coordinate system appropriate to solve Einstein’s equations. For instance, if we are looking for a solution which admits a timelike Killing vector $\bar{\xi}$, it is convenient to choose, at each point of the manifold, the timelike basis vector $\bar{e}_0$ aligned with $\bar{\xi}$, with this choice, the time coordinate lines coincide with the worldlines to which $\bar{\xi}$ is tangent, i.e. with the congruence of worldlines of $\bar{\xi}$, and the components of $\bar{\xi}$ are $\xi^\alpha = (\xi^0,0,0,0)$ If we parameterize the coordinate curves associated to $\bar{\xi}$ in such a way that ξ^0 is constant or equal unity, then $\xi^\alpha = (1,0,0,0)$ and

$$\frac{\partial g_{\alpha\beta}}{\partial x^0} = 0 \tag{3.66}$$

This means that if the metric admits a timelike Killing vector, with an appropriate choice of the coordinate system it can be made independent of time (Poisson (2002); page 8).

A similar procedure can be used if the metric admits a spacelike Killing vector. In this case, by choosing one of the spacelike basis vectors, say the vector $\bar{e}_1$, parallel to $\bar{\xi}$ and by a suitable reparameterization of the corresponding congruence of coordinate lines, one can write $\xi^\alpha = (0,1,0,0)$ and with this choice the metric is independent of

$$\frac{\partial g_{\alpha\beta}}{\partial x^1} = 0 \tag{3.67}$$

If the Killing vector is null, starting from the coordinate basis vectors $\bar{e}_0, \bar{e}_1, \bar{e}_2, \bar{e}_3$ it is convenient to construct a set of new basis vectors $\bar{e}_{\alpha'} = \Lambda^{\beta}_{\alpha'} \bar{e}_{(\beta)}$ such that the vector is a null vector. Then, the vector $\bar{e}_0$ can be chosen to be parallel to $\bar{\xi}$ at each point of the manifold, and by a suitable reparameterization of the corresponding coordinate lines $\xi^\alpha = (1,0,0,0)$ and the metric is independent of x^0.

$$\frac{\partial g_{\alpha\beta}}{\partial x^{0'}} = 0 \tag{3.68}$$

The map $f_t : M \rightarrow M$ under which the metric is unchanged is called an isometry, and the Killing vector field is the generator of the isometry. The congruence of worldlines of the vector $\bar{\xi}$ can be found by integrating the equations $\frac{dx^\mu}{d\lambda} = \xi^\mu(x^\alpha)$.

3.7 Conserved Quantities in Geodesic Motion

In Newtonian mechanics conservation laws are connected to symmetries. To conserve energy, for example, the force must be conserved and derivable from a potential and that potential must be time independent. To conserve linear momentum along a particular direction, the potential must be constant along that direction. To conserve angular momentum, the potential must be spherically symmetric. In short, energy is conserved when there is symmetry under displacements in time, linear momentum is conserved when there is symmetry under displacements in space, and angular momentum is conserved when there is symmetry under rotations.

Conserved quantities for the motion of test particles cannot be expected in a general spacetime that has no special symmetries. A general spacetime metric is time dependent, angle dependent, position dependent, etc.

However, when the spacetime has symmetry, then there is an associated conservation law. For example, if spacetime geometry is independent of time, there is a conserved energy for test particles. How does one tell if spacetime geometry has symmetry? One case is if the metric is not dependent of one of the coordinates, say, then the transformation $x^1 \rightarrow x^1 + const$ leaves the metric unchanged. The vector ξ^α = (0,1,0,0) lies along a direction in which the metric does not change. The vector ξ^α is called the Killing vector associated with the symmetry $x^1 \rightarrow x^1 + const$. A Killing vector is a general way of characterizing symmetry in any coordinate system. Killing vector is used to find constants of the motion for particles following geodesics. This can be seen by considering some geodesic trajectory with 4-velocity $u^\alpha = \frac{dx^\alpha}{d\tau}$, where τ is improper time such that $u^\alpha u_\alpha = -1$. Now, consider

$$\begin{aligned} u^\alpha \nabla_\alpha u^\beta &= u^\alpha(\partial_\alpha u^\beta + \Gamma^\beta_{\alpha\gamma} u^\gamma) = \frac{dx^\alpha}{d\tau}\left(\frac{\partial}{\partial x^\alpha}\frac{du^\beta}{d\tau} + \Gamma^\beta_{\alpha\gamma}\frac{du^\gamma}{d\tau}\right) = 0 \qquad (3.69) \\ &= \frac{dx^\alpha}{d\tau}\frac{\partial}{\partial x^\alpha}\frac{du^\beta}{d\tau} + \Gamma^\beta_{\alpha\gamma}\frac{dx^\alpha}{d\tau}\frac{du^\gamma}{d\tau} = \frac{d^2x^\alpha}{d\tau^2} + \Gamma^\beta_{\alpha\gamma}\frac{dx^\alpha}{d\tau}\frac{du^\gamma}{d\tau} \end{aligned}$$

Here, the last step follows from the geodesic equation. Hence, $u^\alpha \nabla^\beta{}_\alpha = 0$ is the same as the geodesic equation. One of the most useful properties of Killing vector fields is given by the following proposition

Proposition 3.7.1 *Let ξ^α be a Killing vector field and let γ be a geodesic with tangent u^α. Then $u^\alpha \xi_\alpha$ is constant along γ*

Proof 1

$$u_\alpha \nabla_\beta(\xi_\alpha u_\alpha) = u_\beta u_\alpha \nabla_\beta \xi_\alpha + \xi_\alpha u_\beta \nabla_\beta u_\alpha$$

Since the first term of the right hand side vanishes by geodesic equation (3.69) and the second term vanishes by the geodesic equation. Hence, $\xi_\alpha u^\alpha$ is conserved along the trajectory.

Since in general relativity timelike geodesics represent the spacetime motions of freely falling particles and null geodesics represent the path of light rays, proposition 3.7.1 can be interpreted as saying that every one-parameter family of symmetries give rise to a conserved quantity for particles and light rays. This conserved quantity enables one to determine the gravitational red shift in stationary spacetimes and is extremely useful for integrating the geodesic equation when symmetries are present (Carroll (1997); page 140).

In Einstein gravity all event horizons of stationary black holes are Killing horizons. In higher derivative gravity one can show that event horizons are killing horizons if the black hole is static or if it is stationary, axisymmetric and possesses discrete reflection symmetry, called t–φ reflection symmetry. In the following it is understood that event horizons are killing horizons and in particular that stationary black holes in higher derivative gravity are required to be in addition axisymmetric and to have $t - \varphi$ reflection symmetry.

3.7.1 Parallel Transport and Geodesics

Local horizons are usually defined in terms of the properties of null congruence and in particular geodesics congruence. The surface gravity is one of the important properties of the horizon and to define it we will need the idea of a non-affinely parameterized geodesic. We can always assume that geodesics are affinely parameterized, a choice which is always allowed. However, this choice depends on a parameterization of the curve and we show below that certain non-affine parameterizations are appropriate for defining the surface gravity. To see how the inaffinity comes about, we will first consider the derivation of the geodesic equation. A geodesic is defined as a path that parallel transports its own tangent vector, $\bar{V} = \frac{d\bar{x}}{d\lambda}$, i.e. a

curve that satisfies

$$\nabla_{\bar{U}} \bar{U} = 0 \tag{3.70}$$

Let λ be the parameter of the curve and $\{x^i\}$ be any coordinate system in which $U^i = \frac{dx^i}{d\lambda}$ then the component version of equation (3.70) is given by

$$\begin{aligned}\nabla_{\bar{U}} \bar{U} = U^i \nabla_{\bar{e}_i}(U^i \bar{e}_j) = U^i \nabla_{\bar{e}_i} U^j \bar{e}_j + U^i U^j \nabla_{\bar{e}_i} \bar{e}_j = U^i \frac{dU^i}{dx^i} + \Gamma^j_{ik} U^i U_j \\ = \frac{dx^i}{d\lambda} \frac{d}{dx^i}\left(\frac{dx^j}{d\lambda}\right) + \Gamma^j_{ik} U^i U^k = 0\end{aligned} \tag{3.71}$$

$$\frac{d^2 x^j}{d\lambda^2} + \Gamma^j_{ik} \frac{dx^i}{d\lambda} \frac{dx^k}{d\lambda} = 0 \tag{3.72}$$

The geodesic equation (3.72) is invariant under the linear transformation $\mu \to a\lambda + b$. Where a and b are constants, λ is therefore an **affine parameter**. Parallel transport can be defined in analogy with the flat space case of a curve that keeps the components of a tensor the same (Hobson et al. (2006); page 76). In flat space and in Cartesian coordinates, a tensor is constant along a curve $x^{\alpha}(\lambda)$, with parameter λ if its components are constant,

$$\frac{d}{d\lambda}T^{\mu_1\mu_2\ldots\mu_k}_{\nu_1\nu_2\ldots\nu_l} = \frac{dx^{\alpha}}{d\lambda}\nabla_{\alpha}T^{\mu_1\mu_2\ldots\mu_k}_{\nu_1\nu_2\ldots\nu_l} = 0 \tag{3.73}$$

We can define a tensor to be parallel transported along a curve if we write this equation in covariant form as

$$\left(\frac{DT}{d\lambda}\right)^{\mu_1\mu_2\ldots\mu_k}_{\nu_1\nu_2\ldots\nu_l} = \frac{dx^{\alpha}}{d\lambda}\nabla_{\alpha}T^{\mu_1\mu_2\ldots\mu_k}_{\nu_1\nu_2\ldots\nu_l} = 0 \tag{3.74}$$

This definition of parallel transport depends on a choice of connection. It is meaningless to talk about parallel transport without specifying a connection. However, the metric on the spacetime can be used to pick out a unique connection and we say a connection is metric compatible if $\nabla_{\rho}g_{\mu\lambda} = 0$. Thus there is a unique connection that parallel propagates the metric, namely Riemannian connection or Levi-Civita connection or Christoffel connection. For geodesics, we require the tensor being transported to be the tangent to the curve $\frac{dx^{\alpha}}{d\lambda}$. If we change the parameter of the curve by a different parameter μ instead of λ, we have

$$\frac{dx^{\alpha}}{d\lambda} = \frac{d\mu}{d\lambda}\frac{dx^{\alpha}}{d\mu} \tag{3.75}$$

$$\frac{d}{d\lambda}\left(\frac{dx^{\alpha}}{d\lambda}\right) = \frac{d}{d\lambda}\left(\frac{d\mu}{d\lambda}\frac{dx^{\alpha}}{d\mu}\right) \tag{3.76}$$

$$\frac{d^2x^{\alpha}}{d\lambda^2} = \frac{d^2\mu}{d\lambda^2}\frac{dx^{\alpha}}{d\mu} + \frac{d\mu}{d\lambda}\frac{d}{d\lambda}\left(\frac{dx^{\alpha}}{d\mu}\right) \tag{3.77}$$

$$\frac{d^2x^{j}}{d\lambda^2} = \frac{d^2\mu}{d\lambda^2}\frac{dx^{\alpha}}{d\mu} + \frac{d\mu}{d\lambda}\frac{d\mu}{d\lambda}\frac{d}{d\mu}\left(\frac{dx^{\alpha}}{d\mu}\right) = \frac{d^2\mu}{d\lambda^2}\frac{dx^{\alpha}}{d\mu} + \left(\frac{d\mu}{d\lambda}\right)^2\frac{d^2x^{\alpha}}{d\mu^2} \tag{3.78}$$

From equation (3.72)

$$\frac{d^2x^\alpha}{d\lambda^2} = \frac{d^2\mu}{d\lambda^2}\frac{dx^\alpha}{d\mu} + \left(\frac{d\mu}{d\lambda}\right)^2\left(\frac{d^2x^\alpha}{d\mu^2} + \Gamma^\alpha_{\beta\mu}\frac{dx^\beta}{d\mu}\frac{dx^\mu}{d\mu}\right) = 0 \tag{3.79}$$

$$\frac{d^2\mu}{d\lambda^2}\frac{dx^\alpha}{d\mu} + \left(\frac{d\mu}{d\lambda}\right)^2\left(\frac{d^2x^\alpha}{d\mu^2} + \Gamma^\alpha_{\beta\mu}\frac{dx^\beta}{d\mu}\frac{dx^\mu}{d\mu}\right) = 0 \tag{3.80}$$

$$\left(\frac{d^2x^\alpha}{d\mu^2} + \Gamma^\alpha_{\beta\mu}\frac{dx^\beta}{d\mu}\frac{dx^\mu}{d\mu}\right) = \frac{d^2\mu}{d\lambda^2}\frac{dx^\alpha}{d\mu}/\left(\frac{d\mu}{d\lambda}\right)^2 = -\kappa\frac{dx^\alpha}{d\mu} \tag{3.81}$$

Hence, the most general form of the geodesic equation which dose not elect out a particular form of parameterization is

$$\frac{d^2x^\alpha}{d\mu^2} + \Gamma^\alpha_{\beta\mu}\frac{dx^\beta}{d\mu}\frac{dx^\mu}{d\mu} = -\kappa\frac{dx^\alpha}{d\mu} \tag{3.82}$$

where$\kappa = -\frac{d^2\mu}{d\lambda^2}/\left(\frac{d\mu}{d\lambda}\right)^2$. From this equation we see that the new curve is a geodesic, i.e. has the form of equation (3.72), only if the new parameter is related to old parameter by a linear transformation $\mu = a\lambda + b$ (Hobson et al. (2006); page 77). A parameter for which $\kappa = 0$ is called **affine parameter** and for κ 6= 0 is known as **non-affine parameter**. The surface gravity of a black hole is often defined in terms of the value of κ on the horizon for certain null geodesics that define the horizon. This shows that the value of depends on the choice of parameterizations of the null geodesics. If $\kappa = 0$, it means the surface gravity is zero and that the black hole is degenerate.

3.7.2 Killing Vectors in R^3

It is always not simple to solve Killing's equation in any given spacetime but it is frequently possible to write some Killing vectors by inspection. Of course a generic metric has no Killing vectors at all, but to keep things simple we often deal with metrics with high degrees of symmetry. For example, in $\mathbf{R}^3$ with metric $ds^2 = dx^2 + dy^2 + dz^2$, independence of the metric components with respect to x,y, and z immediately yields three Killing vectors

$$x^\mu = (1,0,0),\ y^\mu = (0,1,0),\ z^\mu = (0,0,1) \tag{3.83}$$

These clearly represent the three translations. There are also three rotational symmetries in $\mathbf{R}^3$, which are not quite as simple. These symmetries can be found first by considering the following parametric equations

$$x = r\sin\theta\cos\varphi,\ y = r\sin\theta\sin\varphi,\ z = r\cos\theta \tag{3.84}$$

Where the metric takes the form

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2 \tag{3.85}$$

This metric is manifestly independent of φ so $R = \partial_\varphi$ is a Killing vector. Transforming back to Cartesian coordinates, this becomes

$$R = -y\partial_x + x\partial_y \tag{3.86}$$

The Cartesian components R^μ of all the three rotational Killing vectors are

$$R^\mu = (-y,x,0),\ S^\mu = (z,0,-x),\ T^\mu = (0,-z,y) \tag{3.87}$$

These represent rotations about the z,y and x axis respectively. These equations do solve Killings equation. This can lead to the Killing vectors for the two-sphere S^2 with metric

$$ds^2 = d\theta^2 + \sin^2\theta d\theta^2 \tag{3.88}$$

Since the sphere can be thought of as the locus of points at unit distance from the origin in $\mathbf{R}^3$, and the rotational Killing vectors all rotate such a sphere into itself, they also represent symmetries of S^2. The explicit coordinate-bases representations for the rotational Killing vectors are

$$R = \partial_\varphi,\ S = \cos\varphi\partial_\theta - \cot\theta\sin\varphi\partial_\varphi,\ T = -\sin\varphi\partial_\theta - \cot\theta\cos\partial_\varphi \tag{3.89}$$

We notice that there are no components along ∂, which makes sense for a rotational isometry (Hamilton (2014)). Therefore, the expressions (3.89) for the three rotational Killing vectors in $\mathbf{R}^3$ are exactly the same as those of S^2 in spherical polar coordinates. In $n \geq 2$ dimensions, there can be more Killing vectors than the dimensions. This is because a set of Killing vector fields can be linearly independent, even though at any one point on the manifold the vectors at that point are linearly dependent. Also the commutator of two Killing vector fields is a Killing vector field but it may be possible that the commutator gives a vector field that is not linearly independent or it may simply vanish. So the problem of finding all of the Killing vectors of a metric is very tricky, as it is not always clear when to stop looking.

3.7.3 Killing Vectors on the Schwarzschild Manifold

In a particularly suitable set of coordinates, the line element summarizing the Schwarzschild geometry is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3.90}$$

(i) The Metric is independent of t: there is a Killing vector ξ associated with this symmetry under displacements in the coordinate time which has components $\xi^{\alpha}_{(t)} = (1, 0, 0, 0)$

(ii) Spherically symmetric: the geometry of a two dimensional surface of constant t and r in the four-dimensional geometry (3.83) is given by the line element

$$d\Omega^2 = r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{3.91}$$

This describes the geometry of a sphere of radius of in a flat three-dimensional space. The Schwarzschild geometry thus has the symmetries of a sphere with regards to changes in the angles θ and φ. This is clear in equation (3.91) because the metric is independent of φ and it is invariant under rotations about the z-axis. The Killing vector associated with this symmetry is

$$\xi^{\alpha}_{(t)} = (0, 0, 0, 1) \tag{3.92}$$

Since the geometry is also invariant under rotations about x and y axis, so there are two other Killing vectors:

$$\xi^{\alpha}_{(1)} = \sin\phi\partial\theta + \cot\theta\cos\phi\partial\phi,\ \xi^{\alpha}_{(1)} = -\cos\phi\partial_{\theta} + \cot\theta\sin\phi\partial_{\phi} \tag{3.93}$$

These four Killing vectors are also found in flat spacetime and the corresponding symmetries lead to the conservation of energy and the three components of angular momentum. The existence of these Killing vectors make the definitions of stationary, static and spherical symmetry more concrete.

Definition 3.7.1 *A spacetime is said to be stationary if there exists a one-parameter group of isometries φ_t, whose orbits are timelike curves. This group of isometries expresses the time translation symmetry of the spacetime. Equivalently, a stationary spacetime is one which possesses a timelike Killing vector field ξ^{α} (where we normalize it such that $\xi^2 \to -1$). That is outside a possible horizon, $\xi = \frac{\partial}{\partial t}$ where t is a time coordinate. The general stationary metric in these coordinates is therefore*

$$ds^2 = g_{00}(\bar{x})dt^2 + 2g_{0i}(\bar{x})dtdx^i + g_{ij}(\bar{x})dx^i dx^j \tag{3.94}$$

Definition 3.7.2 *A spacetime is said to be static if it is stationary and if it is also invariant under time-reversal. This requires $g_{0i} = 0$, so the general static metric can be written as*

$$ds^2 = g_{00}(\bar{x})dt^2 + g_{ij}(\bar{x})dx^i dx^j \tag{3.95}$$

for a static spacetime outside a possible horizon.

Definition 3.7.3 *A spherically symmetric spacetime is a spacetime whose isometry group contains a subgroup which is isomorphic to the rotation group SO(3) and the orbits of this group are 2-spheres (ordinary 2-dimensional spheres in 3-dimensional Euclidean space).*

The study of geodesics in the Schwarzschild geometry is considerably aided by the laws of conservation of energy and momentum that hold because the metric is independent of time and it is also spherically symmetric. The first step we will take to understand the Schwarzschild metric more fully is to consider the behavior of geodesics. The nonvanishing Christofel symbols for Schwarzschild metric are

$$\Gamma^1_{00} = \frac{M}{r^3}(r-2M),\ \Gamma^1_{11} = \frac{M}{r(r-2m)},\ \Gamma^0_{00} = \frac{M}{r(r-2M)}$$

$$\Gamma^2_{12} = \frac{1}{r}, \quad \Gamma^1_{11} = -2(r-2M), \quad \Gamma^3_{13} = \frac{1}{r}$$

$$\Gamma^2_{12} = -(r-2M)\sin^2\theta, \quad \Gamma^2_{33} = -\sin\theta\cos\theta, \quad \Gamma^2_{33} = \frac{\cos\theta}{\sin\theta}$$

The geodesic equation therefore gives the following four equations, where is an affine parameter

$$\frac{d^2t}{d\lambda^2} + \frac{2M}{r(r-2M)}\frac{dr}{d\lambda}\frac{dt}{d\lambda} = 0 \tag{3.96}$$

$$\frac{d^2t}{d\lambda^2} + \frac{2M}{r^3}(r-2M)\left(\frac{dt}{d\lambda}\right)^2 - \frac{2M}{r(r-2M)}\left(\frac{dr}{d\lambda}\right)^2 - (r-2M)\left[\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2\right] = 0 \tag{3.97}$$

$$\frac{d^2\theta}{d\lambda^2} + \frac{2}{r}\frac{d\theta}{d\lambda}\frac{dr}{d\lambda} - \sin\theta\cos\theta\left(\frac{d\phi}{d\lambda}\right)^2 = 0 \tag{3.98}$$

$$\frac{d^2\phi}{d\lambda^2} + \frac{2}{r}\frac{d\phi}{d\lambda}\frac{dr}{d\lambda} + 2\frac{\cos\theta}{\sin\theta}\frac{d\theta}{d\lambda}\frac{d\theta}{d\lambda} = 0 \tag{3.99}$$

There does not seem to be much hope for simply solving this set of coupled equations by inspection. Fortunately our task is greatly simplified by the high degree of symmetry of the Schwarzschild metric. We know that there are four Killing vectors: three for the spherical symmetry, and one for time translations. Each of these will lead to a constant of the motion for a free particle; if ξ^μ is a Killing vector, we know that

$$\xi_\mu \frac{dx^\mu}{d\lambda} = const \tag{3.100}$$

There is another constant of the motion that we always have for geodesics; metric compatibility implies that along the path the quantity

$$g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = \epsilon \tag{3.101}$$

is constant. In particular, therefore, one can choose $\epsilon = \pm$ for timelike (spacelike) geodesics, and λ can then be identified with proper time (proper distance), while the choice = 0 sets the initial conditions appropriate to massless particles (for which λ is then not related to proper time or proper distance.

Notice that, the symmetries the four conserved quantities associated with Killing vectors represent are also present in flat spacetime, where the conserved quantities they lead to are very familiar.

Invariance under time translations leads to conservation of energy, while invariance under spatial rotations leads to conservation of the three components of angular momentum. Essentially the same applies to the Schwarzschild metric (Carroll (1997); page 173). We can think of the angular momentum as a three-vector with a magnitude (one component) and direction (two components). Conservation of the direction of angular momentum means that the particle will move in a plane. We can choose this to be the equatorial plane of our coordinate system; if the particle is not in this plane, we can rotate coordinates until it is. Thus, the two Killing vectors which lead to conservation of the direction of angular momentum imply$\theta = \frac{\pi}{2}$. The two remaining Killing vectors correspond to energy and the magnitude of angular momentum. The energy arises from the timelike Killing vector $K = \partial_t$ or

$$K_\mu = \left(-\left(1 - \frac{2M}{r}\right), 0, 0, 0\right) \tag{3.102}$$

The Killing vector whose conserved quantity is the magnitude of the angular momentum is $L = \partial_\varphi$ or

$$L_\mu = (0,0,0,r^2 \sin^2\theta) \qquad (3.103)$$

Since, $\theta = \frac{\pi}{2}$, the two conserved quantities are

$$\left(1 - \frac{2M}{r}\right)\frac{dt}{d\lambda} = E \qquad (3.104)$$

$$r^2\frac{d\phi}{d\lambda} = L \qquad (3.105)$$

For massless particles these can be thought of as the energy and angular momentum; for massive particles they are the energy and angular momentum per unit mass of
the particle.

3.8 Hypersurfaces

Hypersurfaces play important roles in general relativity, appearing in many different contexts, e.g. in the form of hypersurfaces of constant time (for some choice of time coordinate), or as boundaries of space-time regions over which one would like to integrate some quantity. Moreover, some basic familiarity with this subject is required to better understand certain advanced aspects of general relativity like the Hamiltonian formulation of general relativity or the event horizon of the Schwarzschild black hole geometry which turns out to be a null hypersurface.

3.8.1 Description of Hypersurfaces

A hypersurface is an $(n-1)$-dimensional submanifold Σ of an n-dimensional manifold M (Poisson (2002)). One way to specify a hypersurface Σ is by setting a single function to a constant i.e. $f(x) = f_*$.The vector field denoted by

$$\xi^\mu = g^{\mu\nu}\nabla_\nu f \qquad (3.106)$$

will be normal to the surface, in the since that it is orthogonal to all vectors in $T_p\Sigma \subset T_pM$. If ξ^μ is timelike, the hypersurface is said to be spacelike; if ξ^μ is spacelike, the hypersurface is timelike and if ξ^μ is null the hypersurface is also null. Any vector field proportional to a normal vector field

$$\xi^\mu = h(x)\nabla^\mu f \qquad (3.107)$$

in some function $h(x)$, will itself be a normal vector field; since the normal vector is unique up to scaling, any normal vector can be written in this form. For timelike and spacelike hypersurface we can therefore define a normalized vector of the normal vector

$$n^\mu = \frac{\xi^\mu}{(\xi_\mu \xi^\mu)^{1/2}} \qquad (3.108)$$

Then $n^\mu n_\mu = -1$ for spacelike surfaces and $n^\mu n_\mu = +1$ for timelike surfaces; up to an overall orientation, such a normal vector field is unique (Poisson (2002); page 48). For spacelike surfaces the sign is typically chosen so as to make n^μ be future oriented. Normal surfaces have a special feature: they can be divided into a set of null geodesics, called generators of the hypersurface. Let us see how this works. We note that the normal vector ξ^μ is tangent to Σ as well as normal to it, since null vectors are orthogonal to themselves. Therefore the integral curves $x(\lambda)$, satisfying

$$\xi^\mu = \frac{dx^\mu}{d\lambda} \qquad (3.109)$$

will be null curves contained in the hypersurface. These curves $x(\lambda)$ necessarily turn out to be geodesics, although λ might not be an affine parameter (Poisson (2002); page 48). This claim can be verified by recalling that the general form of the geodesic equation can be expressed as

$$\xi^\mu \nabla_\mu \xi_\mu = \eta(\lambda \xi_\nu) \qquad (3.110)$$

where $\eta(\lambda)$ is a function that will vanish if λ is an affine parameter. By substitution of (3.97), we have

$$\begin{aligned} \xi^\mu \nabla_\mu \xi_\mu &= \xi^\mu \nabla_\mu \nabla_\nu f = \xi^\mu \nabla_\nu \nabla_\mu f \\ &= \xi^\mu \nabla_\nu \xi_\mu = \frac{1}{2} \nabla_\nu (\xi^\mu \xi_\mu) \end{aligned} \quad (3.111)\ (3.112)$$

Note that, even though $\xi^\mu \xi_\mu = 0$ on Σ, we cannot be sure that $\nabla_\nu(\xi^\mu \xi_\mu)$ vanishes, since $\xi^\mu \xi_\mu$ might be nonzero off the hypersurface.

Chapter 4

EVENT HORIZONS, KILLING HORIZONS AND TRAPPED SURFACES IN SPACETIMES

4.1 Event horizon

Event horizon defined by Hawking and Ellis (1973) (Page 312) is the boundary of the region from which particles or photons can move out to infinity in the future direction. Hawking in DeWitt and DeWitt (1973), defines event horizon as the boundary of the region from which it is not possible to escape to infinity. (Carroll (2004); Page 240)): defines event horizon as the boundary of the closure of the causal past of future null infinity. By this definition, it is clear that the event horizon is a null hypersurface. A more detailed description of the structure of the event horizon is provided by a theorem proved by Penrose (1965a). According to this theorem, the event horizon is formed by null geodesics (generators) that have no end points in the future (Frolov and Novikov (1997); Page 356). Traditionally, the black hole surface is defined using event horizons. However the event horizon, could never be observed, as it is defined, in any way. But with a known infinite future, the location of the event horizon can be established. Neither in a real physical situation, nor in a numerical evolution of spacetime can any event horizon be located exactly as quoted by (Hayward (1994); Page 1), Ashtekar and Corichi (2000)) The event horizon does not have any physical effect. Such a horizon could be passing through you, gentle reader, at any given instant; no one would notice Krishnan (2013) (Page 16) also stated that: an event horizon could be formed and grown in the room you are reading this article right now, because of the posibility of events which may occur a billion years to come. A typical example is an observer in the flat region of the Vaidya spacetime. The fact that the precise locality of an event horizon at a particular time will not be possible to be determined without complete knowledge of the entire future evolution of spacetime is clear that the event horizon is very teleological and global. However, there may be hints that a black hole has been formed. In this context trapped surfaces (Ashtekar et al., 2001) become interesting.

4.1.1 Killing horizon

Killing horizon is a null hypersurface where a global Killing vector is null (Carroll (2004); Page 244). Event horizon and Killing horizon are closely related. Carter has shown that for static black holes, the event horizon is a Killing horizon for the time translational Killing field (Nielson (2007); Page 45). Hawking and Ellis have shown (Hawking and Ellis (1973); Page 331), that in electrovac General relativity, the event horizon of a stationary black hole must be a Killing horizon. In stationary spacetimes, the event horizon of a black hole is a Killing horizon and foliated by surfaces with vanishing outward null expansion: marginally outer trapped surface (MOTS) (Booth et al. (2017); page 1). However, Killing horizon do not always coincide with event horizon. For example Minkowski space has Killing horizons for non-geodesic observers but no event horizon. In the Boyer-Lindquist form of the Kerr solution, the timelike Killing vector ∂_t becomes null along a timelike hypersurface, the stationary limit surface, called the ergosphere but this is not the event horizon. The Killing horizon occurs where $\partial_t + \Omega_H \partial_\varphi$ becomes null. Hawking and Ellis (Hawking and Ellis (1973); Page 331) have shown that no Killing horizons exist inside the event horizon for linear combinations of the time-translational and axisymmetric Killing vectors. In other words, outer trapped surfaces imply light cannot escape outwards, with a global timelike Killing vector this means they will never reach future null infinity and we have an event horizon.

4.1.2 Killing Horizons as Event Horizons of Stationary Black Hole

The Schwarzschild horizon which serves as a black hole horizon or the event horizon is a null hypersurface whose normal vector is a Killing vector $\xi = \partial_t$. In the same way, the event horizon of the Kerr black hole also has this property, but with respect to a different Killing vector $\xi = \xi + \Omega_h \eta$. The fact that in both cases the event horizons have this property is due to the rigidity theorems in which the global causal notion of an event horizon relates to the local. A Killing horizon is a null hypersurface which has a Killing vector field as normal vector field. These rigidity theorems state that under rather general conditions, and in a variety of circumstances, the event horizon of a stationary black hole must be a Killing horizon (Carroll (2004); Page 244). A very important result we realize is that in the static case the event horizon is a Killing horizon for the asymptotically timelike and hypersurface-orthogonal Killing vector ξ. Killing horizons therefore provide a fairly satisfactory characterization and description of stationary black holes.

4.1.3 Killing Horizons and Surface Gravity κ

A Killing horizon is a null hypersurface which has a Killing vector field as normal vector field. A Killing vector field, ξ^α satisfying the Killing equation $\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0$ defines a Killing horizon, H, of the spacetime $(M, g_{\alpha\beta})$ which is a null hypersurface that is every where tangent to Killing vector field, ξ^α, which becomes null, $\xi^\alpha \xi_\alpha < 0$, in a spacetime region that has H as the boundary. Stationary event horizons in General relativity are Killing horizons: for example, in Schwarzschild geometry, the event horizon $r = 2M$ is also a Killing horizon i.e., a place where the signature of a Killing vector changes and the timelike Killing vector $\xi^\alpha = \left(\frac{\partial}{\partial t}\right)^\alpha$ in the $r > 2M$ region outside the event horizon becomes null at $r = 2M$ and spacelike for $r < 2M$. An event horizon in a locally static spacetime is also a Killing horizon for the Killing vector

$\xi^{\alpha} = \left(\frac{\partial}{\partial t}\right)^{\alpha}$ associated with the time symmetry (Wald (2001); Page 7; Carroll (2004); Page 244). In a wide variety of cases of interest, the event horizon H of a stationary black hole must be a Killing horizon. In stationary spacetimes, the event horizon of a black hole is a Killing horizon (Booth et al. (2017); page 1). Carter (1973), states that for a static black hole the static Killing field $\xi^{\alpha} = \left(\frac{\partial}{\partial t}\right)^{\alpha}$ must be normal to the horizon, whereas for a stationary-axisymmetric black hole with the $t - \varphi$; orthogonality property there exists a Killing field ξ^{α} of the form

$$\xi^{\alpha} = \left(\frac{\partial}{\partial t}\right)^{\alpha} + \Omega_H \left(\frac{\partial}{\partial \phi}\right)^{\alpha} \tag{4.1}$$

Which is a linear combination of the vectors associated with time and rotational symmetries, where is the angular velocity at the horizon and it is normal to the event horizon (Wald (2001); Page 7). Hawking proved in (Hawking and Ellis (1973)), that in vacuum the event horizon of any stationary black hole must be a Killing horizon. A Killing horizon, when present defines a notion of surface gravity. The concept of the Killing horizon is useless in spacetimes that do not admit timelike Killing vectors. Now, let K be any Killing horizon (not necessarily required to be the event horizon H of a black hole), with normal Killing field ξ^{α}. Since $\nabla^{\alpha}(\xi^{\alpha}\xi_{\alpha})$ also is normal to K, these vectors must be proportional at every point on K. Hence, there exists a function, κ on K, known as the surface gravity of K. To every Killing horizon we can associate a quantity called the surface gravity (the gravitational acceleration experienced at the surface of an object). Let ξ^{μ} be a Killing vector field with Killing horizon Σ. Because ξ^{μ} is normal to Σ, along the Killing horizon, it obeys the geodesic equation

$$\xi_{\mu}\nabla_{\mu}\xi_{\nu} = \kappa\xi_{\mu} \tag{4.2}$$

Where the right-hand side arises because the integral curves of ξ^{μ} may not be affinely parameterized. The parameter κ is called the surface gravity; it will be constant over the horizon, except for a bifurcation two-sphere where the Killing vector vanishes and κ can change in sign. This happens, for example at the center of Kruskal diagram in the Schwarzschild solution.

4.2 Raychaudhuri Equation for Timelike Geodesic Congruences and Affine Null Geodesic Congruences

Raychaudhuri-Landau equation categorizes the evolution of systems of non-intersecting geodesics; called geodesic congruences. This allows us to see the evolution of a family of geodesic curves due to their expansion, shear, rotation, and the effect of the stress-energy tensor. It also occurs as a fundamental lemma in the Penrose-Hawking singularity theorems (Hawking and Penrose (1970); Page 12), where, through formalizing the idea of a surface parameterized by geodesic congruences, it governs the evolution and collapse of integral curves of geodesics into “closed trapped surfaces”. The Raychaudhuri equation is intimately related to surface behavior in the membrane paradigm and fluid/gravity correspondence.

Here, we derive an equation for the rate of change of the divergence $\nabla_\alpha u^\alpha$ of a family of geodesics along the geodesics. This simple result, known as the Raychaudhuri equation, has important implications in general relativity, especially in Penrose and Hawking singularity theorems. Thus u^α now denotes a tangent vector field to an affinely parameterized geodesic congruence, $u^\alpha \nabla_\alpha u^\beta = 0$ (and $u^\alpha u_\alpha = -1$ or $u^\alpha u_\alpha = 0$ everywhere for a timelike or null congruence). Introducing the tensor

$B_{\aleph\beta}$

$$B_{\alpha\beta} = \nabla_\beta u_\alpha = 0 \tag{4.3}$$

and therefore only has components in the directions transverse to u^α. Its trace

$$\theta = B^{\beta}_{\alpha} = g^{\alpha\beta} B_{\alpha\beta} \nabla_{\alpha} u^{\alpha} = 0 \tag{4.4}$$

is the divergence of u^{α} and is known as the expansion of the (affinely parameterized) geodesic congruence. The equation governing the evolution of $B_{\alpha\beta}$ along the integral curves of the geodesic vector field is

$$D_{\tau} B^{\beta}_{\alpha} + B^{\beta}_{\alpha} B^{\alpha}_{\beta} = R^{\alpha}_{\delta\beta\gamma} u^{\beta} u^{\delta} \tag{4.5}$$

If the trace of this equation taken, we obtain an evolution equation for the expansion θ given by

$$\frac{d}{d\tau}\theta = -(\nabla_{\alpha} u_{\beta})(\nabla^{\beta} u^{\alpha}) - R_{\alpha\beta} u^{\alpha} u^{\beta} \tag{4.6}$$

For affinely parameterized congruence, $B_{\alpha\beta}$ is automatically a spatial or transverse tensor

$$b_{\alpha\beta} = h^{\beta}_{\alpha} h^{\delta}_{\beta} B_{\gamma\delta} = B_{\alpha\beta} \tag{4.7}$$

Using the elasticity theory, equation (4.7) can be decomposed into its anti-symmetric, symmetric traceless and trace part as

$$b_{\alpha\beta} = \omega_{\alpha\beta} + \sigma_{\alpha\beta} + \frac{1}{3}\theta h_{\alpha\beta} \tag{4.8}$$

where

$$\omega_{\alpha\beta} = \frac{1}{2}(b_{\aleph\beta} - b_{\beta\alpha}),\ \sigma_{\alpha\beta} = \frac{1}{2}(b_{\aleph\beta} + b_{\beta\alpha}) - \frac{1}{3}\theta h_{\alpha\beta},\ \theta = h^{\alpha\beta} b_{\alpha\beta} = g^{\alpha\beta} B_{\alpha\beta} = \nabla_{\alpha} u^{\alpha}$$

(4.9

) The quantities $\omega_{\alpha\beta}$, being the antisymmetric part of the linear map $B_{\alpha\beta}$ measures their rotation, and $\sigma_{\alpha\sigma}$ measures their shear, and θ which measures the average expansion of the infinitesimally nearby surrounding geodesics u^{α} (Wald (1984); Page 217). The expansion θ can be written as

$$\theta = h^{\alpha\beta} b_{\alpha\beta} = h^{\alpha\beta} B_{\alpha\beta} = h^{\alpha\beta} \nabla_{\beta} u_{\alpha} = \frac{1}{2} h^{\alpha\beta} (\nabla_{\beta} u_{\alpha} + \nabla_{\alpha} u_{\beta}) = \frac{1}{2} h^{\alpha\beta} L_{u} g_{\alpha\beta} \tag{4.10}$$

where L_u is the Lie derivative along the vector field u. Substituting $g_{\alpha\beta} = h_{\alpha\beta} - u_\alpha u_\beta$, one finds

$$\theta = \frac{1}{2}h^{\alpha\beta}L_u(h_{\alpha\beta} - u_\alpha u_\beta) = \frac{1}{2}h^{\alpha\beta}L_u h_{\alpha\beta} \tag{4.11}$$

A null geodesic congruence is now considered, with tangent vector field denoted by l^α. These null geodesics are initially chosen to be affinely parameterized in order to

$$l^\alpha l_\alpha = 0, \qquad l^\alpha \nabla_\alpha l^\beta = 0 \tag{4.12}$$

Here n^α is an auxiliary null vector field and $l^\alpha l_\alpha = -1$, as associated projectors. If $b_{\alpha\beta}$ is the projection of the tensor $B_{\alpha\beta} = \nabla_\beta l_\alpha$, then

$$b_{\alpha\beta} = S^\beta_\alpha S^\delta_\beta B_{\gamma\delta} \tag{4.13}$$

The spatial projection $b_{\alpha\beta}$ is given by

$$b_{\alpha\beta} = S_{\alpha\beta} S_{\beta\delta} B_{\gamma\delta} = B_{\alpha\beta} + l_\alpha n_\gamma B_{\gamma\delta} + n_\gamma n_\delta B_{\gamma\delta} \tag{4.14}$$

In this equation the spatial trace of $b_{\alpha\beta}$ with respect to $S_{\alpha\beta}$ is equal to the space-time trace of $B_{\alpha\beta}$ with respect to $g_{\alpha\beta}$

$$g_{\alpha\beta} B_{\alpha\beta} = g_{\alpha\beta} b_{\alpha\beta} = S_{\alpha\beta} b_{\alpha\beta} \tag{4.15}$$

And the square of $b_{\alpha\beta}$ is identical to that of $B_{\alpha\beta}$

$$B_{\alpha\beta} B_{\alpha\beta} = b_{\alpha\beta} b_{\alpha\beta} \tag{4.16}$$

We can decompose $b_{\alpha\beta}$ orthogonally into its trace, symmetric traceless and antisymmetric

$$b_{\alpha\beta} = \frac{1}{2}\theta_l S_{\alpha\beta} + \frac{1}{2}(b_{\alpha\beta} + b_{\beta\alpha} - \theta_l S_{\alpha\beta}) + (b_{\alpha\beta} - b_{\beta\alpha}) = \frac{1}{2}\theta_l S_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta} \tag{4.17}$$

$$\theta_l = \sigma_{\alpha\beta} b_{\alpha\beta} = \sigma_{\alpha\beta} \nabla_\alpha l_\beta = \nabla_\alpha u_\alpha \tag{4.18}$$

The expansion θ_l is given by

$$\theta_l = \frac{1}{2}S^{\alpha\beta} L_l S_{\alpha\beta} \tag{4.19}$$

The quantities θ, $\sigma_{\alpha\beta}$ and $\omega_{\alpha\beta}$ in the null geodesic congruence have the physical interpretation as, respectively, the expansion, shear, and twist of the congruence. The change in the numerical factor in the term $\frac{1}{2}\theta h_{ab}$ as compared with $\frac{1}{3}\theta h_{ab}$ in the timelike case arises simply because the relevant vector space is now two-dimensional

rather than three-dimensional. Equation (4.19) shows that θ_l measures the change in the cross-sectional area element $\sqrt{S}$ of the congruence

$$\theta_l = \frac{1}{\sqrt{S}} L_l \sqrt{S} \tag{4.20}$$

As a typical example, the radial outgoing light rays $l = \partial_v$, $v = t+r$ in Minkowski space have expansion

$$\theta_l = \nabla_\alpha (\partial_v)^\alpha = \frac{1}{r^2}\partial(r^2(\partial_t + \partial_r)^\alpha) = +\frac{2}{r} > 0 \tag{4.21}$$

while the expansion of the radial ingoing light rays $n = \partial_u$, $u = t - r$, is

$$\theta_n = \nabla_\alpha (\partial_u)^\alpha = \frac{1}{r^2}\partial(r^2(\partial_t - \partial_r)^\alpha) = -\frac{2}{r} < 0 \tag{4.22}$$

This shows that the outgoing light rays expand and the ingoing light rays contract. As $r \to 0$, both expansions diverge, but this does not mean a pathology of Minkowski space-time.

The expansions of null vectors play important role in the singularity theorems (where trapped surfaces are characterized by negative expansions for both ingoing and outgoing null vectors) and in the study of black holes and the laws governing the evolution of their event horizons. In particular, in the latter case the Raychaudhuri equation is a very important ingredient in the proof of Hawkins's theorem (that under reasonable conditions the cross-sectional area of the event horizon of a black hole cannot decrease).

4.3 Principal Null Vectors of the Schwarzschild and the Kerr Metrics

Here, we give the principal null vectors of both the Schwarzschild spacetime and the Kerr spacetime. These two examples will serve as a testing ground for various ideas when searching for apparent horizons defined in terms of the expansions of vectors. In the Schwarzschild coordinates, the metric is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \qquad (4.23)$$

The components of the two future directed radial null vectors can be given as

$$l^\mu \propto \left(1, 1 - \frac{2M}{r}, 0, 0\right) \qquad (4.24)$$

$$n^\mu \propto \left(1, -\left(1 - \frac{2M}{r}\right), 0, 0\right) \qquad (4.25)$$

With this, one can easily show that

$$C_{abcd} l^b l^d = -\frac{2M}{r^3} l_a l_c \qquad (4.26)$$

$$C_{abcd} n^b n^d = -\frac{2M}{r^3} n_a n_c \qquad (4.27)$$

The eigenvalues are the same since the eigenvalues are just $C_{abcd} l^a n^b l^c n^d$ and their value does not depend on the normalization of either l^α or n^α. In the boyerLindquist coordinates, we have

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 - \frac{4Mar\sin^2\theta}{\rho^2} dt d\phi + \left(r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\rho^2}\right)\sin^2\theta d\phi^2 + \frac{\rho^2}{\triangle} dr^2 + \rho^2 d\theta^2 \qquad (4.28)$$

The components of the two future directed radial null vectors can be given as

$$l^{\mu} = \left(1, \frac{\triangle}{r^2 + a^2}, 0, \frac{a}{r^2 + a^2}\right) \quad (4.29)$$

$$n^{\mu} = \left(1, -\frac{\triangle}{r^2 + a^2}, 0, \frac{a}{r^2 + a^2}\right) \quad (4.30)$$

And one can easily show that

$$C_{abcd}l^b l^d = \frac{2Mr(3a^2 \cos^2 \theta^2 - r^2)}{\rho^6} l_a l_c \quad (4.31)$$

$$C_{abcd}n^b n^d = \frac{2Mr(3a^2 \cos^2 \theta^2 - r^2)}{\rho^6} n_a n_c \quad (4.32)$$

Since both l^{α} and n^{α} are repeated principal null directions, both the Schwarzschild and the Kerr metrics are double null principal directions.

4.4 Causal Structure

The idea that future events can be understood as consequences of initial conditions plus the laws of physics is causality (Carroll (2004); Page 78). The causal structure of spacetime is illustrated by figure 4.1 below. Associated with each event *p* in spacetime is a light cone. Half of the cone is labelled future and the other half past. Those events that can be reached by a material particle from *p* lie in the interior of the future light cone; these comprise the chronological future of *p*. The chronological future of p together with events lying on the cone itself comprises the causal future of *p*; physically, it represents events which in principle can be influenced by a signal emitted from *p*. The causal structure in general relativity is locally of the same qualitative nature as in flat spacetime of special relativity. But the main differences can occur globally because of nontrivial topology, spacetime singularity or the twisting of the directions of light cones as moves from point to point (Wald (1984); Page 188).

Definition 4.4.1 *A causal curve is any smooth curve that is nowhere spacelike i.e. it is timelike or null everywhere (Carroll (2004); Page 79).*

Definition 4.4.2 *Given any subset S of a manifold M, the causal future of S denoted $J^+(S)$, is the set of points that can be reached from S by following a futuredirected causal curve (Carroll (2004); Page 79).*

Definition 4.4.3 *The chronological future $I^+(S)$ is the set of points that can be reached by following a future-directed timelike curve (Wald (1984); Page 189; Hawking and Ellis (1973); Page 182; Carroll (2004); Page 79).*

A curve of zero length is achronal but not causal; so a point p will always be in its own causal future $J^+(p)$, but not necessarily its own chronological future $I^+(p)$. The causal past $J^+(p)$ and chronological past I^- are defined analogously.

A subset $S \subset M$ is called achronal if no two points in S are connected by a timelike curve; for example, any edgeless spacelike hypersurface in Minkowski spacetime is achronal (Carroll (2004); Page 79).

4.5 Local Characterizations of Black Holes

This section discusses some of the definitions of local horizons that have appeared in the Literature. Characterizing black holes by means of classical event horizon is a global concept which has the following drawbacks; it is a teleological concept, i.e. the knowledge of the whole spacetime is needed in order to locate event horizon and black hole region, the event horizon can enter into flat spacetime regions. This has made local characterization of black holes very important. In this context,

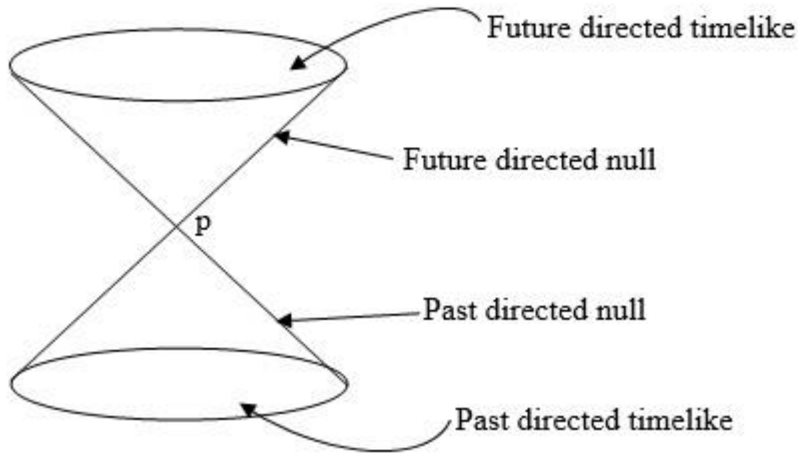

Figure 4.1: Light cone at p

the seminal notion of trapped surface plays a crucial role, capturing the idea that all light rays emitted from the surface locally converge (Jaramillo (2011); Page 2). The presence of a closed trapped surface is a very useful criterion that provides a sufficient indication that a region of spacetime lies within a black hole. Such a local criterion is valuable view of the awkward teleological nature of the black hole horizon, i.e. the fact that its precise locality at a given time on a given spacelike hypersurface cannot be determined without complete knowledge of the entire future evolution of spacetime (Hawking and Israel, 1979). It is for this reason that Hawking (Hawking and Ellis (1973); Page 320) refers to the outer boundary of the region containing closed trapped surfaces on a given spacelike hypersurface as the apparent horizon relative to that hypersurface: it may not coincide with the intersection of the true horizon with the hypersurface but it is at least guaranteed not to lie outside
it.

Definition 4.5.1 (Gourgoulhon and Jaramillo 2008; page 1) *defined a black hole as* $B := M - J^{-}(p^{+})$*. Where* M *is a 4-dimensional manifold endowed with a Lorentzian metric* g *such that* (M,g) *is asymptotically flat,* p^{+} *is the future null infinity and* $J^{-}(p^{+})$ *is the causal past of* p^{+}*. In other words, a black hole in asymptotically flat spacetime is defined as a region such that no causal signal from it can reach future null infinity* p^{+}

and the event horizon is the boundary of B (Gourgoulhon and Jaramillo (2008); page 1)

4.5.1 Trapped Surface

A two-dimensional surface S in a four dimensional spacetime has two null directions normal to the surface at each point. Thus we can distinguish two future directed families of null geodesics emerging from the surface. If we denote the in-going and out-going null normals to the surface S by l^α and n^α respectively, then θ_l and θ_n are their respective expansions. For a sphere in flat space, the out-going light rays are diverging and the ingoing ones are converging, i.e. $\theta_l > 0$ and $\theta_n < 0$. The surface S is said to be trapped if both expansions are negative: $\theta_l < 0$ and $\theta_n < 0$ or a trapped surface S according to Penrose is a compact, space-like 2-dimensional submanifold of space-time on which $\theta_l\theta_n > 0$, where l^α and n^α are the two null normals to S. Trapped surfaces are 2-dimensional spacelike surfaces whose area decrease locally along any future directions. The notion of trapped surface due to Penrose singularity theorem (Penrose (1965b); Page 211) captures the idea that in a strong gravitational field like the gravitational collapse, the outgoing light rays converge. For stationary black holes, the event horizon and the Killing horizon are equal and characterizes the boundary of the region which contains trapped surfaces (Booth et al., 2017). In dynamical black holes, trapped surfaces are generally dissociated from the event horizon and located inside the apparent horizon (Ben-Dov (2007); Page 3). These trapped surfaces are very important in the singularity theorem and their presence indicates the formation of singularity and therefore black holes. (Senovilla (2011); Page 2). It is therefore important to explore the relationship between The rate of increase of an infinitesimal transverse 2-dimensional crosssectional area δA carried along with the geodesics

$$\theta = \frac{1}{\delta A}\frac{d\delta A}{dt}\,, \tag{4.33}$$

is the expansion of a congruence of null geodesics.
The traditional Black Hole solutions in General relativity, constituted by the Kerr-Newman family of metrics, have closed trapped surfaces in the region inside the Event Horizon (Senovilla (2011); Page 2).

4.5.2 Marginally Trapped Surface

The surface is said to be marginally trapped (MTS) if $\theta_l = 0$ and $\theta_n < 0$. The singularity theorems of (Penrose (1965a); Hawking and Penrose (1970); Senovilla (1998)) found in (Senovilla (2011); Page 2) have shown that the signature of a spacetime containing a black hole is the presence of such surfaces. (Hayward (1994)), page 5; defines marginal surface as a spatial 2-surface S on which one null expansion vanishes. Note however that this is not necessarily a signature of strong gravitational field; they are present even for large black holes which have correspondingly small tidal forces at the horizon. It can be shown that trapped surfaces must lie inside the event horizon, and that cross-sections of the event horizon for stationary black holes are MTSs.

An outer trapped surface defined by Hawking as a compact spacelike 2-dimensional submanifold in (M,g_{ab}) such that the expansion of the outgoing null geodesics orthogonal to the surface is non-positive (Ashtekar and Krishnan (2004); Page 20. This definition does not matter whether the ingoing null geodesics are converging or not but it includes for convenience the case $\theta = 0$.

Trapped Region: Hawking defines the trapped region $T(M)$ in a surface M as the set of all points in M, through which there passes an outer-trapped surface, lying entirely in M. The spacetime region T containing trapped surface is called the trapped region (Ashtekar and Krishnan (2004); Page 20). (Schnetter et al. (2006); Page 2), defines a trapped region as the region where trapped surfaces exist, it can be in the full spacetime or on a Cauchy surface. (Hayward (1994); Page

4), defines a trapped region as a subset of space-time through each point of which there passes a trapped surface. (Hayward (1994); Page 4) defines trapping boundary as a connected component of the boundary of an inextendible trapped region. Under certain assumptions (which appear to be natural intuitively but technically are quite strong), he was able to show that the trapping boundary is foliated by marginally trapped surfaces (MTSs), i.e., compact, space-like 2-dimensional submanifolds on which the expansion of one of the null normals, say l^{μ} vanishes and that of the other, n^{μ} is everywhere non-positive, (Ashtekar and Krishnan, 2004;

Page 21). A trapping horizon is defined as a hypersurface of M foliated by spacelike 2-surfaces S such that the expansion scalar θ_l of one of the two families of null geodesics orthogonal to S vanishes. A trapping horizon can be either spacelike or null (Hayward (1994); Page 1).

4.5.3 Apparent Horizon (AH)

Given a spacelike 3-surface, outer boundary of region containing outer trapped surfaces that lie in the 3-surface is called the apparent horizon (Ben-Dov (2007), page 2). On a given spatial hypersurface, all (marginally) outer trapped surfaces can be found. Here, the outermost marginally outer trapped surface on the spatial slice is called the apparent horizon. In the practice of numerical relativity, the apparent horizon serves as the definition of the boundary of a black hole. The importance of trapped surfaces in numerical relativity thus constitutes a strong motivation for the study of these.

Both of the above definitions of apparent horizon are highly dependent on the given spatial slicing of the spacetime. There may exist trapped surfaces lying not in one of the given spatial slices that extend beyond the apparent horizon. For instance, there are slicings of the Schwarzschild spacetime, reaching the singularity, which fail to include a trapped surface in any spatial slice (Wald and Iyer, 1991), even though the whole interior of the Schwarzschild black hole is filled with trapped surfaces. In general, the apparent 3-horizon is neither unique nor continuous because a different foliation of the spacetime into spacelike surfaces can result in a different location of the apparent horizon through the spacetime (Ben-Dov (2007), page 2).

4.5.4 Marginally (outer) Trapped Tube (MOTT)

Many important horizons are built from marginally (outer) trapped surfaces. A hypersurface foliated by marginally (outer) trapped surfaces is referred to as a marginally (outer) trapped tube (Jakobsson (2017); page 28). It is defined by (Hawking and Ellis (1973); Page 319), as a component of the boundary of the trapped region. It is a two-surface where the expansion of the outgoing null geodesic normal to the surface is zero. (Hayward (1994); Page 5), calls it a two-surface on which one expansion vanishes a marginal surface without it necessarily being outgoing. (Schnetter et al. (2006); Page 2), a marginally trapped surface is a closed two-surface for which one, or both of the future directed null normals has zero expansion, without the other necessarily being specified.

4.5.5 Isolated Horizon (IH)

An isolated horizon by (Ashtekar and Krishnan (2004); Page 14), is a null hypersurface foliated by marginally outer trapped surfaces, with extra conditions imposed. It is isolated in the sense that it does not interact with its surroundings. In a dynamical situation the area of a black hole is expected to grow. Thus, the notion of an isolated horizon may be complemented by that of a dynamical horizon (Hayward (1994); Page 7), which is intended to model an evolving black hole. The event horizons of the stationary Schwarzschild and Reissner-Nordstr¨om solutions are isolated horizons (Jakobsson (2017); page 29). In fact, every Killing horizon with the required topology is an isolated horizon. A Killing horizon is not necessarily a black hole horizon, and neither is an isolated horizon. The concept of isolated horizons thus applies to a wider class of horizons, not only event horizons.

4.5.6 Dynamical Horizon (DH)

A dynamical horizon is a spacelike marginally trapped tube. According to (Ashtekar and Krishnan (2004); page 20), a dynamical horizon has the following properties

i. it is a three-dimensional, spacelike hypersurface that can be foliated by closed spacelike 2-surface.

ii. The expansion of one null normal to the foliations n^{α} is negative $\theta_n < \theta$. iii. The expansion of the other null normal l^{α} vanishes i.e. $\theta_l = 0$

From this definition, it basically tells us that a dynamical horizon is a space-like hypersurface which is foliated by closed, marginally trapped two-surface. The requirement of the expansion of the incoming null normal to be strictly negative is because we want to study a black hole (future horizon) rather than a white hole. Ashtekar uses a 2+1 decomposition on a three dimensional space-like surface. The Cauchy data on the dynamical horizon must satisfy the scalar and vector constraints. After doing the decomposition, (Ashtekar and Krishnan (2004); page 26), obtained the energy flux cross the dynamical horizon. Their fluxes are local and the energy flux is positive. The change in the horizon area is related to these fluxes. However, this kind of approach does not tell us the gravitational free data near horizon when considering the full 4-dimensional space-time.

4.5.7 A Trapping Horizon

A **trapping horizon** defined by (Hayward (1994); Page 3), as the closure of a three-surface foliated by marginal surfaces, for which $\theta_l = 0$, and that also satisfies the non-degeneracy conditions θ_n 6= 0 and $L_n\theta_l$ 6=. It is either spacelike or null.

A trapping horizon may be future trapped or past trapped, depending on if it is foliated by future or past marginally trapped surfaces. If the congruence of light rays having zero expansion on the horizon diverges just outside the horizon and converges just inside, the trapping horizon is said to be outer, and vice versa for an inner trapping horizon. Applying these concepts to the Reissner-Nordstr¨om solution, we find that the event horizon is a future outer trapping horizon; the inner horizon is a future inner trapping horizon, while the white hole horizons are past outer/inner trapping horizons. The existence of a black hole could very well be defined by the presence of a future outer trapping horizon.

4.5.8 Future Outer Trapping Horizon

A future, outer, trapping horizon (FOTH) defined by (Ashtekar and Krishnan (2004); Page 20), is a smooth 3-dimensional sub-manifold H of space-time, foliated by closed 2-manifolds S, such that

i. the expansion of one future directed null normal to the foliation, say l^{α}, vanishes, $\theta_l = 0$

ii. the expansion of the other future directed null normal n^{θ} is negative i.e. $\theta_n < 0$ (to distinguish between white holes and black holes) (Krishnan (2013); page 20)

iii. the directional derivative of θ_l along n^{α} is negative, $L_n\theta_l < 0$ (to distinguish between inner and outer horizons of, the Kerr solution).

4.6 Existence of Trapped and Marginally Trapped Surfaces using Gauss's Divergence Theorem

The divergence theorem only applies to closed surface *S*. By a closed surface *S*, we mean a surface consisting of one connected piece which does not intersect itself and which completely encloses a single finite region *D* of space called its interior. The closed surface S is then said to be the boundary of D; we include S in D. A sphere, a cube and a torus are examples of closed surfaces.

4.6.1 Gauss's Divergence Theorem

This theorem states that the integral of the divergence of a vector field over a region V equals the flux of the field through the surface S bounding V provided the field is suitably smooth inside V and S (Borisenko and Tarapov (1979); page 157). Consider region V, in which a vector field $\bar{g}$ is continuous and differentiable, the divergence of this vector field is given by

$$\int_V \nabla \cdot g d\bar{V} = \oint_S \bar{g} n d\bar{S} \tag{4.34}$$

Where the surface S is a closed surface that completely surrounds a very small region V at point r.

Definition 4.6.1 *The flux of a vector field measures "how much" vector field crosses a given surface. The divergence theorem relates the total flux of a vector field out of a closed surface S to the integral of the divergence of the vector over the enclosed volume V.*

Definition 4.6.2 *The divergence basically indicates the amount of vector field $\bar{g}$ that is converging to or diverging from a given point. For example, consider the following vector fields in the region of a specific point.*

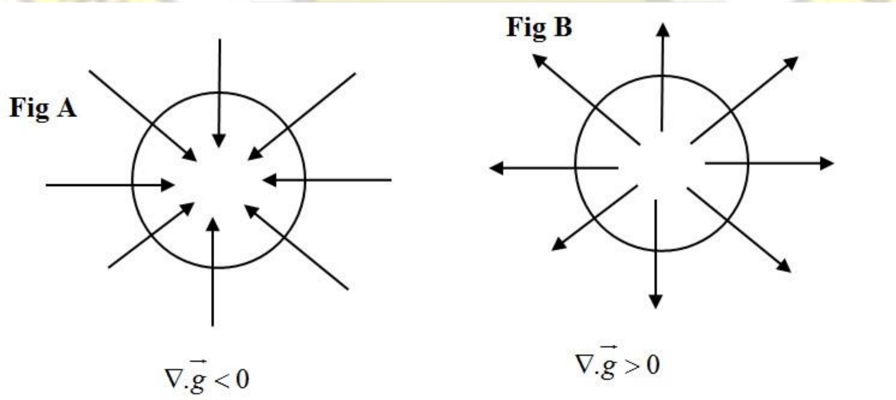

Figure 4.2: Showing positive and negative divergence

The field **fig A** is converging to a point and therefore the divergence of a vector field at that point is negative. Conversely, the vector field **fig B** is diverging from a point. As a result, the divergence of the vector field at that point is positive.

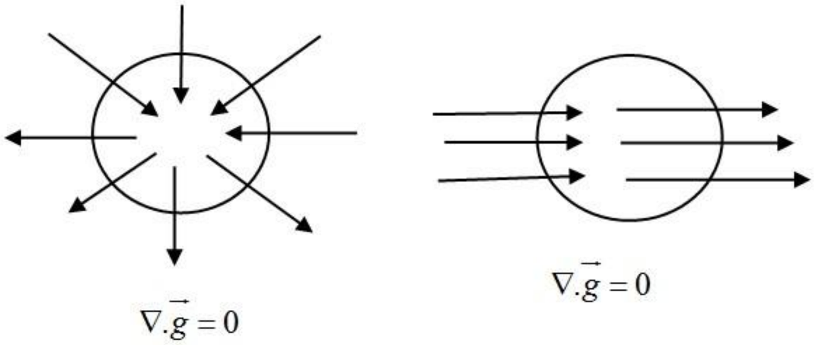

Figure 4.3: Showing zero divergence

For each of these vector fields, the divergence is zero. Over some portions of the surface, the normal components is positive, where as on the other portions, the normal component is negative. But if there is a massive source inside the surface, its gravitational field has an attractive or converging effect. Close enough to a massive source; the outgoing null vectors converge and the divergence becomes negative i.e. $\nabla \cdot \bar{l} < 0$. A surface *S* is said to be trapped if and only if both divergences are negative: $\nabla \cdot \bar{l} = \theta_l < 0$ and $\nabla \cdot \bar{n} = \theta_n < 0$. The surface where the divergence of outgoing null vector becomes zero and the divergence of ingoing null vector is less than zero is said to be marginally trapped (MTS) i.e. $\nabla \cdot \bar{l} = \theta_l = 0$ and $\nabla \cdot \bar{n} = \theta_n = 0$. Since the divergence theorem applies to close surface, it can also be used to proof the existence of trapped and marginally surfaces in black holes.

4.7 Gravitational Singularity

Gravitational field of a point mass is

$$\bar{g} = -\frac{GM}{r^2}\hat{r} \tag{4.35}$$

This field is defined everywhere except at $r = 0$ where it blows up to infinity; there is a singularity there. How do we treat this singularity? According to Gauss's law; if you draw a spherical surface or any closed shape around a mass M and measure the flux through that surface, the flux will be the same no matter how the mass inside is distributed, as it only depends on the quantity of mass enclosed. As far as real world measurement is concern, it does not matter that a point mass has an infinite gravitational field at the center or at the distance point of measurement, the gravitational field is finite and the same as if the mass were diffused into a cloud without a singularity. The singularity matters in showing how the usual mathematical approach fails, thereby indicating a more sophisticated approach is needed to handle that case. This may be done by applying Gauss's theorem to the gravitational field to derive the gravitational flux. Equation (4.34) presents us with an interesting paradox when we consider the vector field equation (4.35). On one hand, the divergence of this vector field is given by

$$\nabla \cdot \bar{g} = -GM \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\bar{r}}{r^2} \right) \right) = 0 \qquad (4.36)$$

From equation (4.34), the left hand side vanishes i.e. $\int_V \nabla \cdot \bar{g} dV = 0$. On the other hand, choosing V to be a sphere of radius r and denoting its surface as S, we have

$$\oint_S \bar{g} \cdot \bar{n} dA = -GM \int_0^{2\pi} d\phi \int_0^{2\pi} r^2 \sin\theta \hat{r} \cdot \frac{\hat{r}}{r^2} = -4\pi GM \qquad (4.37)$$

We immediately notice that, according to the divergence theorem, we have a big problem i.e. $0 = -4\pi GM$. This paradox is resolved by noting that $\nabla \cdot \left(-GM \frac{\hat{r}}{r^2}\right) = 0$ is valid only at $r \neq 0$. To reconcile the two sides of the divergence theorem, we therefore introduce a singular function known as the delta distribution $\delta^3(r)$ defined by the identity

$$\delta^3(r) = \nabla \cdot \left(-GM\frac{\hat{r}}{r^2} \right) \tag{4.38}$$

with the property that

$$\delta^3(r) = \begin{cases} 0, & r \neq 0 \\ \infty, & r = 0 \end{cases} \tag{4.39}$$

Additional property of the delta distribution is

$$\int_V f(r)\delta(r-a)dV = \begin{cases} f(a) & \text{,if } a \text{ is located inside } V \\ 0, & \text{if } a \text{ is located outside } V \end{cases} \tag{4.40}$$

Using the delta distribution $\delta^3(r)$, we can write

$$\nabla \cdot \bar{g} = -4\pi GM\delta^3(r) \tag{4.41}$$

And thus,

$$\int_V \nabla \cdot \bar{g}dV = -4\pi GM \int_V \delta^3(r)dV = -4\pi GM \tag{4.42}$$

4.8 Discussion of result

We can now investigate the various definitions of black holes that have appeared in the Literature. (Gourgoulhon and Jaramillo (2008); page 1) defined a black hole as $B := M - J^-(p^+)$. Where M is a 4-dimensional manifold endowed with a Lorentzian metric g such that (M, g) is asymptotically flat, p^+ is the future null infinity and $J^-(p^+)$ is the causal past of p^+ In other words, a black hole in asymptotically flat spacetime is defined as a region such that no causal signal from it can reach future null infinity p^+ and the event horizon is the boundary of B (Gourgoulhon and Jaramillo (2008); page 1). The region outside the black hole is called the domain of outer communications. This is the most common definition of a black hole and captures the essential part of a region of spacetime that cannot ever causally influence the region outside of itself. However, if we consider the processes that are possible during the formation of a black hole or in the course of its subsequent evolution, it becomes clear that this definition actually does not describe quite what it was meant to do (Frolov and Novikov (1997); page 356). The boundary of the black hole, H^+, is thus determined not only by some specifics of the spacetime at a given moment (say, a strong field in some region) but also by the entire future history. The problem of finding the event horizon H^+ is a problem with final, not initial, conditions. This property is usually referred to as the **teleological nature of the horizon**. The boundary H^+ thus bounds not so much a region with a especially strong gravitational field (although this field is certainly necessary, or H^+ would not appear at all) but rather a region with very specific global properties; namely, no rays escape from this region to infinity. It is this property - the invisibility from infinity, the impossibility for particles and light rays to escape - that justifies the name "**black hole**" for this region. In addition, the event horizon is formed by null geodesics, for which a number of strong theorems can be formulated.

Chapter 5

MAIN RESULT 1: GLOBAL AND LOCAL CHARACTERIZATION OF SCHWARZSCHILD BLACK HOLE

5.1 Spherically Symmetric Black Holes

A black hole usually refers to a region of no escape. In physical terms a region of spacetime where gravity is so strong that any particle or light ray entering that region can never escape from it (Wald (1984); Page 299). To the more mathematical relativist, a black hole is characterized by the existence of an event horizon which is a boundary of future null infinity or surface inside which no point is connected to future infinity by photon trajectories. The term black hole was coined by Wheeler in 1960's as understanding of these exotic objects grew. A black hole has two singularities; a physical singularity is present at $r = 0$ and a coordinate singularity at $r = 2M$ (where M is the mass of the spherically symmetric body)in the Schwarzchild spacetime. In 1960's Kruskal and Szekeres independently examined the mathematical structure of the Schwarzschild spacetime and found a coordinate which removed coordinate singularities and revealed the nature of the solution as having two asymptotically flat universes, one of which is not accessible in standard Schwarzschild coordinates (Poisson (2002); page 126).

5.2 Schwarzschild Black Holes

The simplest of all stationary black-hole solutions of the source-free Einstein equations is that for the static spherically symmetric spacetime, asymptotically flat at spatial infinity, described in the Literature by Schwarzschild (1916). This solution can be derived by making full use of the symmetries and their connection to Killing vectors. In view of the spherical symmetry we introduce polar coordinates (r,θ,φ) in three-dimensional space and add a time coordinate t, measuring asymptotic Minkowski time at $r \to \infty$. The Schwarzschild solution is obtained by requiring that there exists at least one coordinate system parameterized like this in which the following two conditions hold: the metric components are independent on time t and the line element is invariant under the three-dimensional rotation group SO(3) acting on three vectors **r** in the standard linear way. The metric can be derived as follows (Carroll (2004); page 194)

The Schwarzschild metric is a metric which is static and spherically symmetric. Because it is a spherically symmetric spacetime, it can be put in the form

$$ds^2 = g_{aa}(a,b)da^2 + g_{ab}(a,b)(dadb + dbda) + g_{bb}(a,b)db^2 + r^2(a,b)d\Omega^2 \qquad (5.1)$$

where $r(a,b)$ is an undetermined function to which we have merely given a suggestive label.

Now, we can change the coordinates from (a,b) to (a,r) by inverting $r(a,b)$ unless r were a function of a alone; in this case we could just as easily switch to (b,r).
The metric (5.1) is then given by

$$ds^2 = g_{aa}(a,b)da^2 + g_{ar}(a,b)(dadr + drda) + g_{rr}(a,r)db^2 + r^2(a,r)d\Omega^2 \qquad (5.2)$$

Next, we want to find a function $t(a,r)$ such that in the (t,r)coordinates system, there will be no cross terms in the metric. Now, since

$$dt = \frac{\partial t}{\partial a}da + \frac{\partial t}{\partial r}dr \qquad (5.3)$$

We can therefore write

$$dt^2 = \left(\frac{\partial t}{\partial a}\right)^2 da^2 + \left(\frac{\partial t}{\partial a}\right)\left(\frac{\partial t}{\partial r}\right)(dadr + drda) + \left(\frac{\partial t}{\partial r}\right)^2 dr^2 \qquad (5.4)$$

Replacing the first three terms in the metric (5.2) by

$$mdt^2 + ndr^2 \qquad (5.5)$$

for some functions m and n. This is equivalent to the requirements

$$m\left(\frac{\partial t}{\partial a}\right)^2 = g_{aa} \qquad (5.6)$$

$$n + m\left(\frac{\partial t}{\partial r}\right)^2 = g_{rr} \qquad (5.7)$$

and

$$m\left(\frac{\partial t}{\partial a}\right)\left(\frac{\partial t}{\partial r}\right) = g_{ar} \qquad (5.8)$$

This means, we have three equations for the three unknowns $t(a,r)$, $m(a,r)$ and $n(a,r)$ just enough to determine precisely up to initial conditions for t. They are determined in terms of the unknown functions g_{aa},g_{ar} and g_{rr} which are still undetermined. Therefore the metric can be put in the form

$$ds^2 = m(t,r)dt^2 + n(t,r)dr^2 + r^2 d\Omega^2 \qquad (5.9)$$

Here, the difference between the two coordinates t and r is that, r has been chosen to be the one that multiplies the metric for the two sphere. This choice was motivated by what we know about the metric for the flat Minkowski space which is written as

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 \qquad (5.10)$$

Notice that as $r \to \infty$, the (5.9) becomes Minkowskian so the metric under consideration is indeed asymptotically flat spacetime (this is known as pseudo-Riemannian).

We therefore choose $m(t,r)$ to be negative. With this choice, we can replace the functions m and n with new functions α and β and put the metric in the form

$$ds^2 = -e^{2\alpha(t,r)}dt^2 + e^{2\beta(t,r)}dr^2 + r^2 d\Omega^2 \tag{5.11}$$

This is what can be done for a general metric in a spherically symmetric spacetime. The next step is to solve Einstein's equation for $\alpha(t,r)$ and $\beta(t,r)$. Using the metric (5.11), the non-vanishing Christoffel symbols are as follows

$$\begin{aligned}
&\Gamma^t_{tt} = \partial^\alpha_t, \qquad \Gamma^t_{tr} = \partial_r\alpha, \qquad \Gamma^t_{rr} = e^{2(\beta-\alpha)}\partial_t\beta, \\
&\Gamma^r_{tt} = e^{2(\beta-\alpha)}\partial_r\alpha, \qquad \Gamma^r_{tr} = \partial_t\beta, \qquad \Gamma^r_{rr} = \partial_t\beta, \\
&\Gamma^\theta_{r\theta} = \tfrac{1}{r}, \qquad \Gamma^r_{t\theta} = -re^{-2\beta}, \qquad \Gamma^\theta_{r\theta} = \tfrac{1}{r} \\
&\Gamma^r_{\phi\phi} = -re^{-2\beta}\sin^2\theta, \qquad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \qquad \Gamma^\phi_{\theta\phi} = \cot\theta
\end{aligned} \tag{5.12}$$

Using these non-vanishing Christoffel symbols, the non-vanishing components of the Riemann tensor are

$$\begin{aligned}
&R^t_{rtr} = e^{2\beta-\alpha}\left[\partial^2_t\beta + (\partial_t\beta)^2 - \partial_t\alpha\partial_t\beta\right] + \left[\partial_r\alpha\partial_r\beta - \partial^2_r\beta - (\partial_r\alpha)^2\right] \\
&R_{\theta t\theta} = -re^{-2\beta}\partial_r\alpha \\
&R^t_{\phi t\phi} = -re^{-2\beta}\sin^2\theta\partial_r\alpha \\
&R^t_{\theta r\theta} = -re^{-2\alpha}\partial_t\beta \\
&R^t_{\phi r\phi} = -re^{-2\alpha}\sin^2\theta\partial_t\beta \\
&R^r_{\theta r\theta} = re^{-2\beta}\partial_r\beta \\
&R^\theta_{\phi\theta\phi} = (1-e^{-2\beta})\sin^2\theta
\end{aligned} \tag{5.13}$$

and we have the Ricci tensors as

$$\begin{aligned}
&R_{tt} = \left[\partial^2_t\beta + (\partial_t\beta)^2 - \partial_t\alpha\partial_t\beta\right] + e^{2(\alpha-\beta)}\left[\partial^2_r\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \tfrac{2}{r}\partial_r\alpha\right] \\
&R_{rr} = -\left[\partial^2_r\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta - \tfrac{2}{r}\partial_r\beta\right] + e^{2(\beta-\alpha)}\left[\partial^2_t\beta + (\partial_t\alpha)^2 - \partial_t\alpha\partial_t\beta\right] \\
&R_{tr} = \tfrac{2}{r}\partial_t\beta \\
&R_{\theta\theta} = e^{-2\beta}\left[r(\partial_r\beta - \partial_r\alpha) - 1\right] + 1 \\
&R_{\phi\phi} = R_{\theta\theta}\sin^2\theta
\end{aligned} \tag{5.14}$$

Now, we need to solve Einstein's equation in vacuum, $R_{\mu\nu} = 0$. From $R_{tr} = 0$, we have

$$\partial_t \beta = 0 \tag{5.15}$$

If we take the time derivative of $R_{\theta\theta} = 0$ and using (5.15), we get

$$\partial_t \partial_r \alpha = 0 \tag{5.16}$$

which can be written as

$$\beta = \beta(r) \tag{5.17}$$
$$\alpha = f(r) + g(t) \tag{5.18}$$

(5.19)

The first term in the metric (5.11) is thus $-e^{2f(r)}e^{2g(t)}dt^2$. But we can redefine our time coordinate by replacing $dt \rightarrow e^{-g(t)}dt$; in other words, we can choose t such that $g(t) = 0$, hence $\alpha(t,r) = f(r)$. We can therefore write

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2 \tag{5.20}$$

We notice from metric (5.11) that none of the metric components is dependent on the time coordinate t. We have therefore proven an important result; any spherically symmetric vacuum metric possesses a timelike Killing vector. This is a very interesting property and it gets its own name; a metric that possesses a Killing vector that is timelike near infinity is called stationary. There is also a more restrictive property: a metric is called static if it possesses a timelike Killing vector which is orthogonal to a family of hypersurfaces. A hypersurface in an n-dimensional manifold is simply an $(n - 1)$ dimensional sub-manifold. The metric (5.20) is not only stationary but also static. The Killing vector ∂_t is orthogonal to the surfaces $t = const$ since there are no cross terms such as $dtdr$ and so on. We can therefore say that a static metric is a metric in which nothing is moving, while a stationary metric allows things to move but only in a symmetric way. For instance, the static spherically symmetric metric (5.20) will describe non-rotating stars or black holes while rotating systems which keep rotating in the same way at all times will be described by metrics that are stationary but not static.

Now, since bothR_{tt} and R_{rr} vanish, we can write

$$0 = e^{2(\beta-\alpha)} R_{tt} + R_{rr} = \frac{2}{r}(\partial_t \alpha + \partial_r \beta) \tag{5.21}$$

which implies $\alpha = -\beta + c$ where c is a constant. We can set this constant c equal to zero by rescaling our coordinates by $t \rightarrow e^{-c}t$, so we have

$$\alpha = -\beta \tag{5.22}$$

Next using $R_{\theta\theta} = 0$, we have

$$e^{2\alpha}(2r\partial_r\alpha + 1) = 1 \tag{5.23}$$

This is equivalent to $\partial_r(re^{2\alpha}) = 1$ which is solved to obtain

$$e^{2\alpha} = 1 + \frac{R_s}{r} \tag{5.24}$$

where R_s is some undetermined constant. Using (5.22) and (5.24) our metric becomes

$$ds^2 = -\left(1 + \frac{\mu}{r}\right) dt^2 + \left(1 + \frac{\mu}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{5.25}$$

The only thing left to do is to find the constant R_s. However, fortunately, it is straightforward to check that for any value of this metric solves the two equations $R_{tt} = 0$ and $R_{rr} = 0$.

Lastly, we assign a physical interpretation to the constant R_s. The most important use of a spherically symmetric vacuum solution is to represent the spacetime outside a star or planet. In that case, we apply the weak field limit as $r \to \infty$. In this limit (5.25) implies

$$g_{tt}(r \to \infty) = -(1 + \tfrac{\mu}{r})$$
$$g_{rr}(r \to \infty) = (1 - \tfrac{\mu}{r}) \qquad (5.26)$$

On the other hand, the weak field limit is

$$g_{tt}(r \to \infty) = -(1 + 2\varphi)$$
$$g_{rr}(r \to \infty) = (1 - 2\varphi) \qquad (5.27)$$

where the potential $\phi = -\frac{GM}{r}$. Hence, the metric do agree in this limit, setting $R_s = -\frac{2GM}{r}$, our final result is the celebrated Schwarzschild metric given by

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\Omega^2 \qquad (5.28)$$

which can be written as

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dr^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \qquad (5.29)$$

where M is a constant (typically interpreted as the mass of the spherically symmetric body), and $d\Omega^2 = d\theta^2 + \sin^2 d\varphi^2$ is the usual metric on S^2. Notice that as $r \to \infty$

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 \qquad (5.30)$$

the Minkowski metric, so dS^2 is indeed asymptotically flat spacetime. In equation (5.29) we have chosen units such that $C = G = 1$. Stationary spacetimes $(M;g)$ are defined to be spacetimes which have a time-like Killing vector field K. This means that observers moving along the integral curves of K do not notice any change. This definition implies that we can introduce coordinates in which the components $g_{\mu\gamma}$ of the metric do not depend on time. To see that, suppose we choose a spacelike hypersurface $\Sigma \subset M$ and construct the integral curves of K through Σ. A spacetime is said to be static if it is stationary and if in addition, there exists a spacelike hypersurface Σ which is orthogonal to the orbits of the isometry. Furthermore, the spatial hypersurfaces Σ are expected to be spherically symmetric. This means that the group SO(3) (i.e. the group of rotations in three dimensions) must be an isometry group of the metric $d\Omega^2$. The orbits of SO(3) are two-dimensional spacelike surfaces on Σ. Thus, $SO(3)$ isometries may then be interpreted physically as rotations and thus spherically symmetric spacetime is one whose metric remains invariant under rotations (Wald (1984); page 120).

The striking feature of the Schwarzschild solution is that the metric components become singular at both $r = 2M$ and $r = 0$. This singular behaviour of the components could be due to (i) a breakdown of the coordinates used to obtain the general form of the metric equation (5.29), because the Killing vector field $K = 0$ or $\nabla_a r = 0$(or K and $\nabla_a r$ become collinear) or (ii) a true singularity of the spacetime structure. The singularity at $r = 2M$ is caused by a breakdown of the coordinates while the singularity at $r = 0$ is a true physical singularity (Wald (1984); page 132). The metric is well-behaved for $r > 2M$ (exterior Schwarzschild) and $r = 2M$ is called the Schwarzschild radius.

One way to understand the geometry of a spacetime is to explore its causal structure, as defined by light cones. We therefore consider radial null curves, those for which θ

and φ are constant and $ds^2 = 0$. From equation (5.29), we have $\frac{dt}{dr} = \pm\left(1 - \frac{2M}{r}\right)^{-1}$.

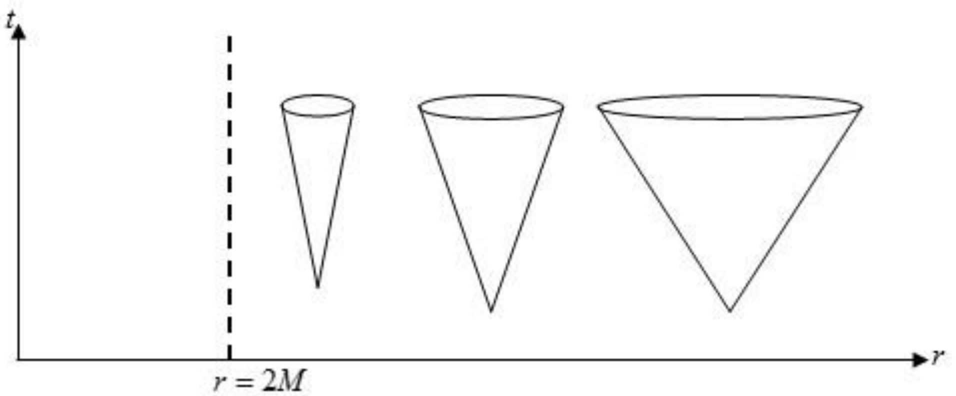

Figure 5.1: In Schwarzschild coordinates, light cones appear to close up as we approach $r = 2M$

This measures the slope of the light cones on a spacetime diagram of $t - r$ plane. For large r the slope is ±1, as it will be in flat space, while as we approach $r = 2M$, we get $\frac{dt}{dr} = \pm\infty$ and the light cones close up as shown in figure (5.1). Thus a light ray that approaches $r = 2M$ never seems to get there, at least in this coordinate system; instead it seems to asymptote this radius (Carroll (2004); page 218). At $r = 2M$ then the metric becomes degenerate since the dt term disappears.

5.3 The Singularities of Schwarzschild Solution

When we examine the metric (5.29) in some more detail, we immediately see that there is a problem when $r \to 2M$: $g_{00} \to 0$ and $g_{rr} \to \infty$. Moreover, when $r \to 0$, $g_{00} \to \infty$ and $g_{rr} \to 0$. In both cases we say that there is a singularity, but of a different nature. In order to check whether a singularity is a genuine curvature singularity, we should compute the scalars which we can construct from the Riemann tensor and see if they diverge. To check whether the Riemann tensor is well-behaved is not enough, in fact for the Schwarzschild metric the components of $R_{\theta\gamma\delta\alpha}$ are

$$R^t_{rtr} = -\frac{2M}{r^3}\left(1 - \frac{2M}{r}\right)^{-1}, \; R^t_{\theta t\theta} = \frac{1}{\sin^2\theta} R^t_{\phi t\phi} = \frac{M}{r^3}$$
$$R^\theta_{\phi\theta\phi} = \frac{2M}{r^5}\sin^2\theta, \; R^r_{\theta r\theta} = \frac{1}{\sin^2\theta} R^r_{\phi r\phi} = -\frac{M}{r^5}$$

and they diverge both at $r = 0$, and at $r = 2M$. However, if we compute the scalar invariants, like $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{48M^2}{r^6}$, we find that they diverge only at $r = 0$. We conclude that $r = 0$ is a true curvature singularity, while $r = 2M$ is only a *coordinate singularity*, due to an inappropriate choice of the coordinates (Carroll (1997); page 172).

5.4 Birkhoff's Theorem

Birkhoff's theorem establishes that Schwarzschild metric is the unique solution to the Einstein field equations that describes the vacuum spacetime outside a spherically symmetric body of mass M. This theorem implies that a spherical mass distribution cannot emit gravitational waves (Poisson (2002); Page 125). Spherical symmetry guarantees that we can introduce coordinates r and t such that the surfaces of constant r and t have the structure of a sphere with radius r. On one such surface we can introduce colatitude and longitude coordinates θ and θ. The (θ,φ) coordinates can be extended in a natural way to other values of r by choosing the radial lines to lie in the direction of the covariant derivative vector $\nabla_a r$, and this ensures that the metric will not have any nonvanishing terms in $drd\theta$ or $drd\varphi$, which could only arise if our choice had broken the symmetry between positive and negative values of $d\theta$ and $d\varphi$.

5.5 Kruskal-Szekeres Coordinates

The difficulties of the Schwarzschild metric because of its singular nature at $r = 2M$, the coordinates (t,r,θ,φ) are not as useful for understanding the nature of the event horizon of a black hole. However, a non singular coordinate system like the KruskalSzekeres (KS) coordinates introduced in this section is used to remove the coordinate singularity at the horizon (Hartle (2003); Page 269). Such a representation was constructed independently by Kruskal (1960) and Szekeres (1960). The KruskalSzekeres (KS) coordinates discuss the continuation of the Schwarzschild solution across the event horizon and produce a metric that is manifestly regular at $r = 2M$ (Poisson (2004); Page 164). Introducing Kruskal-Szekeres coordinates (v,u,θ,φ), where θ and φ are the same as the Schwarzschild polar angles but the new variables u and v are defined according to the following coordinates transformations:

$$u = \left(\frac{r}{2M} - 1\right)^{\frac{1}{2}} e^{\frac{r}{4M}} \cosh\left(\frac{t}{4M}\right) \quad (5.31)$$
$$v = \left(\frac{r}{2M} - 1\right)^{\frac{1}{2}} e^{\frac{r}{4M}} \sinh\left(\frac{t}{4M}\right) \qquad r > 2M$$

$$u = \left(1 - \frac{r}{2M}\right)^{\frac{1}{2}} e^{\frac{r}{4M}} \sinh\left(\frac{t}{4M}\right) \quad (5.32)$$
$$v = \left(1 - \frac{r}{2M}\right)^{\frac{1}{2}} e^{\frac{r}{4M}} \cosh\left(\frac{t}{4M}\right) \qquad r < 2M$$

Differentiating any of the two pairs $r > 2M$ or $r < 2M$, first with respect to t

$$du = \frac{1}{4M}\left(\frac{r}{2M} - 1\right)^{1/2} e^{\frac{r}{4M}} \sinh\left(\frac{t}{4M}\right) dt \quad (5.33)$$
$$dv = \frac{1}{4M}\left(\frac{r}{2M} - 1\right)^{1/2} e^{\frac{r}{4M}} \cosh\left(\frac{t}{4M}\right) dt \quad (5.34)$$

Squaring, we obtain

$$du^2 = \frac{1}{16M^2}\left(\frac{r}{2M} - 1\right) e^{\frac{r}{2M}} \sinh^2\left(\frac{t}{4M}\right) dt^2 \quad (5.35)$$
$$dv^2 = \frac{1}{16M^2}\left(\frac{r}{2M} - 1\right) e^{\frac{r}{2M}} \cosh^2\left(\frac{t}{4M}\right) dt^2 \quad (5.36)$$

Subtracting (5.35) from (5.36), we obtain

$$16M^2\left(\frac{r}{2M} - 1\right)^{-1} e^{-\frac{r}{2M}} du^2 - 16M^2\left(\frac{r}{2M} - 1\right)^{-1} e^{-\frac{r}{2M}} dv^2 = \left(\cosh^2\left(\frac{t}{4M}\right) - \sinh^2\left(\frac{t}{4M}\right)\right) dt^2 \quad (5.37)$$

Thus

$$dt^2 = 16M^2\left(\frac{r}{2M} - 1\right)^{-1} e^{-\frac{2}{2M}} (du^2 - dv^2) \quad (5.38)$$

Also differentiating with respect to r

$$du = [\frac{1}{4M}(\frac{r}{2M} - 1)^{\frac{-1}{2M}} e^{\frac{r}{4M}} \cosh(\frac{t}{4M}) + \frac{1}{4M}(\frac{r}{2M} - 1)^{\frac{1}{2}} e^{\frac{r}{4M}} \cosh(\frac{t}{4M})]dr \quad (5.39)$$

$$du = [\frac{1}{8M} e^{\frac{r}{4M}} \cosh(\frac{t}{4M})(\frac{r}{2M} - 1)^{\frac{-1}{2}}]dr \quad (5.40)$$

Squaring and arranging, we have

$$64M^2e^{-\frac{r}{2M}}(\frac{r}{2M}-1)du^2 = \cosh^2(\frac{t}{4M})dr^2 \tag{5.41}$$

Similarly

$$64M^2e^{-\frac{r}{2M}}(\frac{r}{2M}-1)dv^2 = \sinh^2(\frac{t}{4M})dr^2 \tag{5.42}$$

Subtracting (5.42) from (5.41) we have

$$dr^2 = 0 \tag{5.43}$$

Substituting (5.37) and (5.43) into the Schwarzchild metric equation (5.29), the line element obtained for carrying out these transformations is

$$ds^2 = \left(\frac{32M^3}{r}\right)e^{-\frac{r}{2M}}(-dv^2+du^2)+r^2(d\theta^2+\sin^2\theta d\phi^2) \tag{5.44}$$

Where $r = (u,v)$ defined implicitly by the relation

$$\left(\frac{r}{2M}-1\right)e^{\frac{r}{2M}} = u^2 - v^2 \tag{5.45}$$

This metric is totally regular at r = 2M. There is simply no trace of singular behaviour on equation (5.44) which allows us to finally conclude that the Killing horizon is merely a coordinate singularity in the original Schwarzschild coordinates (Hartle (2003); Page 270). However, the curvature singularity at r = 0 persist.

No coordinate transformation can reduce the divergent behaviour of the curvature scalars there. A very important property of Kruskal-Szekeres coordinates is that

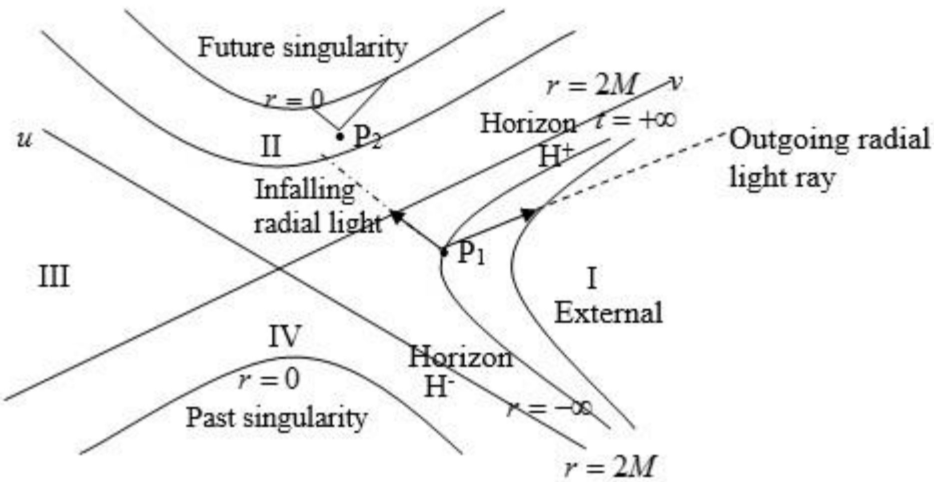

Figure 5.2: The analytic extension of the Schwarzschild spacetime by Kruskal coordinates.

light rays and timelike trajectories always lie within a two-dimensional light cone bounded by 45^0 lines. A radial moving light ray travels on a trajectory v = constant or u = constant. A non-radially directed light ray or timelike trajectory always lies inside the two-dimensional light cone. By these properties, the causal properties of the black hole geometry can easily be understood.

Considering a point P_1 in Region I, a radially outgoing light ray from P_1 will escape falling into the singularity as shown in Figure (5.2). An infalling light ray from P_1 will eventually cross the horizon H^+ and then hit the future singularity. This means an observer in Region I can send messages to infinity as well as into Region II. Consider Region II. From any point P_2 any signal must eventually hit the singularity. Furthermore, no signal can ever escape to Region I. Thus no observer who stays outside $r = 2M$ can ever be influenced by events in Region II. For this reason Region II is said to be behind the horizon. From Region III no signal can ever get to Region I, and so it is also behind the horizon. Points in Region IV can communicate with Region I but Region I however cannot communicate with Region

IV. We can therefore say that Regions II and III are behind the future horizon while Regions III and IV are behind the past horizon (Suskind and Lindesay (2005); Page 13). We had in spherical coordinates the isometry $t \rightarrow t + c$ with Killing vector $\xi = \left(\frac{\partial}{\partial t}, 0, 0, 0\right)$; this expresses the stationary character of the Schwarzschild solution. In Kruskal coordinates this Killing vector field becomes $\xi = \frac{1}{4M}\left(V\frac{\partial}{\partial U} - U\frac{\partial}{\partial}\right)$

We also see that the point $(U,V) = (0,0)$ is a fixed point of ξ and this point corresponds to a 2-sphere. One could think that the geodesically incomplete character of solutions could be due to the spherically symmetric collapse, but (Hawking and Ellis (1973); Page 258), showed with their singularity theorems that this incompleteness is a general feature of gravitational collapse.

5.6 Eddington-Finkelstein Coordinates

The Kruskal-Szekeres approach has several drawbacks. First, the explicit construction of the coordinates is relatively complicated, and must be carried out in a fairly long series of steps. Second, the fact that r is only implicitly defined in terms of these coordinates makes working with them rather difficult. Third, the manifold covered by these coordinates, with its two copies of each surface r = constant, is unnecessarily large for most practical applications; while the extension across the event horizon is desirable, the presence of another asymptotic region (for which $r = 2M$) often is not. Despite these drawbacks the KS coordinates are not to be dismissed out of hand because they do play important role in black-hole physics: we would advocate, for pedagogical purposes, the construction of simpler coordinate systems for extending the Schwarzschild spacetime across the event horizon.

A useful alternative are the Eddington-Finkelstein (EF) coordinates systems (v,r,θ,φ) that is regular on the Schwarzschild horizon. It is a pair of coordinate systems which are adapted to radial null geodesics for Schwarzschild geometry. It is rarely necessary to employ coordinates that cover all four regions of the Kruskal diagram although it is often desirable to have coordinates that are well behaved at $r = 2M$.
In such a situation we choose v and r or u and r as coordinates. These coordinate systems are called ingoing and outgoing Eddington-Finkelstein coordinates respectively (Poisson (2004); page 167).

For radial light rays, we have $d\theta = d\varphi = 0$ and $ds^2 = 0$, the Schwarzschild metric equation (5.29) takes the form

$$0 = ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 \tag{5.46}$$

Simplifying, we have

$$dt^* = \pm\frac{dr}{1-\frac{2M}{r}} = \pm\frac{\frac{r}{2M}dr}{\frac{r}{2M}-1} = \pm\left(1+\frac{1}{\frac{r}{2M}-1}\right)dr \quad (5.47)$$

$$t^* = \pm\left(r + 2M\ln\left(\frac{r}{2M}-1\right)\right) \quad (5.48)$$

We have a change of coordinates to derive a new metric. Here, we keep *r,θ* and *φ* but replace the time component *t* with

$$t = v - t^* = v - r - 2M\ln\left(\frac{r}{2M}-1\right) \quad (5.49)$$

and differentiating and squaring, we have

$$dt^2 = dv^2 - 2\left(1-\frac{2M}{r}\right)^{-1} dr dv + \left(1-\frac{2M}{r}\right)^{-2} dr^2 \quad (5.50)$$

where *r,t,* and *M* have their usual meanings in the Schwarzschild metric, and θ and φ are assumed to be unchanged. For either $r > 2M$ or $r < 2M$, insertion into the standard Schwarzschild line element (5.29) gives

$$ds^2 = -\left(1-\frac{2M}{r}\right)dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi) \quad (5.51)$$

Equation (5.51) is the ingoing Eddington-Finkelstein coordinates. The Schwarzschild metric expressed in this new coordinate is manifestly non-singular at $r = 2M$. The singularity at $r = 0$ remains. Thus the singularity at the Schwarzschild radius is a coordinate singularity that can be removed by a new choice of coordinate systems (Hartle (2003); Page 258). Figure (5.3) is a spacetime diagram showing the world lines of the Schwarzschild geometry's radial light rays plotted in EddingtonFinkelstein coordinates.

Null lines of constant *v* have been plotted 45^0 angle as they would usually be in flat space using $\tilde{t} = v - r$ as vertical coordinate. The light rays at $r = 2M$ are indicated by heavy solid line. Future light cones at a few intersections indicated. These tip further and further toward $r = 0$ as that radius approached. Radial light rays behave qualitatively differently outside the Schwarzschild radius $r = 2M$ than inside it. At every point with $r > 2M$, one radial ray (the $v = const$) is moving inward to smaller and smaller values of r. The other radial ray is moving outward to larger and larger values of r. In contrast, for $r > 2M$, both radial light rays are moving inward to smaller and smaller values of r and eventually to the singularity at $r = 0$. At the boundary $r = 2M$ separating the two regions, one radial ray moves inward while the other remains stationary, hovering at the Schwarzschild radius. The surface $r = 2M$ divides spacetime into two regions: the region outside $r = 2M$ from which light can escape to infinity and the region inside $r = 2M$, where gravity is so strong that not even light can escape. This is the defining feature of black hole geometry. The surface is called the *event horizon* or in short the horizon of the black hole (Hartle (2003); Page 258). No event that occurs inside the horizon can ever be seen by an observer that is outside. The properties of the event horizon are: It is a null hypersurface which is generated by null geodesic segments which have no future end points but which do have past end points (at the point of emission of the flushes); the divergence of these null geodesic generators is positive during the collapse phase and is zero in the final time independent state (DeWitt and DeWitt (1973); Page 9). It is never negative: the area of a 2-dimensional cross-section of the horizon increases monotonically from zero to a final value $16\pi r^2$. The event horizon at $r = 2M$ may also be described as a *Killing horizon* in the sense that it is a null hypersurface on which the norm of the Killing vector ∂_t vanishes. In this case it is timelike on one side and spacelike on the other. It has constant area given by integrating the square root of the determinant of the metric over all θ and φ. That is $\sqrt{r^4 \sin^2 \theta}$. Therefore $A = \int_0^{2\pi} d\varphi \int_0^{\pi} r^2 \sin\theta d\theta = 2\pi r^2 [-\cos]_0^{\pi} = 4\pi r^2 = 16\pi m^2$.

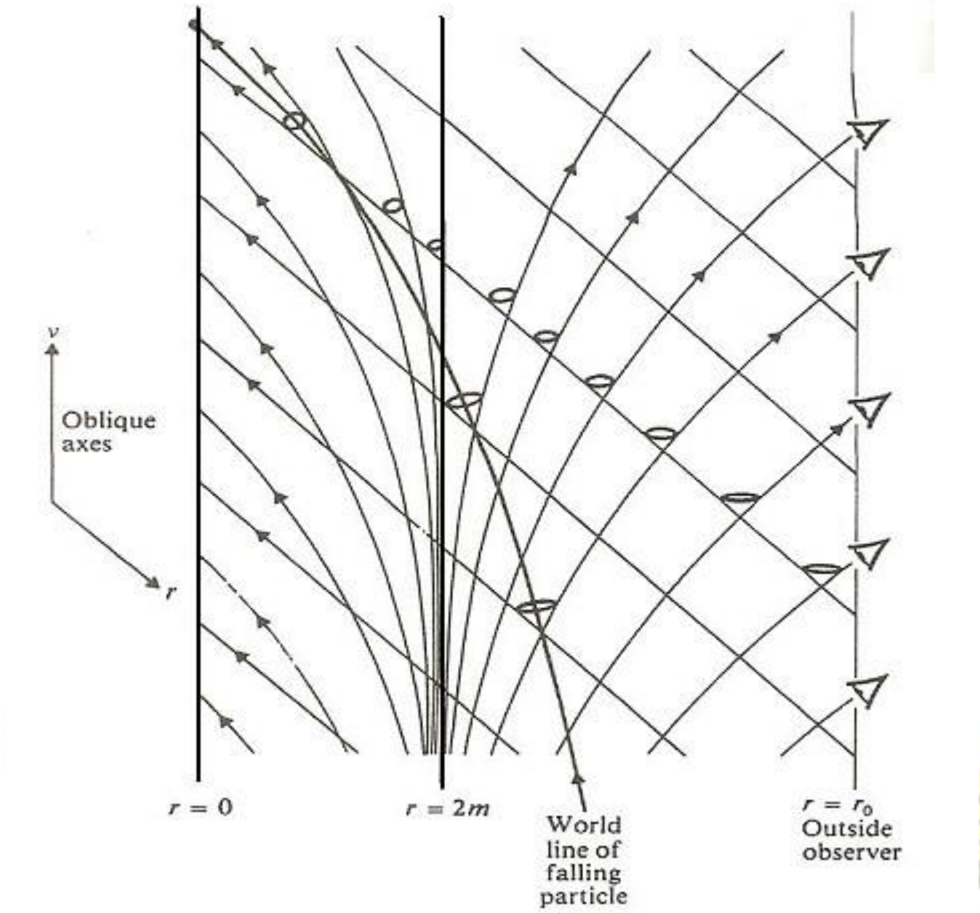

Figure 5.3: Radial null geodesic in Eddington-Finkelstein coordinates with θ and φ constant. Ingoing null geodesics are represented by lines on which $\bar{t} + r$ = const., while null geodesics propagating in the opposite direction have increasing values of r for $r > 2M$, but decreasing values for $r < 2M$

5.7 Collapse of a Spherically Symmetric Star

The basic features of such a collapsing spherically symmetric homogeneous dust cloud configuration are summarized in Figure (5.4). The gravitational collapse starts when the star surface is outside its Schwarzschild radius $r = 2M$, and a light ray emitted from the surface of the star can escape to infinity. However, once the star has collapsed below $r = 2M$, *a black hole*, that is a region of no escape, develops in the spacetime, which is bounded by the event horizon at $r = 2M$. Any point in this empty region below the surface $r = 2M$ represents a *trapped surface* (which is a two-dimensional sphere in spacetime) in that both the outgoing and in going families of null geodesics emitted from this point converge, and hence no light ray comes out of this region bounded by $r = 2M$ (Hawking and Ellis (1973); page 30). Then, the collapse to an infinite density and curvature singularity at the origin becomes inevitable in a finite proper time, as measured by an observer on the surface of the star. In this case, the black hole region in the resulting vacuum Schwarzschild spacetime is given by $0 < r < 2M$ and the outer boundary of this region, $r = 2M$, is called the *event horizon*. On the event horizon, only the radial outwards photons stay where they are, but all the rest of the photons move inwards. No information from this black hole region can propagate outside the $r = 2M$ boundary to any outside observer. In the Schwarzschild geometry, for a source situated outside $r = 2M$, part of the photon trajectories emitted with decreasing r values will go towards the black hole and fall into the singularity (Hawking and Ellis (1973); page 29). All the other null geodesics will escape to infinity and they intersect the future null infinity. If a source is located below $r = 2M$, no null geodesic would come out of the black hole and they necessarily end up in the singularity in the future. The final state of a complete gravitational collapse, either spherically symmetric or otherwise, could possibly be a vacuum spacetime that incorporates the rotation, and possibly also the electromagnetic fields associated with the object.

5.8 Penrose-Carter Diagram of the Schwarzschild Spacetime

In this thesis the spacetimes we will be working with are typically of dimensions higher than two, it is therefore important to compactify them into a two-dimensional picture that captures the causal structure of the original spacetime by using the so

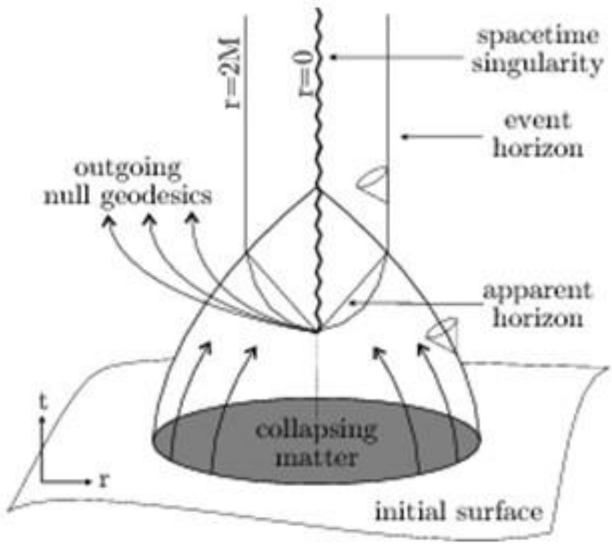

Figure 5.4: Homogeneous dust cloud collapse. The trapped surfaces form when the star enters $r = 2M$ radius. The event horizon forms prior to the singularity, creating a black hole as the collapse end state.

called Penrose-Carter diagram. This means that it is possible to attach a boundary to this picture which captures the idea of asymptotically flat spacetimes. For such spacetimes we can get what is sometimes a useful global picture of the causal structure (i.e. which events can be causally connected by using the Penrose-Carter diagrams). These make it possible to draw pictures of what is happening at infinity by changing the coordinates to bring infinity to a finite coordinate value. This distortion is carried out in such a way that the relationship between light rays is maintained. We say that these diagrams show the causal structure of spacetimes (Raine and Edwin (2009); Page 57).

Penrose diagrams are a useful way to represent the causal structure of spacetimes, especially if they have spherical symmetry, like the Schwarzschild black hole. They represent the geometry of a two-dimensional surface of fixed angular coordinates. Furthermore they "compactify" the geometry so that it can be drawn in total on the finite plane. We first consider Penrose diagram for Minkowski space.

5.8.1 The Penrose-Carter Diagram of Minkowski 2-dimensional Space

We start with the metric in polar coordinates

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega \tag{5.52}$$

Where $d\Omega = d\theta^2 + \sin^2\theta\varphi^2$ is a metric on a unit two-sphere and ranges of timelike and spacelike coordinates are: $-\infty < t < \infty$, $0 \le r < \infty$

In order to get coordinates with finite ranges, let us switch to null coordinates:

$$v = t + r = \tan\frac{1}{2}(T + X) \text{ and } u = t - r = \tan\frac{1}{2}(T - X) \tag{5.53}$$

With corresponding ranges $-\infty < u < \infty$, $-\infty < v < \infty$, $u \le v$

By means of straightforward calculations we find that in the new variables the flat metric becomes

$$ds^2 = \frac{-dT^2 + dX^2}{4\cos^2\frac{1}{2}(T+X)\cos^2\frac{1}{2}(T-X)} + r^2 d\Omega^2 \tag{5.54}$$

The conformal factor of the new metric, $4\cos^2\frac{1}{2}(T+X)\cos^2\frac{1}{2}(T-X)$ blows up at $|T \pm X| = \pi$ which makes the boundary of the compact (T,X) space-time infinitely far away from any of its internal point. This fact allows one to map the compact (T,X) space-time onto the non-compact (t,r) space-time. Furthermore, we can see that equality $dt^2 - dr^2 = 0$ implies also that $dT^2 - dX^2 = 0$ and vise versa. Hence, the conformal factor is irrelevant in the study of the properties of the light-like world-lines - those which obey, $ds^2 = 0$. Then, we drop off the conformal factor and draw the compact (T,X) space-time as shown on the Figure 5.5. On this diagram we show light-like rays by arrowed straight lines. The arrows on them show the direction of the light propagation, as $t \to +\infty$. Furthermore, on this diagram $I^{\pm}$ represent the entire space, $r \in (-\infty,+\infty)$ at $t = \pm\infty$. These are space-like past and future infinities. Also I^0 is the entire time line, $r \in (-\infty,+\infty)$ at $t = \pm\infty$, i.e. this is time-like space infinity. And finally $J^{\pm}$ are light-like past and future infinities, i.e. these are the curves on which light-like world-lines originate and terminate, correspondingly.

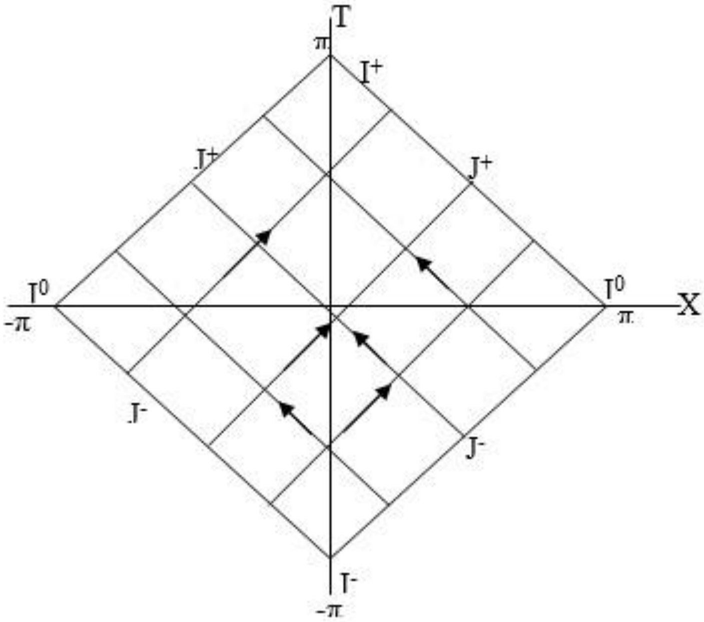

Figure 5.5: The Penrose-Carter diagram of Minkowski 2-dimensional space

5.8.2 Penrose-Carter Diagram for Schwarzschild Spacetime

One advantage of the double-null Kruskal coordinates is the fact that they make the causal structure of the Schwarzschild spacetime very clear. Another useful set of double-null coordinates is obtained by applying the transformation

$$\tilde{U} = \arctan U, \qquad \tilde{V} = \arctan V \tag{5.55}$$

By this transformation, the Kruskal metric becomes

$$ds^2 = -\frac{32M^3}{r}e^{-\frac{r}{2M}}\sec^2\tilde{U}\sec^2\tilde{V}d\tilde{U}d\tilde{V} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{5.56}$$

The rescaling of the null coordinates does not affect the appearance of radial light rays which propagate at 45^0 in the spacetime diagram based on the new coordinates, figure (5.6). However, while the range of the initial coordinates was infinite i.e.

$-\infty < U < \infty$, $-\infty < V < \infty$ it is finite for the new coordinates $-\frac{\pi}{2} < \tilde{U} < \frac{\pi}{2}$, $-\frac{\pi}{2} < \tilde{V} < \frac{\pi}{2}$. The entire spacetime is therefore mapped into a finite domain of the $\tilde{U} - \tilde{V}$ plane. Even though, this Compactification of manifold introduces bad coordinate singularities at the boundaries of the new coordinate system, these are of no concern when the purpose is simply to construct a compact map of the entire spacetime (Poisson (2004); Page 169).

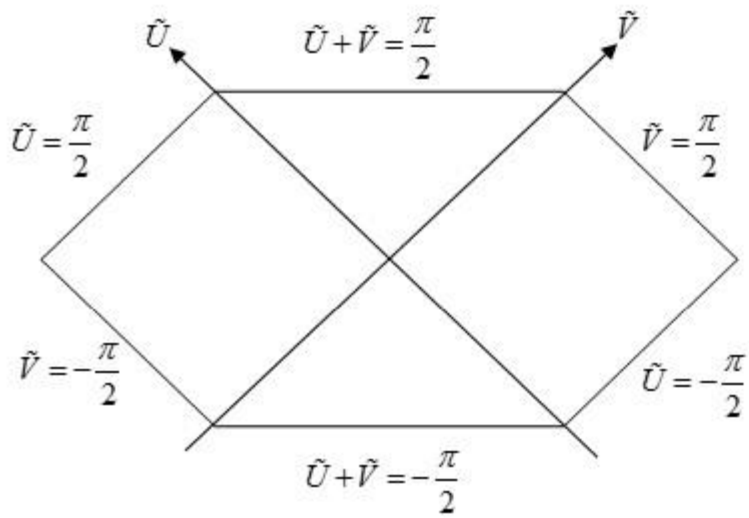

Figure 5.6: Compactify coordinates for the Schwarzschild spacetime

In the new coordinates, the surfaces $r = 2M$ are located at $\tilde{U} = 0$ and the singularities at $r = 0$, or $UV = 1$ which is equivalent to $\tan\tilde{U}\,\tan\tilde{V}$ and using $\tan(U+V) = \frac{\tan U - \tan V}{1-\tan U \tan V}$, we obtain $U + V = \tan^{-1}(\pm\infty) = \pm\frac{\pi}{2}$ The space-time is also bounded by the surfaces $\tilde{U} = \pm\frac{\pi}{2}$ and $\tilde{V} = \pm\frac{\pi}{2}$. The four points $(\tilde{U}, \tilde{V}) = \left(\pm\frac{\pi}{2}, \pm\frac{\pi}{2}\right)$ are singularities of the coordinate transformation. It is very important to assign names to the various boundaries of the compactified spacetime figure (5.7). The surfaces $\tilde{U} = \frac{\pi}{2}$ and $\tilde{V} = \frac{\pi}{2}$ are called future null infinity and labeled $\mathscr{I}^+$. The diagram makes it clear that $\mathscr{I}^+$ contains the future endpoints of all outgoing null geodesics (those along which r increases). Similarly, the surfaces

$\tilde{U} = -\frac{\pi}{2}$ and $\tilde{V} = -\frac{\pi}{2}$ are called past null infinity and are labeled ℓ^{+}. These contain the past end points of all ingoing null geodesics (those along which *r* decreases).

The points at which ℓ^{+} and ℓ^{-} meet are called spacelike infinity and are labeled i^0. These contain the end points of all spacelike geodesics. The points $(\tilde{U}, \tilde{V}) = (0, \frac{\pi}{2})$ and $(\tilde{U}, \tilde{V}) = (\frac{\pi}{2}, 0)$ are called future timelike infinity and are labeled i^+. These contain the future endpoints of all timelike geodesics that do not terminate at $r = 0$. Finally, the points $(\tilde{U}, \tilde{V}) = (0, -\frac{\pi}{2})$ and $(\tilde{U}, \tilde{V}) = (-\frac{\pi}{2}, 0)$ are called past time-like infinity and are labeled i^-. These contain the past endpoints of all timelike geodesics that do not originate at $r = 0$. Compactified maps such as the one displayed in figure (5.7) are called Penrose-Carter diagrams. They display, at a glance, the complete causal structure of the spacetime under consideration (Poisson (2004); Page 169).

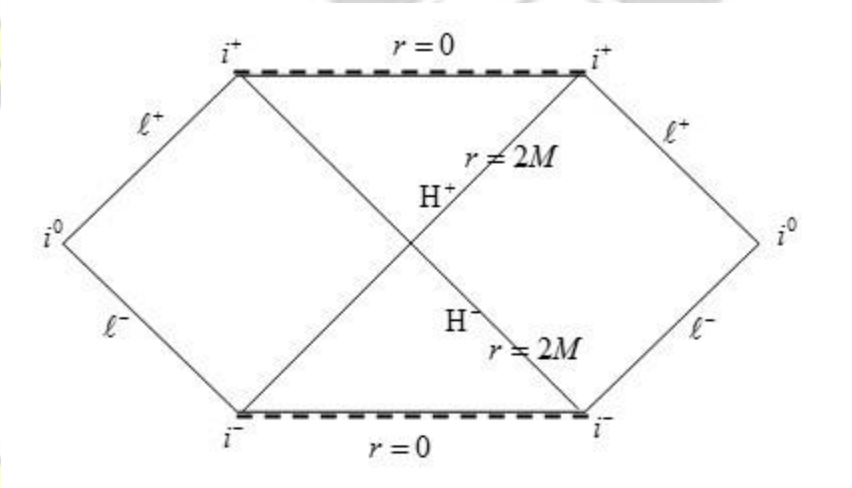

Figure 5.7: Penrose-Carter diagram of the Schwarzschild spacetime

5.9 Reissner-Nordström Solutions

The Reissner-Nordstrom (RN) solution represents the space-time outside a spherically symmetric charged body which carries an electric charge without spin or magnetic dipole so this is not a good representation of the field outside an electron. Therefore the energy-momentum tensor is that of the electromagnetic field in the space-time which results from the charge on the body. It is the unique spherically symmetric asymptotically flat solution of the Einstein-Maxwell equations which is locally similar to the Schwarzschild solution (Hawking and Ellis (1973); page 156). It has coordinates in which the metric takes the form

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}dr^2 + r^2 d\Omega^2 \tag{5.57}$$

Where M is the total Arnowitt, Deser, Misner (ADM) mass of spacetime and Q is the electric charge of the black hole. The electromagnetic-field tensor takes the form

$$F^{tr} = \frac{Q}{r^2} \tag{5.58}$$

From equation (5.57), $g(\partial_t \partial_t) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} = 0$ has zeros at $r = r_\pm$, where $r_\pm = M \pm \sqrt{M^2 - Q^2}$. The roots are both real and the RN spacetime truly contains a black hole, when $Q^2 \leq M^2$. The metric has singularities at r_+ and r_- where $r_\pm = M \pm \sqrt{M^2 - Q^2}$; it is regular in the regions defined by $r_+ < r < \infty$, $r_- < r < r_+$ and $0 < r < r_-$. The special case of a black hole with $Q^2 = M^2$ referred to as an extreme RN black hole is regular in the regions $r_+ < r < \infty$ and $0 < r < r_-$. The Reissner-Nordström (RN) geometry has two horizons, the outer horizon r_+ and an inner horizon r_-. The Reissner-Nordström time coordinate t is timelike outside the outer horizon, $r > r_+$, spacelike between the horizons $r_- < r < r_+$, and again timelike inside the inner horizon $r < r_+$. Conversely, the radial coordinate r is spacelike outside the outer horizon, $r > r_+$, timelike between the horizons $r_- < r < r_+$, and spacelike inside the inner horizon $r < r_-$. In the special case that the charge and mass are equal, $Q = M$, the inner and outer horizons merge into one, $r_+ = r_- = M$ and the metric becomes

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dt^2 + \left(1 - \frac{M}{r}\right)^{-2} dr^2 + r^2 d\Omega^2 \tag{5.59}$$

This special case describes the extremal Reissner-Nordstro¨m geometry. The extremal Reissner-Nordstro¨m geometry proves to be of particular interest in quantum gravity because its Hawking temperature is zero, and in string theory because extremal black holes arise as solutions under certain duality transformations. The singularities of the RN metric may be removed by introducing suitable coordinates extending the manifold to obtain a maximal analytic extension as in the Schwarzschild case.

We also observe that when $Q^2 > M^2$, $r_\pm$ turn imaginary and there would be no horizon. This case is not much important to our study because we are only interested in spacetimes with horizons. Just as for the Schwarzschild metric, the singularity at $r = 0$ is the curvature singularity. The Killing vector fields for the RN black hole are the same as that of the Schwarzschild black hole. The major differences that arise are due to the existence of two zeros in the factor in front of dt^2 in equation (5.57), rather than one as in the Schwarzschild case (Hawking and Ellis (1973); page 156). In particular this implies that the regions $r_+ < r < \infty$ and $0 < r < r_-$ are both static, whereas the region $r_+ < r < r_-$ (when it exists) is spatially homogeneous but is not static.

5.10 Trapped and Marginally Trapped Surfaces in Schwarzschild Black Hole.

The behaviour of light rays for $r = 2M$ and $r < 2M$ can be written invariantly and geometrically in terms of θ_l of a null geodesic congruence which measures the change in the cross-sectional area element $\sqrt{S}$ of the congruence by introducing the null vector.

$$n = -\partial_r \qquad l = \partial_v + \frac{1}{2}f(r)\partial_r \tag{5.60}$$

From the vector fields l^α and n^α defined in equation (5.60), we see that they are manifestly orthogonal to the constant (v,r) spheres and their expansions must involve area elements on these spheres so it is easy to calculate their expansions. These two future-pointing vector fields are both null, i.e. $n_\alpha n^\alpha = l_\alpha l^\alpha$ and are cross-normalized to $n_\alpha l^\alpha = -1$.

Analysis of trapped and marginally trapped surfaces in Schwarzschild black hole applying covariant divergence of a vector field

√

With $S = r^2 \sin\theta$ it is easy to calculate the covariant divergences as follows Covariant divergence of outgoing null vector is given by

$$\theta_l = \nabla \cdot \bar{l} = \frac{l'}{\sqrt{g}}\frac{\partial}{\partial r}\sqrt{g} = \frac{1}{2}\frac{f(r)}{2r^2 \sin\theta}\frac{\partial}{\partial r}(r^2 \sin\theta) = \frac{f(r)}{r} = \frac{r-2M}{r^2} \tag{5.61}$$

Covariant divergence of Ingoing null vector is given by

$$\theta_n = \nabla \cdot \bar{n} = \frac{n'}{\sqrt{g}}\frac{\partial}{\partial r}\sqrt{g} = -\frac{1}{2r^2 \sin\theta}\frac{\partial}{\partial r}(r^2 \sin\theta) = \frac{-2}{r} \tag{5.62}$$

From (5.61) and (5.62), we have,

$$\theta_l = \frac{r-2M}{r^2} \implies \begin{cases} \theta_l > 0 & r > 2M \\ \theta_l = 0 & r = 2M \\ \theta_l < 0 & r < 2M \end{cases} \tag{5.63}$$

$$\theta_l = \nabla \cdot \bar{n} = \frac{-2}{r} \tag{5.64}$$

In general, the divergence of a vector field results in a scalar field that is positive in some regions in space, negative in other regions and zero elsewhere. In flat space, the outgoing null vectors diverge so the divergence is positive $\nabla\cdot \bar{l} = \theta_l > 0$ whereas the ingoing null vectors converge so the divergence is negative $\nabla \cdot \bar{n} = \theta_n < 0$. But if there is a massive source inside the surface, its gravitational field has an attractive or converging effect. Close enough to a massive source; the outgoing null vectors converge and the divergence becomes negative i.e. . The surface S where both divergences are negative i.e. $\nabla \cdot \bar{l} = \theta_l < 0$ and $\nabla \cdot \bar{n} = \theta_n < 0$ are said to be trapped.

The surface where the divergence of outgoing null vector becomes zero and the divergence of ingoing null vector is less than zero is said to be marginally trapped (MTS) i.e. $\nabla \cdot \bar{l} = \theta_l = 0$ and $\nabla \cdot \bar{n} = \theta_n < 0$. This marginal surface is defined by (Hayward (1994); page 5), as a spatial 2-surface S on which one null expansion vanishes. The singularity theorems of (Penrose (1965a); Hawking and Penrose (1970); (Senovilla (2011); Page 6) have shown that the presence of such surfaces is the signature of a spacetime containing a black hole. These definitions clearly constitute a local concept and are related to very strong gravitational fields, since for weak fields, one has clearly $\theta_l = \nabla \cdot \bar{l} > 0$. As we have just seen, for a Schwarzschild black hole, all the natural definitions of the surface of a black hole agree. Thus, the $r = 2M$ surface is both the boundary of the trapped region and also the event horizon. This means the event and apparent horizons of the Schwarzschild spacetime coincide. This coincidence, however, is a consequence of the fact that the spacetime is stationary (Poisson (2004); Page 172). Matters are however not so simple in dynamical situations which will be looked at below and it is perhaps the simplest example of a dynamical black hole, namely the spherically symmetric Vaidya spacetime (Poisson (2004); Page 172).

5.10.1 Trapped and Marginally trapped surfaces in Schwarzschild black hole applying the flux of a vector field

The flux of outgoing null vector.

The field $\bar{g} = \frac{GM}{r^2}\hat{r}$ is radial and orthogonal to the surface and $\bar{l} = \frac{1}{2}f(r)\hat{r}$. The flux

$$\phi = \oint_S \bar{g} \cdot \bar{l} dA = \oint \frac{GM}{r^2}\hat{r} \cdot \frac{1}{2}f(r)\hat{r} dA = \frac{GM}{r^2}\frac{1}{2}f(r)r^2(4\pi)$$

$$= 2\pi GMf(r) = 2\pi GM\left(\frac{r-2M}{r}\right) \tag{5.65}$$

The flux of ingoing null vector

The field $\bar{g} = \frac{GM}{r^2}\hat{r}$ is radial and orthogonal to the surface and $n = -\hat{r}$. The flux

$$\phi = \oint_S \bar{g} \cdot \bar{n} dA = -\oint \frac{GM}{r^2} \hat{r}\hat{r} dA = -\frac{GM}{r^2} r^2 (4\pi) = -4\pi GM \tag{5.66}$$

From equations (5.65) and (5.66), we have the flux of outgoing null vector l

$$\phi_l = 2\pi GM \frac{r - 2M}{r} \implies \begin{cases} \phi_l > 0 & r > 2M \\ \phi_l = 0 & r = 2M \\ \phi_l < 0 & r < 2M \end{cases} \tag{5.67}$$

Flux of ingoing vector field n $\varphi_n = -4\pi GM$.

From these equations, for $r > 2M$, i.e. in flat space, the flux of S along the outgoing null vector is positive: $\varphi_l > 0$ whereas that along ingoing null vector is negative: $\varphi_n < 0$.

However, in the black hole region i.e. $r < 2M$, we get both fluxes to be negative. Such surfaces are known as trapped surfaces and play a fundamental role in black hole theory and, in particular, in the singularity theorems.

For $r = 2M$ hypersurface, $\varphi_l = 0$, $\varphi_n < 0$, is called marginally trapped surfaces. Thus, we see that the $r = 2M$ hypersurface separates the region where the spherically symmetric trapped surfaces live and are a signature of a black hole spacetime. Since Gauss's divergence theorem applies to closed surfaces, it means that we can use it to define trapped and marginally surfaces in Schwarzschild black hole.

5.11 Vaidya Spacetime

We have so far considered stationary balck holes, where the event and apparent horizon coincide in the Schwarzschild spacetime (Poisson (2004); Page 173). For more general black hole spacetimes the event and apparent horizons are distinct hypersurfaces. We therefore consider dynamical balck holes such as the Vaidya spacetime. The ingoing Vaidya metric is obtained by replacing M by $M(v)$ in the ingoing Eddington-Finkelstein metric as follows (Poisson (2004); Page 173).

$$ds^2 = -f dv^2 + 2dvdr + r^2 d\Omega, \qquad f = 1 - \frac{2M(v)}{r} \tag{5.68}$$

The stress-energy tensor for this metric

$$8\pi T_{\alpha\beta} = \frac{2}{r^2}\frac{dM}{dv}\partial_\alpha v \partial_\beta v \tag{5.69}$$

Now in this metric, the vector

$$k_\alpha = -\partial_\alpha v,\ k^\alpha = -\partial_r,\ k \cdot k = 0,\ k^\alpha \nabla_\alpha k^\beta = 0 \tag{5.70}$$

is tangent to ingoing null geodesics. Hence the Vaidya stress-energy tensor is

$$T_{\alpha\beta} = \frac{2}{4\pi r^2}\frac{dM}{dv} k_\alpha k_\beta \tag{5.71}$$

If k^α were timelike, we would interpret this as the stress-energy tensor of dust with density $\rho = \frac{1}{4\pi r^2}\frac{dM}{dv}$. However, k^α is actually null so we say that the ingoing Vaidya metric is sourced by radially infalling null dust (Poisson (2004); Page 173). We require that the null dust has positive density: $\rho > 0$ implies $\frac{dM}{dr} > 0$. Since $v = t - r$, this implies that the black-hole mass increases in time. Conversely, working with the outgoing Eddington-Finkelstein coordinates with $M = M(u)$, we can derive the outgoing Vaidya metric sourced by radially outgoing null dust. The positivity of the density in that case requires a decreasing black hole mass $\frac{dM}{dr} < 0$.

Trapped and Marginally trapped surfaces in Vaidya spacetime

The null normal vectors orthogonal to the sphere

$$l = \frac{\partial}{\partial v} + \frac{1}{2}\left(1 - \frac{2M(v)}{r}\right)\frac{\partial}{\partial r}, \qquad n = -\frac{\partial}{\partial r} \tag{5.72}$$

and their covariant divergences are respectively given by (5.73) and (5.74)

$$\theta_l(v,r) = \nabla \cdot \bar{l} = \frac{n^r}{\sqrt{g}}\frac{\partial}{\partial r}\sqrt{q} = \frac{r-2M(v)}{r^2} \tag{5.73}$$

$$\theta_n(v,r) = \nabla \cdot \bar{n} = \frac{n^r}{\sqrt{g}}\frac{\partial}{\partial r}\sqrt{q} = \frac{-2}{r} \tag{5.74}$$

Trapped and marginally trapped surfaces in Vaidya spacetime applying covariant divergence of a vector field

$$\theta_l(v,r) = \nabla \cdot \bar{l} = \frac{l^r}{\sqrt{g}}\frac{\partial}{\partial r}\sqrt{g} = \frac{1}{2}\frac{f(r)}{2r^2\sin\theta}\frac{\partial}{\partial r}(r^2\sin\theta) = \frac{f(r)}{r} = \frac{r-2M(v)}{r^2} \tag{5.75}$$

Covariant divergence of Ingoing null vector is given by

$$\theta_n(v,r) = \nabla \cdot \bar{n} = \frac{n'}{\sqrt{g}}\frac{\partial}{\partial r}\sqrt{g} = -\frac{1}{2r^2\sin\theta}\frac{\partial}{\partial r}(r^2\sin\theta) = \frac{-2}{r} \tag{5.76}$$

Thus, just like the Schwarzschild metric, the $r = 2M(v)$ is marginally trapped surface. However, with $r < 2M(v)$ (the black hole region), both divergences are negative and the surface is said to be trapped.

From (5.75) and (5.76) , the field $\bar{g} = \frac{GM}{r^2}\hat{r}$ is radial and orthogonal to the surface

$$l = \frac{\partial}{\partial v} + \frac{1}{2}\left(1 - \frac{2M(v)}{r}\right)\frac{\partial}{\partial r} \tag{5.77}$$

The flux

$$\begin{aligned}\phi_l &= \oint_S \bar{g} \cdot \bar{l} dA = \oint_S \frac{GM}{r^2}\hat{r}\frac{1}{2}\left(1 - \frac{2M(v)}{r}\right)\hat{r}dA \\ &= \frac{GM}{r^2}\frac{1}{2}\left(1 - \frac{2M(v)}{r}\right)(4\pi r^2) = 2\pi GM\left(\frac{r-2M(v)}{r}\right)\end{aligned} \tag{5.78}$$

$$n = -\frac{\partial}{\partial r}$$

$$\phi_n = \oint_S \bar{g} \cdot \bar{n} dA = \oint_S \frac{GM}{r^2}\hat{r} \cdot (-\hat{r})dA = -\frac{GM}{r^2}(4\pi r^2) = -4\pi GM \tag{5.79}$$

In the same way like the Schwarzschild metric, for $r > 2M(v)$, i.e. in flat space, the flux of S along the outgoing null vector is positive: $\varphi_l > 0$ whereas that along ingoing null vector is negative: $\varphi_n < 0$. But if $r < 2M(v)$ (the black hole region), both fluxes are negative. Such surfaces are known as trapped surfaces. For $r = 2M(v)$ hypersurface, $\varphi_l = 0$, $\varphi_n < 0$ is called marginally trapped surface.

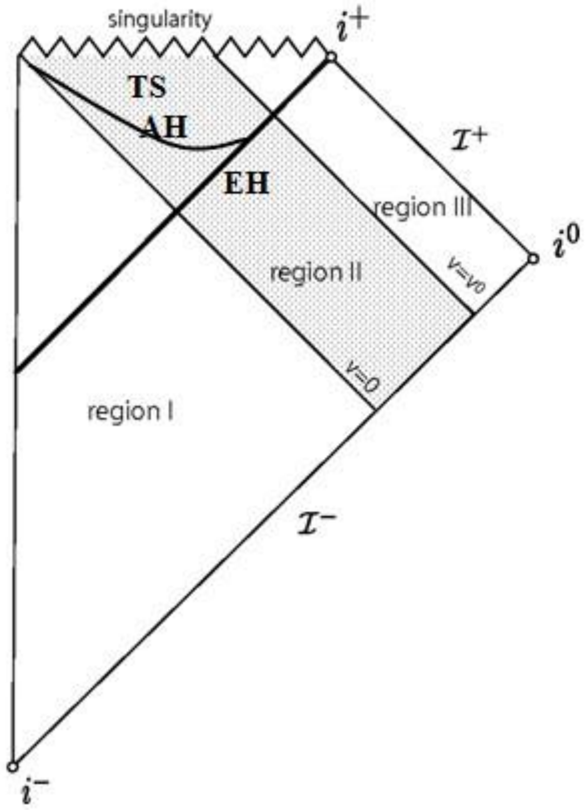

Figure 5.8: Penrose-Carter conformal diagram for Vaidya spacetime. Region I is flat and II is the Vaidya spacetime region. Region III is the Schwarzschild spacetime region. The event horizon EH is seen to be distinct from the $r = 2M(v)$ surface. The two agree only in the final Schwarzschild region. The Apparent Horizon (Marginally trapped tube) AH is described by $r = 2M(v)$. Trapped surface lies inside the apparent horizon.

5.12 Discussion of Result

From the above analysis, we notice that local horizon can be found easily in spherical spacetimes. In the Schwarzschild spacetime, the event horizon and apparent horizon coincide. This is due to the fact that the spacetime is stationary. For a more general spacetime like the Vaidya spacetime (the dynamic spacetime), the event horizon and apparent horizon are distinct hypersurfaces. From figure

(5.8), the two are equal in the Schwarzschild region. The three dimensional boundary of the region of the spacetime that contains trapped surfaces - the trapped region is the trapping horizon and its two dimensional intersection with a spacelike hypersurface is called the apparent horizon. The apparent horizon is therefore a marginally trapped surface- a closed two surface on which one of the congruences has a zero expansion. Unless the null energy condition is violated, the apparent horizon always lies in the event horizons in the dynamic situations like the Vaidya spacetime (Poisson (2002); page 155). This implies the boundary of the region containing trapped surfaces is not in general, the event horizon as shown by (Ben-Dov (2007); Page 27).

Chapter 6

MAIN RESULT 2: GLOBAL AND LOCAL CHARACTERIZATION OF THE KERR BLACK HOLE

6.1 Axially Symmetric Spacetime

So far we have studied two exact solutions of the Einstein equations which describe black holes (Schwarzschild metric and the Reissner-Nordstr¨om metric)

We therefore look at the Kerr metric which is stationary and axially symmetric. In general, astronomical bodies are rotating and so one would not expect the solution outside them to be exactly spherically symmetric. The standard relativistic model of the gravitational field of a rotating star is the Kerr spacetime. This spacetime is fully revealed only when the star collapses, leaving a black hole- otherwise the bulk of the star blocks exploration (O'Neill (1995); page 55). The importance of the Kerr metric stems from the black hole uniqueness theorems, which establish uniqueness of Kerr black holes under suitable global conditions (Chrusciel et al. (2012); Page

20). Axisymmetry is also likely to play a key role in describing black hole formation. Investigating axisymmetric spacetimes will enable us to see what properties of the spherically symmetric case are unique to axisymmetric case.

6.2 Kerr metric in Boyer-Lindquist Coordinates

The Kerr solutions are the only known family of exact solutions which could represent the stationary axisymmetric asymptotically flat field outside a rotating massive object (Hawking and Ellis (1973); Page 161). The Kerr metric which describes a rotating black hole depends on two parameters, its mass M and its rate of rotation (angular momentum J). The most interesting case is the slow rotation. It reduces entirely to the Schwarzschild spacetime if rotation stops (O'Neill (1995); Page 55). The metric looks very simple in the Boyer-Lindquist coordinate system. These coordinates are particularly useful in that they minimize the number of off-diagonal components of the metric. These coordinates help particularly in analyzing the asymptotic behaviour and in trying to understand the key difference between an "event horizon" and an "ergosphere". In these coordinates the metric has only one off-diagonal component and takes the form

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right) dt^2 - \frac{4Mar\sin^2\theta}{\rho^2} dt d\phi + \frac{\Sigma}{\rho^2}\sin^2\theta\phi^2 + \frac{\rho^2}{\triangle} dr^2 + \rho^2 d\theta^2 \tag{6.1}$$

$$= -\frac{\rho^2\triangle}{\Sigma} dt^2 + \frac{\Sigma}{\rho^2}\sin^2\theta(d\phi - \omega dt)^2 + \frac{\rho^2}{\triangle} dr^2 + \rho^2 d\theta^2 \tag{6.2}$$

where $a = J/M$, $\rho^2 = r^2 + a^2\cos^2\theta$, $4 = r^2 - 2Mr + a^2$, $\Sigma = (r^2 + a^2)^2 - a^2 4 \sin^2\theta$, $\omega = -\frac{g_{t\phi}}{g_{\phi\phi}} = \frac{2Mar}{\Sigma}$

The components of the inverse metric are

$$g^{tt} = \frac{-\sum}{\rho^2\triangle},\; g^{t\phi} = -\frac{2Mar}{\rho^2\triangle},\; g^{\phi\phi} = \frac{\triangle - a^2\sin^2\theta}{\rho^2\triangle\sin^2\theta},\; g^{rr} = \frac{\triangle}{\rho^2},\; g^{\theta\theta} = \frac{1}{\rho^2}$$

The coordinates (t,r,θ,φ) are called Boyer-Lindquist coordinates. The parameter a, termed the *Kerr parameter*, has units of length in geometrized units just like the mass. The parameter J will be interpreted as angular momentum and the parameter M as the mass for the black hole. The Kerr metric has the following properties. It is not static implies it is not invariant for time reversal. It is stationary and does not depend explicitly on time t. It is axisymmetric and does not depend explicitly on φ. This metric form is clearly invariant under simultaneous inversion of t and φ, i.e. under the transformation $t \to -t, \varphi \to -\varphi$ although it is not invariant under inversion of t alone (except when $a = 0$). It is a vacuum solution of the Einstein equations, valid in the absence of matter. If the black hole is not rotating i.e. $a = J/M = 0$, the Kerr spacetime reduces to the Schwarzschild spacetime (Chandrasekhar (1983); Page 289; O'Neill (1995); Page 58) then;

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta + \sin\theta^2 d\theta^2) \qquad (6.3)$$

The Kerr family thus includes the Schwarzschild black hole in the special case of zero angular momentum.

In the limit $M \to 0$ with a 6= 0, the metric reduces to

$$ds^2 = -dt^2 + \frac{r^2 + a^2\cos^2\theta}{r^2 + a^2}dr^2 + (r^2 + a^2\cos^2\theta)d\theta^2 + (r^2 + a^2)\sin^2\theta d\phi \qquad (6.4)$$

This is flat Minkowski space in so-called "oblate spheroidal" coordinates, and you can relate them to the usual Cartesian coordinates of Euclidean 3-space by defining (Carroll (2004); Page 262).

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \qquad (6.5)$$

where

$\sqrt{\ \ \ \ \ }$ $\sqrt{\ \ \ \ \ }$

$$x = \quad r^2 + a^2 \sin\theta \cos\varphi, \; y = \quad r^2 + a^2 \sin\theta \sin\varphi, \; z = r\cos\theta \tag{6.6}$$

6.3 Symmetries of the Kerr metric

Being stationary and axisymmetric, the Kerr metric admits two Killing vector fields, both of which are manifest; since the metric coefficients are independent of t and φ, both $K\partial_t$ and $r = \partial_\varphi$ are Killing vectors. The Killing vector R^μ expresses the axial symmetry of the solution (Carroll, 2004; Page 262). The vector K^μ is not orthogonal to t = *const* hypersurfaces, and in fact is not orthogonal to any hypersurfaces; this means the metric is stationary but not static. This makes sense because the black hole is spinning and so not static. It is stationary because it is spinning at exactly the same way at all times. In other words, the metric cannot be static because it is not time-reversal invariant, since that would reverse the angular momentum of the black hole (Carroll (2004); Page 262).

6.4 Singularity of the Kerr metric

The Kerr metric (6.1) is singular for 4 = 0 and for $\rho = 0$. By computing the curvature invariants

$$R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = \frac{48M^2(r^2 - a^2\cos^2\theta)(\rho^4 - 16a^2r^2\cos^2\theta)}{\rho^{12}}$$

one finds that they are regular at 4 = 0, and singular at $\rho = 0$. Thus $\rho = 0$ is a true, curvature singularity of the manifold, whereas 4 = 0 is a coordinate singularity (Gourgoulhon (2017); page 174). Notice that in the Schwarzschild limit ($a = 0$) $\rho = r^2 = 0$ gives the curvature singularity, while (for r 6= 0) 4 = $r(r - 2M) = 0$ gives the coordinate singularity at the horizon. The metric takes the form

$$g_{\mu\nu} = \begin{pmatrix} g_{tt} & 0 & 0 & g_{t\phi} \\ 0 & \frac{\rho^2}{\triangle} & 0 & 0 \\ 0 & 0 & \rho & 0 \\ g_{t\phi} & 0 & 0 & g_{\phi\phi} \end{pmatrix} \tag{6.7}$$

where

$$\begin{aligned} g_{tt} &= -\left(1 - \tfrac{2Mr}{\rho^2}\right), \\ g_{t\phi} &= -\tfrac{2Mr}{\rho^2} a \sin^2\theta \\ g_{\phi\phi} &= \left[r^2 + a^2 \tfrac{2Mr}{\rho^2} a^2 \sin^2\theta\right] \sin^2\theta = \tfrac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)^2 - a^2\sin^2\theta\triangle\right] \end{aligned} \tag{6.8}$$

The inverse of this metric is

$$g^{\mu\nu} = \begin{pmatrix} g^{tt} & 0 & 0 & g^{t\phi} \\ 0 & \frac{\triangle}{\rho^2} & 0 & 0 \\ 0 & 0 & \frac{1}{\rho} & 0 \\ g^{t\phi} & 0 & 0 & g^{\phi\phi} \end{pmatrix} \tag{6.9}$$

where

$$\begin{aligned} g^{tt} &= -\tfrac{1}{\triangle}\left[r^2 + a^2 \tfrac{2Mra^2}{\rho^2}\sin^2\theta\right] \\ g^{t\phi} &= -\tfrac{2Mra}{\rho^2\triangle} \\ g^{\phi\phi} &= \left[r^2 + a^2\tfrac{2Mr}{\rho^2}a^2\sin^2\theta\right]\sin^2\theta = \tfrac{\triangle - a^2\sin^2\theta}{\rho^2\triangle\sin^2\theta} \end{aligned} \tag{6.10}$$

The curvature singularity at

$$\rho^2 = r^2 + a^2\cos^2\theta = 0 \tag{6.11}$$

occurs only in the equatorial plane $\theta = \frac{\pi}{2}$ at r= 0. If we interpret the Boyer-Lindquist coordinates (t,r,θ,φ) as spherical polar coordinates, like in the Schwarzschild spacetime, we are restricted to only $\theta = \frac{\pi}{2}$ at r= 0 but what if $\theta \neq \frac{\pi}{2}$ at r= 0 This has no meaning in polar coordinates so we need a coordinate system which has not the coordinate singularity r = 0 to be able to distinguish and analyze the curvature singularity. The Kerr-Schild coordinates can make sense of this statement (Chandrasekhar (1983); Page 308). In order to understand the singularity structure, we now change coordinate frame, to the so-called Kerr-Schild coordinates, which are well defined in $r = 0$. The metric in Kerr coordinates $(t,r,\theta,\bar{\varphi})$ is given by

$$ds^2 = -d\bar{t}^2 + dr^2 + \rho^2 dd\theta^2 + (r^2+a^2)\sin^2\theta d\bar{\phi}^2 - 2a\sin^2\theta dr d\bar{\phi} + \frac{2Mr}{\rho^2}(d\bar{t} + dr - a\sin^2\bar{\phi})^2 \tag{6.12}$$

The Kerr-Schild coordinates $(t,x,y,\bar{z})$ are defined by

$$\begin{aligned} x &= \sqrt{x^2+a^2}\sin\theta\cos\left(\bar{\theta} + \arctan\tfrac{a}{r}\right) \\ y &= \sqrt{x^2+a^2}\sin\theta\sin\left(\bar{\theta} + \arctan\tfrac{a}{r}\right) \\ z &= r\cos\theta \end{aligned} \tag{6.13}$$

From these relations, we have

$$x^2 + y^2 = (r^2 + a^2)\sin^2\theta, \qquad z^2 = r^2\cos^2\theta \tag{6.14}$$

Thus

$$\frac{x^2+y^2}{r^2+a^2} + \frac{z^2}{r^2} = 1 \tag{6.15}$$

The surfaces with constant r are ellipsoids figure (6.1) and

$$\frac{x^2+y^2}{a^2\sin^2\theta} - \frac{z^2}{a^2\cos^2\theta} = 1 \tag{6.16}$$

Then the surfaces with constant θ are half-hyperboloids figure (6.2)

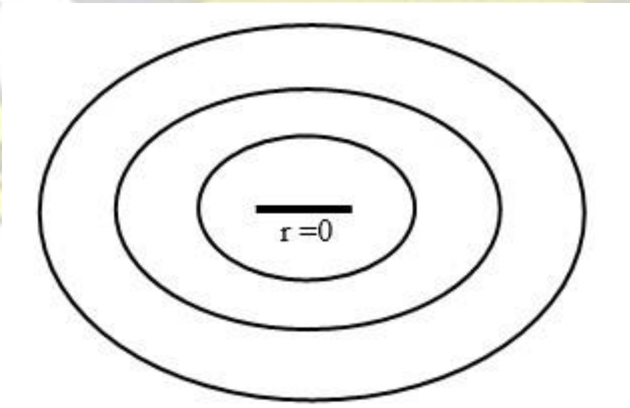

Figure 6.1: $r = const$ ellipsoidal surfaces in the Kerr-Schild frame; the thick line represents the $r = 0$ disk

Concerning the singularity in the Kerr space-time, the non-vanishing components of the Riemann tensor diverge only for $r = 0$ and $\theta = \pi/2$ and it is the only singularity that we have there. The divergence at $r = 0$ occurs only for $\theta = \pi/2$ and it is clear that its nature cannot be the same as the singularity $r = 0$ of the Schwarzschild and Reissner-Nordstrom space-times (Chandrasekhar (1983); Page 308). The real nature of the singularity of the Kerr spacetime can be well understood by first eliminating the inherent ambiguity in the coordinate system (r,θ,φ) at $r = 0$. This ambiguity can be abolished by the choice of the Cartesian coordinate system (x,y,z). In these coordinates, r^2 is implicitly defined in terms of x,y and z as $(x^2 + y^2) = (r^2 + a^2)\sin^2\theta$. The surfaces of constant r are confocal

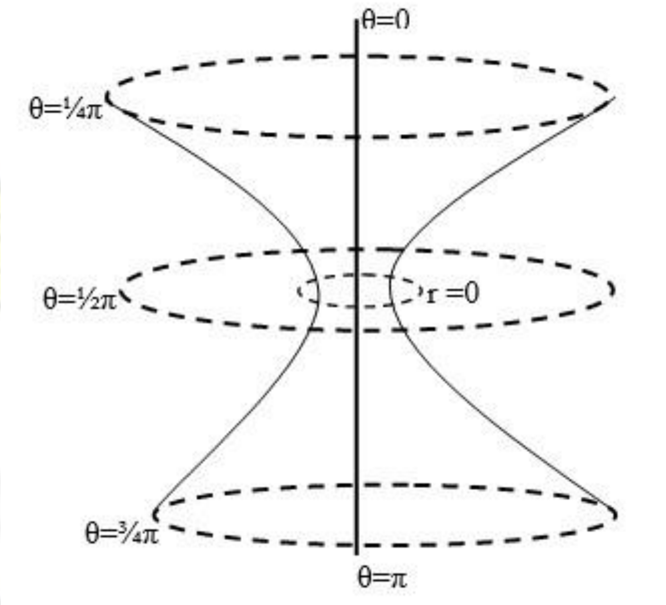

Figure 6.2: θ = *const* half-hyperboloidal surfaces in the Kerr-Schild frame; the thick ring represents the $r = 0$, $\theta = \pi/2$ singularity

ellipsoids whose principal axes coincide with the coordinate axes. These ellipsoids degenerate, for $r = 0$, to the disc $x^2 + y^2 \leq a^2$, $z = 0$. The point, ($r = 0$, $\theta = \pi/2$), corresponds to the ring $x^2 + y^2 = a^2$, $z = 0$ and the singularity along this ring is the only singularity of the Kerr space-time (Chandrasekhar (1983); Page 309). See figure (6.2).

6.5 The Event Horizon of the Kerr metric

The Kerr metric is stationary and axially symmetric, unlike the Schwarzschild metric which is static or spherically symmetric The Kerr metric is the important exact solution of the Einstein equations in astrophysics. Kerr metric is stationary because it has a time-translation Killing vector $\xi = \partial_t$, and axially symmetric because it has rotational Killing vector $\eta\partial_\varphi$. In general the Killing vector of the Kerr metric is of the form $K = a\xi + b\eta$ (Carroll (2004); Page 269). In the Schwarzschild and Reissner-Nordstro¨m solutions, the Killing vector K^α which is timelike at large values of r is timelike everywhere in the region I, (figure (6.3) below) becoming null on the surfaces $r = 2M$ and $r = r_+$ respectively. These surfaces are null. This means that a particle which crosses one of these surfaces in the future direction cannot return again to the same region. They are the boundary of the region of the solution from which particles can escape to the infinity and are called the event horizons (Carroll (2004); Page 258).

The event horizon of the Kerr spacetime is a null 3-dimensional surface. Its spatial slices have the geometry of a 2-dimensional distorted sphere. The rotating black hole exists for $a \leq M$. For $a > M$, the Kerr solution does not have a horizon and it describes a naked singularity. It is generally believed that such a singularity does not arise in real physical processes, like gravitational collapse, (Frolov and Novikov (1997); Page 248). The collapse with a formation of a black hole is possible when the system loses enough of its angular momentum so that the condition $a/M \leq 1$ is satisfied. The event horizon is the inner boundary of the ergosphere. The infinite redshift surface is located outside the horizon and touches it only at two points, the north and south poles.

6.6 Types of Rotations and their Event Horizons in the Kerr Spacetime

If we consider the mass $M > 0$ to be constant and vary the angular momentum per unit mass a, the following types of rotations are found:

1. $a^2 = 0$ gives Schwarzschild spacetime
2. $0 < a^2 < M^2$ gives slow rotating Kerr spacetime (slow Kerr)
3. $a^2 = M^2$ gives extreme Kerr spacetime
4. $a^2 > M^2$ gives rapidly rotating Kerr spacetime (fast Kerr)

The difference between these rotation types is given by the horizon function

$4 = r^2 - 2Mr + a^2$:

For the Schwarzschild spacetime 4 has two roots 0 and $2M$. For the slow Kerr, 4

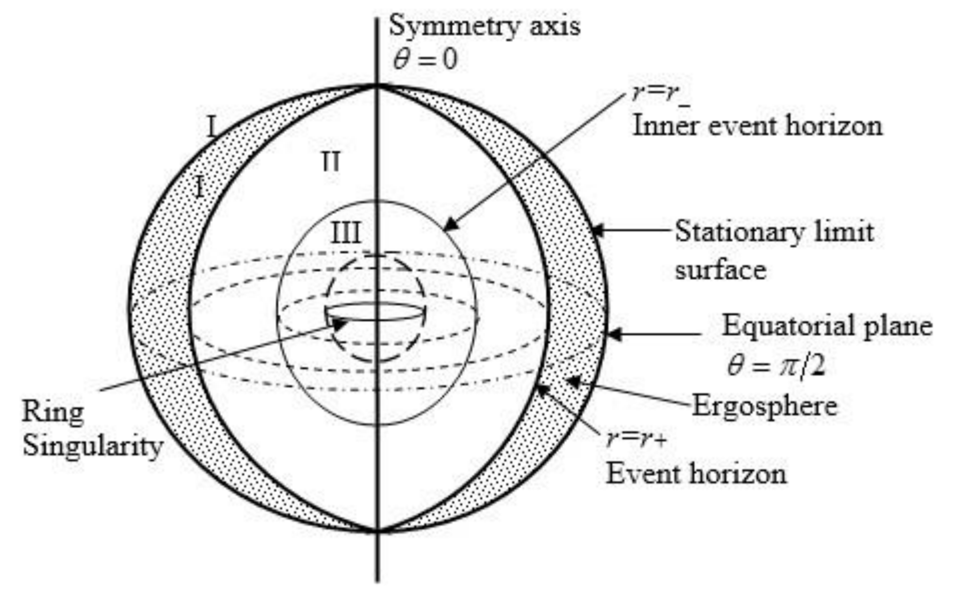

Figure 6.3: In the Kerr solution with $0 < a^2 < m^2$, the ergosphere between the stationary limit surface and the horizon at $r = r_+$ is a region in which it is possible to enter and leave again but not to remain stationary. Particles can escape to infinity from region I (outside the event horizon $r = r_+$) but not from region II (between $r < r_+$ and $r < r_-$) and region III ($r < r_-$; this region contains the ring singularity) (O'Neill (1995); Page 63

)

has two roots $r_\pm = M \pm (M^2 - a^2)^{\frac{1}{2}} < 2M_-$. For extreme Kerr, 4 = $(r - M)^2$ so $r = M$ is a double root. For fast Kerr, 4 has no real root.

These zeros of 4 give horizons which are crucial features of relativistic gravitation On the horizon H: 4 = 0. The cases above show that, for fast Kerr spacetime, there is no horizon; for extreme Kerr spacetime, there is a single horizon H: $r = M$; and for slow Kerr spacetime there are slow two horizons H_+: $r = r_+$ and H_-: $r = r_-$ (Chandrasekhar (1983); Page 62).

6.7 Null directions

The standard way of treating problems in the general theory of relativity used to be to consider the Einstein's field-equation in a local coordinate basis adapted to the problem on hand. But in recent years, it has appeared advantageous, in some contexts, to proceed somewhat differently by choosing a suitable tetrad basis of four linearly independent vector-fields, projecting the relevant quantities on to the chosen basis and considering the equations satisfied by them. A tetrad is a field which consists of a set of four orthonormal vectors at each point of spacetime. In the applications of the tetrad formalism, the choice of the tetrad basis depends on the underlying symmetries of the space-time we wish to grasp and is, to some extent, a part of the problem. Besides, it is not always clear what the relevant equations are, and what the relations among them may be. Considering the axisymmetric spacetime, the orbits of the isometry group are spacelike curves rather than spacelike two surfaces. Because any spacelike two-surface is a candidate for a horizon, one may ask whether there are two-surfaces selected by null vectors, expansions of which will select out the event horizon in the Kerr solution, which we can use to select out horizons in dynamical situations. The only place to start in the Kerr metric is the two principal null vectors of the Kerr solution.

In the Boyer-Lindquist coordinates (t,r,θ,φ), (Chandrasekhar (1983); Page 299) gives real null vectors **i** and **n** of the Newman Penrose formalism in terms of null geodesics and adjoining to them a complex null-vector **m**, orthogonal to them and thus well pose at the past horizon. The principal null congruences are geodesic and shear-free. In Boyer-Lindquist coordinates these geodesics are defined by the equations

$$\frac{dt}{d\lambda} = \frac{r^2+a^2}{\triangle}, \ \frac{dr}{d\lambda} = \pm 1, \ \frac{d\theta}{d\lambda} = 0, \ \frac{d\phi}{d\lambda} = \frac{a}{\triangle}$$

where λ is an affine parameter and the null vectors are given by

$$l^\mu = \frac{r^2+a^2}{\triangle}\left(\frac{\partial}{\partial t}\right)^\mu + \left(\frac{\partial}{\partial r}\right)^\mu + \frac{a}{\triangle}\left(\frac{\partial}{\partial \phi}\right)^\mu \quad \text{OR}$$

$$l^\mu = \frac{1}{\triangle}(r^2+a^2, \triangle, 0, a) \tag{6.17}$$

$$n^\mu = \frac{r^2+a^2}{2\rho^2}\left(\frac{\partial}{\partial t}\right)^\mu - \frac{\triangle}{2\rho^2}\left(\frac{\partial}{\partial r}\right)^\mu + \frac{a}{2\rho^2}\left(\frac{\partial}{\partial \phi}\right)^\mu \quad \text{OR}$$

$$n^\mu = \frac{1}{2\rho^2}(r^2+a^2, -\triangle, 0, a) \tag{6.18}$$

$$m^\mu = \frac{1}{\bar{\rho}\sqrt{2}}\left[ia\sin\theta\left(\frac{\partial}{\partial t}\right)^\mu + \left(\frac{\partial}{\partial \theta}\right)^\mu + \frac{i}{\sin\theta}\left(\frac{\partial}{\partial \phi}\right)^\mu\right] \quad \text{OR}$$

$$m^\mu = \frac{1}{\bar{\rho}\sqrt{2}}(ia\sin\theta, 0, 1, i\cos\mathrm{ec}\theta) \tag{6.19}$$

Where $\bar{\rho} = r + ia\cos\theta$, $\bar{\rho}^* = r - ia\cos\theta$ The correct normalization for the null vectors $l^\alpha l_\alpha = n^\alpha n_\alpha = 0$, $l^\alpha n_\alpha = -1$ and $m^\mu \bar{m}_\mu = 1$. The covariant forms of the basis vectors are

$$l_\mu = \frac{1}{\triangle}(\triangle, -\rho^2, 0, -a\triangle\sin^2\theta) \ (6.20) \ ) \qquad (6.21)$$

$$n_\mu = \frac{1}{2\rho^2}(\triangle, +\rho^2, 0, -a\triangle\sin^2\theta) \qquad (6.22)$$

The only $m_\mu = \frac{1}{\bar{\rho}\sqrt{2}}(ia\sin\theta, 0, -\rho^2, -i(r^2+a^2)\sin\theta$ nonvanishing Weyl scalar is

$$\psi_2 = -C_{abcd} l^a m^b n^c \bar{m}^d = -\frac{M}{(r - ia\cos\theta)^3} \tag{6.23}$$

This implies that r = 0 and $\pi/2$ is a true curvature singularity whose shape is a ring with coordinate radius a. It also follows from equation (6.23) that the Kerr solution belongs to the type D metric in the algebraic classification of the Weyl tensor (Chandrasekhar (1983); Page 299).

6.8 Kerr in Advanced Eddington-Finkelstein Coordinates

The system of Boyer-Lindquist coordinates (t,r,θ,φ) breaks down when 4 = 0 i.e. on a possible horizon. In order to express the Kerr metric in a system of coordinates which remains regular there, we define a generalization of the familiar advanced time (or ingoing) Eddington-Finkelstein coordinates (v,r,θ,φ) which are well behaved on the future horizon but singular on the past horizon (Gourgoulhon (2017); page 175). We introduce new coordinates

$$dv = dt + \frac{r^2 + a^2}{\triangle} dr, \qquad d\psi = d\phi + \frac{a}{\triangle} dr \tag{6.24}$$

Squaring and expanding, we have

$$dt^2 = \left(dv - \frac{r^2 + a^2}{\triangle} dr \right)^2 = dv^2 - 2\frac{r^2 + a^2}{\triangle} dv dr + \frac{r^2 + a^2}{\triangle^2} dr^2 \tag{6.25}$$

$$d\phi^2 = \left(d\psi - \frac{a}{\triangle} dr \right)^2 = d\psi^2 - \frac{2a}{\triangle} dr d\psi + \frac{a^2}{\triangle^2} dr^2 \tag{6.26}$$

$$dt d\phi = \left(dv - \frac{r^2 + a^2}{\triangle} dr \right) \left(d\psi - \frac{a}{\triangle} dr \right) = dv d\psi - \frac{a}{\triangle} dr dv - \frac{(r^2 + a^2)}{\triangle} dr d\psi + \frac{a(r^2 + a^2)}{\triangle^2} dr^2 \tag{6.27}$$

Inserting equations (6.25, 6.26, and 6.27) into (6.1) we obtain the following line element for the improved Kerr metric in ingoing Eddington-Finkelstein coordinates

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right) dv^2 + 2dvdr - 2a\sin^2\theta drd\phi - \frac{4Mar\sin^2\theta}{\rho^2} dvd\phi + \frac{\Sigma}{\rho^2}\sin^2\theta\theta d\phi^2 + \rho^2 d\theta^2 \tag{6.28}$$

These coordinates produce an extension of the Kerr metric across the future horizon.
The corresponding explicit form of the metric is

1 0

$$g_{\mu\nu} = \begin{bmatrix} -\left(1 - \frac{2Mr}{\rho^2}\right) & & & -\frac{2Mra\sin^2\theta}{\rho^2} \\ 1 & 0 & 0 & -a\sin^2\theta \\ 0 & 0 & \rho^2 & 0 \\ -\frac{2Mr}{\rho^2}a\sin^2\theta & -a\sin^{\theta} & 0 & (r^2+a^2)\sin^2\theta + \frac{2Mr}{\rho^2}a^2\sin^4\theta \end{bmatrix} \tag{6.29}$$

Carter has given two principal null vectors of the Weyl tensor for the Kerr metric in Eddington-Finkelstein coordinates as shear-free null geodesic congruences as

$$l^\mu = \left(r^2 + a^2, \frac{\triangle}{2}, 0, a\right) \tag{6.30}$$

$$n^\mu = (0, -1, 0, 0) \tag{6.31}$$

The normalization for null vectors are $l^\alpha l_\alpha = n^\alpha n_\alpha = 0$ and $n^\alpha l_\alpha = -\rho^2$

Here, an extra factor $-\frac{1}{\rho^2}$ is needed. Using

$$n^\mu = (0,-1,0,0) \tag{6.32}$$

$$n_\mu = (-1, 0, 0, a\sin^2\theta) \tag{6.33}$$

$$l^\mu = \left(\frac{r^2+a^2}{\rho^2}, \frac{\triangle}{2\rho^2}, 0, \frac{a}{\rho^2}\right) \tag{6.34}$$

$$l_\mu = \left(-\frac{1}{2}\frac{\triangle}{\rho^2}, 1, 0, \frac{1}{2}\frac{a\triangle\sin^2\theta}{\rho^2}\right) \tag{6.35}$$

(6.36
)

We find $l^\alpha l_\alpha = n^\alpha n_\alpha = 0$ and $n^\alpha l_\alpha = -1$ The important differences from the spherically symmetric space are as follows: the hypersurfaces of constant v are not in general null hypersurfaces since the normal one-form to these surfaces n_α with components $n_\aleph$ = (1,0,0,0) has norm

$$n^\mu n_\mu = \frac{a^2 \sin^2\theta}{\rho} \tag{6.37}$$

This only vanishes on the axis of rotation $\theta = 0$ and $\theta = \pi$. This norm is positive so the hypersurfaces of constant v are in general timelike hypersurfaces. This means that the n^α given above is tangent but not normal to the hypersurfaces of constant v. Furthermore, neither l^α nor n^α are normal to the orbits of the isometry φ^α since

$$\begin{aligned} l^\alpha \phi_\alpha &= \frac{1}{2}\frac{a\triangle \sin^2\theta}{\rho^2} && (6.38) \\ l^\alpha \phi_\alpha &= a \sin^2\theta && (6.39) \end{aligned}$$

although l^α is normal to the isometry orbits on the horizon 4 = 0. However, if we calculate the expansion of l^α and n^α, we have

$$\begin{aligned} \theta_l &= \frac{\triangle r}{\rho^4} && (6.40) \\ \theta_n &= -\frac{2r}{\rho^2} && (6.41) \end{aligned}$$

So the expansion of the ingoing congruence θ_n will be negative everywhere and the expansion of the outgoing congruence θ_l will change sign at the horizon 4 = 0. Thus the surface 4 = 0 will be a marginal outer trapped surface. In the region for $r_- < r < r_+$, θ_l for 4 < 0 and $\theta_n < 0$.This implies that trapped surfaces exist for Kerr in advanced Eddington-Finkelstein coordinates.

6.9 Surface Gravity of the Kerr Spacetime

Wald (1984; 313) and Poisson Eric (2004; Page 187) use the Killing vector ξ^{α} and the fact that the horizon is a Killing horizon to define the surface gravity. This vector is null at the event horizon and is in fact tangent to the horizon's null generators. If ξ^{α} does not coincide with the stationary Killing field t^{α}, we obtain an axial Killing field ψ^{α} in the spacetime by taking a linear combination of ξ^{α} and t^{α}. This can be written as

$$\xi_{\alpha} = t_{\alpha} + \Omega_{H}\psi_{\alpha} \tag{6.42}$$

Where Ω_H is called the angular velocity of the horizon. Since the horizon is a null surface and the vector ξ^{α} is normal to the horizon, we have $\xi^{\alpha}\xi_{\alpha} = 0$ on the horizon, so, in particular, $\xi^{\alpha}\xi_{\alpha}$ is constant on the horizon. Hence $\nabla^{\alpha}(\xi^{\alpha}\xi_{\beta})$ also is normal to the horizon, so on the horizon there exists a function κ such that

$$\nabla^{\alpha}(\xi^{\alpha}\xi_{\beta}) = -2\kappa\xi^{\alpha} \tag{6.43}$$

Taking the Lie derivative of equation (6.43) with respect to the Killing field ξ^{α}, we find

$$L_{\xi}\kappa = 0 \tag{6.44}$$

This shows that κ is a constant on the orbits of ξ^{α}. In fact, the gravity is constant over the horizon. In other words, its value does not change from orbit to orbit. Using equation (6.43), we can calculate the surface gravity by taking the norm of ξ^{α} as

$$\xi^{\alpha}\xi_{\beta} = \frac{\Sigma \sin^2\theta}{\rho^2}(\Omega_H - \omega)^2 - \frac{\rho^2\triangle}{\Sigma} \tag{6.45}$$

and differentiating yields

$$(\xi^{\alpha}\xi_{\beta})_{;\alpha} = \frac{\rho^2\triangle_{,\alpha}}{\Sigma} \tag{6.46}$$

on the horizon, at which $\omega = \Omega_H$ and $4 = 0$

We have $4_{,\alpha} = 2(r_{+} - M)\partial_{\alpha}r$ and $\xi^{\alpha} = (1 - a\Omega_H \sin^2\theta)\partial_{\alpha}r$ on the horizon and the surface gravity can be calculated as

$$\kappa = \frac{r_+ - M}{r_+^2 - a^2} = \frac{\sqrt{M^2 - a^2}}{2M(M + \sqrt{M^2 - a^2})} \tag{6.47}$$

We notice that $\kappa = 0$ for extreme Kerr black hole and in the limit $a \to 0$, κ reduces to $\frac{1}{4M}$, this is the Schwarzschild case. We also notice that in the general case κ does not depend on θ and the surface gravity is uniform on the event horizon.

6.10 Trapped Surface and Marginally Trapped Surface for Kerr in Doran Coordinates

Introduced by (Doran (2000); Page 3), here we obtain another coordinate system for Kerr metric which reduces Schwarzschild geometry in Painleve-Gullstrand form when $a = 0$. The metric in Doran coordinate is given by

$$ds^2 = -dt^2 + \rho^2 d\theta^2 + (r^2 + a^2)\sin^2\theta d\phi^2 + \frac{\rho^2}{r^2 + a^2}\left[dr + \frac{\sqrt{2Mr(r^2 + a^2)}}{\rho^2}(dt^2 - a\sin^2\theta d\phi^2)\right]^2 \tag{6.48}$$

This is obtained from the advanced Eddington-Finkelstein form via the coordinate transformation

$$dt = dv - \frac{dr}{1 + \left(\frac{2Mr}{r^2+a^2}\right)^{\frac{1}{2}}} \tag{6.49}$$

$$d\phi = d\psi - \frac{adr}{r^2 + a^2 + (2Mr(r^2 + a^2))^{\frac{1}{2}}} \tag{6.50}$$

This transformation is well-defined for all r, though the integrals involved do not appear to have a single closed form. The key features of the line element (6.48) are

i. As $a \to 0$ one obtains

$$ds^2 = -dt^2 + \left(dr + \sqrt{\frac{2M}{r}}dt\right)^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{6.51}$$

Which simply is Schwarzschild geometry in Painleve-Gullstrand form

ii. As $M \to 0$ one obtains

$$ds^2 = -dt^2 + \frac{\rho^2}{r^2+a^2}dr^2 + \rho^2 d\theta^2 + (r^2+a^2)\sin^2\theta d\phi^2 \tag{6.52}$$

This is flat Minkowski space in oblate spheroidal coordinates. The metric (6.48) lends itself very naturally to the tetrad formation.

Trapped surface and marginally trapped surface for Kerr in Doran coordinates applying the covariant divergence of a vector field

The null tetrad expressed in (t,r,θ,φ) coordinates in Doran (2000) is

$$l^\mu = \frac{1}{r^2+a^2}\left(r^2+a^2, (r^2+a^2) - (2Mr(r^2+a^2))^{\frac{1}{2}}, 0, a\right) \tag{6.53}$$

$$n^\mu = \frac{1}{2\rho^2}\left(r^2+a^2, -(r^2+a^2) - (2Mr(r^2+a^2))^{\frac{1}{2}}, 0, a\right) \tag{6.54}$$

$$m^\mu = \frac{1}{\sqrt{2}(r+ia\cos\theta)}(ia\sin\theta, 0, 1, i\cos\text{ec}\theta) \tag{6.55}$$

This gives the correct normalization $l^\alpha l_\alpha = n^\alpha n_\alpha = 0$, $n^\alpha l_\alpha = -1$ The covariant divergence of the outgoing null vector field $l = \frac{(r^2+a^2)-(2Mr(r^2+a^2))^{\frac{1}{2}}}{r^2+a^2}\hat{r}$ is given by

$$\begin{aligned}
\theta_l &= \frac{1}{\rho^2\sin\theta}\left(\frac{(r^2+a^2)-(2Mr(r^2+a^2))^{\frac{1}{2}}}{r^2+a^2}\right)\frac{\partial}{\partial r}(\rho^2\sin\theta) \\
&= \frac{1}{\rho^2\sin\theta}\left(\frac{(r^2+a^2)-(2Mr(r^2+a^2))^{\frac{1}{2}}}{r^2+a^2}\right)\frac{\partial}{\partial r}(r^2+a^2\cos\theta)\sin\theta \\
&= \frac{1}{\rho^2\sin\theta}\left(\frac{(r^2+a^2)-(2Mr(r^2+a^2))^{\frac{1}{2}}}{r^2+a^2}\right)2r\sin\theta = \frac{2r(\sqrt{r^2+a^2}-\sqrt{2Mr})}{\rho^2\sqrt{r^2+a^2}}
\end{aligned}$$

$$\theta_l = \frac{2r(r^2+a^2-2Mr)}{\rho^2\sqrt{r^2+a^2}(\sqrt{r^2+a^2}+\sqrt{2Mr})} = \frac{2r\triangle}{\rho^2\sqrt{r^2+a^2}(\sqrt{r^2+a^2}+\sqrt{2Mr})} \tag{6.56}$$

From (6.54) we also have

$$n^\mu = \frac{r^2+a^2}{2\rho^2}\left(\frac{\partial}{\partial t}\right)^\mu - \frac{(r^2+a^2)-(2Mr(r^2+a^2))^{\frac{1}{2}}}{2\rho^2}\left(\frac{\partial}{\partial}\right)^\mu + \frac{a}{2\rho^2}\left(\frac{\partial}{\partial\phi}\right)^\mu \tag{6.57}$$

The covariant divergence of the ingoing null vector field $n = \frac{-\left((r^2+a^2)-(2Mr(r^2+a^2))^{\frac{1}{2}}\right)}{2\rho^2}\hat{r}$ is given by

$$
\begin{aligned}
\theta_n &= -\frac{1}{\rho^2 \sin\theta}\left(\frac{(r^2+a^2)-(2Mr(r^2+a^2))^{\frac{1}{2}}}{2\rho^2}\right)\frac{\partial}{\partial r}(\rho^2 \sin\theta) \\
&= -\frac{1}{\rho^2 \sin\theta}\left(\frac{(r^2+a^2)-(2Mr(r^2+a^2))^{\frac{1}{2}}}{2\rho^2}\right)\frac{\partial}{\partial r}(r^2 + a^2\cos\theta)\sin\theta \\
&= -\frac{1}{\rho^2 \sin\theta}\left(\frac{(r^2+a^2)-(2Mr(r^2+a^2))^{\frac{1}{2}}}{2\rho^2}\right)2r\sin\theta \\
&= -\frac{r\sqrt{r^2+a^2}(\sqrt{r^2+a^2}-\sqrt{2Mr})}{\rho^4}
\end{aligned}
$$

$$
\theta_n = -\frac{r\sqrt{r^2+a^2}(\sqrt{r^2+a^2}-\sqrt{2Mr})}{\rho^4} \tag{6.58}
$$

Equation (6.56) has the property of vanishing when 4 = 0, positive for 4 > 0 and negative when if we always choose the positive roots which follows from (6.53) and the requirement that l^α should be outgoing. Thus the surface 4 = 0 will be a marginal trapped surface. In the region $r_- < r < r_+$, $\theta_l < 0$ for 4 < 0 and $\theta_n < 0$ is always negative. This implies that trapped surfaces exist for Kerr in Doran coordinates.

Trapped surface and marginally trapped surfaces for Kerr in Doran coordinates applying the flux of a vector field

The field $\bar{g} = \frac{GM}{r^2}\hat{r}$ is radial and orthogonal to the surface and The flux of outgoing null vector $l = \frac{r^2+a^2-(2Mr(r^2+a^2))^{\frac{1}{2}}}{r^2+a^2}\hat{r}$ is given by

$$
\begin{aligned}
\theta_l &= \oint_S \bar{g}\cdot \bar{l}\,dA = \oint_S \frac{GM}{r^2}\hat{r}\cdot\left(\frac{r^2+a^2-(2Mr(r^2+a^2))^{\frac{1}{2}}}{r^2+a^2}\right)\hat{r}\,dA \\
&= \frac{GM}{r^2}\left(\frac{r^2+a^2-(2Mr(r^2+a^2))^{\frac{1}{2}}}{r^2+a^2}\right)(4\pi r^2) \\
&= 4\pi GM\left(\frac{r^2+a^2-(2Mr(r^2+a^2))^{\frac{1}{2}}}{r^2+a^2}\right) \\
&= 4\pi GM\left(\frac{r^2+a^2-2Mr}{\sqrt{r^2+a^2}(\sqrt{r^2+a^2}+\sqrt{2Mr})}\right) = \frac{4\pi GM\triangle}{\sqrt{r^2+a^2}(\sqrt{r^2+a^2}+\sqrt{2Mr})}
\end{aligned}
$$

The flux of ingoing null vector field $n = -\frac{(r^2+a^2)-(2Mr(r^2+a^2))^{\frac{1}{2}}}{2\rho^2}\hat{r}$ is given by

$$\begin{aligned}
\theta_n &= \oint_S \bar{g}\cdot\bar{n}dA = -\oint_S \frac{GM}{r^2}\hat{r}\cdot\left(\frac{(r^2+a^2)-(2Mr(r^2+a^2))^{\frac{1}{2}}}{2\rho^2}\right)\hat{r}dA \\
&= -\frac{GM}{r^2}\left(\frac{(r^2+a^2)-(2Mr(r^2+a^2))^{\frac{1}{2}}}{2\rho^2}\right)(4\pi r^2) \\
&= -4\pi GM\left(\frac{(r^2+a^2)-(2Mr(r^2+a^2))^{\frac{1}{2}}}{2\rho^2}\right)
\end{aligned}$$

The surface 4 = 0 is the marginally trapped surface. For 4 < 0, $\theta_l < 0$ and $\theta_n < 0$. This implies applying the flux of a vector field; trapped surface exist in the region $r_- < r < r_+$ in the Doran coordinates.

6.11 Trapped surface and Marginally Trapped Surface in Kerr Vaidya Solution

The Kerr-Vaidya solution is an explicitly dynamic solution to Einstein's equations that can either model a rotating radiating star or a rotating collapsing null fluid (Nielsen (2009), page 129). The Kerr-Vaidya solution is normally given in terms of Eddington-Finkelstein coordinates (v,r,θ,φ) as

$$g_{\mu\nu} = \begin{bmatrix} -\left(1-\frac{2M(v)r}{\rho^2}\right) & & & -\frac{2M(v)ra\sin^2\theta}{\rho^2} \\ 1 & 0 & 0 & -a\sin^2\theta \\ 0 & 0 & \rho & 0 \\ -\frac{2M(v)r}{\rho^2}a\sin^2\theta & -a\sin^2\theta & 0 & (r^2+a^2)\sin^2\theta+\frac{2M(v)r}{\rho^2}a^2\sin^4\theta \end{bmatrix} \quad (6.59)$$

1 0

$$g^{\mu\nu} = \begin{bmatrix} \frac{a^2\sin^2\theta}{\rho^2} & \frac{r^2+a^2}{\rho^2} & 0 & \frac{a}{\rho^2} \\ \frac{r^2+a^2}{\rho^2} & \frac{\triangle(v,r)}{\rho^2} & 0 & \frac{a}{\rho^2} \\ 0 & 0 & \frac{1}{\rho^2} & 0 \\ \frac{a}{\rho^2} & \frac{a}{\rho^2} & 0 & \frac{1}{\rho^2\sin^2\theta} \end{bmatrix} \quad (6.60)$$

The only difference is that M is a function of v and $4(v,r) = r^2 + a^2 - 2M(v)r$. The solution still has two principal null vectors whose vanishing expansion can be used to locate the horizon which is located at $4(v,r) = 0$. These null vectors are given by

$$l^\mu = \left(\frac{r^2+a^2}{\rho^2}, \frac{\triangle(v,r)}{2\rho^2}, 0, \frac{a}{\rho^2}\right) \tag{6.61}$$

$$n^\mu = (0,-1,0,0) \tag{6.62}$$

Trapped surface and marginally trapped surfaces in Kerr Vaidya solution applying the covariant divergence of a vector field

Applying the null vectors (6.61) and (6.62) their covariant divergences are given by

$$\theta_l = \frac{l^r}{\sqrt{g}}\frac{\partial}{\partial r}\sqrt{g} = \frac{\triangle(v,r)}{2\rho^2}\frac{1}{\rho^2\sin\theta}\frac{\partial}{\partial r}(\rho^2\sin\theta) = \frac{\triangle(v,r)r}{\rho^4}$$

$$\theta_n = \frac{n^r}{\sqrt{g}}\frac{\partial}{\partial r}\sqrt{g} = -\frac{1}{\rho^2\sin\theta}\frac{\partial}{\partial r}(\rho^2\sin\theta) = -\frac{2r}{\rho^2}$$

Here, we have a marginally trapped surface when $4(v,r) = 0$. In the region $r_- < r < r_+$, $\theta_l < 0$ for $4(v,r) < 0$ and $\theta_n < 0$ is always negative. This im-

plies that trapped surfaces and marginally trapped surfaces exist in Kerr-Vaidya solution.

Trapped surface and marginally trapped surfaces in Kerr Vaidya solution applying the flux of a vector field

The flux of outgoing null vector field is given by The field $\bar{g} = \frac{GM}{r^2}\hat{r}$ is radial and orthogonal to the surface and $\bar{l} = \frac{\triangle(v,r)}{2\rho^2}\hat{r}$

$$\theta_l = \oint_S \bar{g}\cdot\bar{l}dA = \oint_S \frac{GM}{r^2}\hat{r}\cdot\frac{\triangle(v,r)}{2\rho^2}\hat{r}dA = \frac{GM}{r^2}\frac{\triangle(v,r)}{2\rho^2}(4\pi r^2) = \frac{2\pi GM\triangle(v,r)}{\rho^2} \tag{6.63}$$

The flux of ingoing null vector field is given by

The field $\bar{g} = \frac{GM}{r^2}\hat{r}$ is radial and orthogonal to the surface and $\bar{n} = -\hat{r}$

$$\theta_n = \oint_S \bar{g}\cdot\bar{n}dA = \oint_S \frac{GM}{r^2}\hat{r}\cdot(-\hat{r})dA = -\frac{GM}{r^2}(4\pi r^2) = -4\pi GM \tag{6.64}$$

Thus the surface $4(v,r) = 0$ will be a marginally trapped surface.

For $4(v,r) < 0$, $\theta_l < 0$ and $\theta_n < 0$. This implies trapped surfaces exist in the region $r_- < r < r_+$ in the Kerr Vaidya black hole.

6.12 Trapped Surface and Marginally Trapped Surface in the Kerr Spacetime

In a general spacetime $(M, g_{\mu\nu})$ with the metric $g_{\mu\nu}$ having signature (-+++), one can define two future directed null vectors n^μ and l^μ whose expansion scalars are given by

$$\theta_l = q^{\mu\nu}\nabla_\mu l_\nu = \frac{l^\alpha}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x^\alpha}, \qquad \theta_n = q^{\mu\nu}\nabla_\mu n_\nu = \frac{n^\alpha}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x^\alpha} \tag{6.65}$$

where $q_{\mu\nu} = g_{\mu\nu} + l_\mu n_\nu + m_\mu l_\nu$ is the metric induced by $g_{\mu\nu}$ on the two dimensional spacelike surface formed by spatial foliation of the null hypersurface generated by l^μ and n^ν. Then (i) a two dimensional spacelike surface S is said to be a trapped surface if both $\theta_l < 0$ and $\theta_n < 0$; (ii) S is said to be a marginally trapped surface if one of the two null expansions vanishes i.e. $\theta_l = 0$ or $\theta_n = 0$.

Trapped surface in Kerr black hole applying the covariant divergence of a vector field

Krishnan (2013); Page 25, gives a suitable choice of the ingoing and outgoing future directed null vectors for non-extremal Kerr black hole in his paper by

$$l^\mu\triangle_\mu = \frac{\partial}{\partial v} + \frac{\triangle}{2(r^2+a^2)}\frac{\partial}{\partial r} + \frac{a}{r^2+a^2}\frac{\partial}{\partial\phi} \tag{6.66}$$

$$l^\mu\triangle_\mu = -\left(\frac{r^2+a^2}{\rho^2}\right)\frac{\partial}{\partial r} \tag{6.67}$$

The covariant versions are

$$l_\mu = -\frac{\triangle}{2(r^2+a^2)}\left(\frac{\partial}{\partial v}\right)^\mu + \frac{\rho^2}{r^2+a^2}\left(\frac{\partial}{\partial r}\right)^\mu + \frac{\triangle a\sin^2\theta}{2(r^2+a^2)}\left(\frac{\partial}{\partial\phi}\right)^\mu \tag{6.68}$$

$$n_\mu = \frac{r^2+a^2}{\rho^2}\left(-\frac{\partial}{\partial v} + a\sin^2\theta\frac{\partial}{\partial\phi}\right) \tag{6.69}$$

$\sqrt{}$

where 4 = $(r - r_+)(r - r_-)$ and $r_\pm = M \pm$ $M^2 - a^2$ The null vectors satisfy the following conditions $l^\mu n_\mu = -1$, $l^\mu l_\mu = n^\mu n_\mu = 0$.

Covariant divergence of outgoing null vector field

$$\begin{aligned} l &= \partial_v + \frac{\triangle}{2(r^2+a^2)}\partial_r + \frac{a}{r^2+a^2}\partial_\phi \\ \theta_l &= \nabla \cdot \bar{n} = \frac{l^r}{\sqrt{g}}\frac{\partial}{\partial r}\sqrt{g} = \frac{\triangle}{2(r^2+a^2)}\frac{1}{\rho^2 \sin\theta}\frac{\partial}{\partial r}(\rho^2 \sin\theta) = \frac{\triangle r}{\rho^2(r^2+a^2)} \quad (6.70) \end{aligned}$$

Covariant divergence of ingoing null vector field

$$\begin{aligned} \bar{n} &= -\left(\frac{r^2+a^2}{\rho^2}\right)\partial_r \\ \theta_n &= \nabla \cdot \bar{n} = \frac{n^r}{\sqrt{g}}\frac{\partial}{\partial r}\sqrt{g} = -\frac{r^2+a^2}{\rho^2}\frac{1}{\rho^2 \sin\theta}\frac{\partial}{\partial r}(\rho^2 \sin\theta) = -\frac{2r(r^2+a^2)}{\rho^4} \quad (6.71) \end{aligned}$$

So the expansion of the ingoing congruence θ_n will be negative everywhere and the expansion of the outgoing congruence θ_l will change sign at the horizon 4 = 0. Thus the surface 4 = 0 will be a marginally trapped surface. In the region $r_- < r < r_+$, $\theta_l < 0$ for 4 < 0 and $\theta_n < 0$. This implies that trapped surfaces exist for non extreme Kerr black hole in this region. In contrast, for the extreme Kerr black hole i.e. when $a = M$, we have the outgoing and ingoing expansions to be

$$\theta_l = \frac{r(r-M)^2}{\rho^2(r^2+a^2)}, \quad \theta_n = -\frac{2(r^2+a^2)}{\rho^4} \tag{6.72}$$

Here inside or outside extremal horizon $r < M$ or $r > M$, $\theta_l > 0$ and $\theta_n < 0$. This implies that there are no trapped surfaces for extremal Kerr black hole beyond the event horizon.

Trapped surface and marginally trapped surface in Kerr black hole applying the flux of a vector field

The flux of outgoing null vector field

The field $-g = \frac{GM}{r^2}\hat{r}$ is radial and orthogonal to the surface and $l = \frac{\triangle}{2(r^2+a^2)}\hat{r}$

$$
\begin{aligned}
\phi &= \oint_S \bar{g}\cdot \bar{l}dA = \oint_S \frac{GM}{r^2}\hat{r}\cdot\frac{\triangle}{2(r^2+a^2)}\hat{r}dA = \frac{GM}{r^2}\frac{\triangle}{2(r^2+a^2)}r^2(4\pi) \quad (6.73)\\
&= \frac{2\pi GM\triangle}{(r^2+a^2)}
\end{aligned}
$$

The flux of ingoing null vector field

The field $\bar{g} = \frac{GM}{r^2}\hat{r}$ is radial and orthogonal to the surface and $n = -\frac{(r^2+a^2)}{\rho^2}\hat{r}$

$$
\begin{aligned}
\phi &= \oint_S \bar{g}\cdot \bar{n}dA = -\oint_S \frac{GM}{r^2}\hat{r}\cdot\frac{r^2+a^2}{\rho^2}\hat{r}dA = -\frac{GM}{r^2}\frac{(r^2+a^2)}{\rho^2}r^2(4\pi) \quad (6.74)\\
&= -\frac{4\pi GM(r^2+a^2)}{\rho^2}
\end{aligned}
$$

From equations (6.74) and (6.75),

The flux of outgoing null vector field $l,\ \phi_l = \frac{2\pi GM\triangle}{(r^2+a^2)}$

The flux of ingoing null vector field $n,\ \phi_n = -\frac{4\pi GM(r^2+a^2)}{\rho^2}$

From these equations, the flux of the ingoing null vector φ_n is negative everywhere and that of outgoing null vector φ_l will change sign at the horizon 4 = 0. Thus the surface 4 = 0 will be a marginally trapped surface.

For $\theta < 0$, θ_l and $\theta_n < 0$. This implies trapped surfaces exist in the region $r_- < r < r_+$ for the Kerr black hole. In contrast, for the extreme Kerr black hole i.e. when $a = M$, the fluxes of outgoing null vector and ingoing null vector are respectively given by

$$
\phi_l = \frac{2\pi GM(r-M)^2}{(r^2+a^2)} \quad (6.75)
$$

$$
\phi_n = -\frac{4\pi GM(r^2+a^2)}{\rho^2} \quad (6.76)
$$

Here inside or outside extremal horizon that is $r < M$ or $r > M$, $\varphi_l > 0$ and $\varphi_n < 0$. This implies that there are no trapped surfaces for extremal Kerr black hole beyond the event horizon.

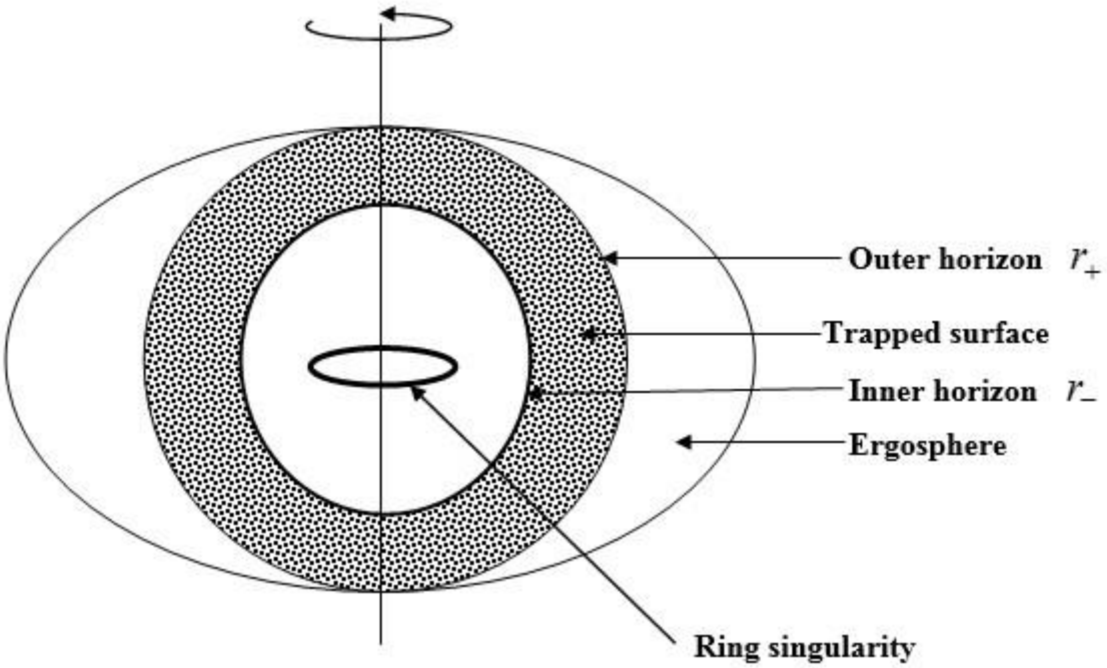

Figure 6.4: This figure shows the location of the event horizons, the ring singularity, ergosphere and trapped surface which lies in the region $r_- < r < r_+$ in Kerr black hole

Chapter 7

CONCLUSION AND RECOMMENDATION

7.1 Conclusion

We have investigated trapped and marginally trapped surfaces in stationary and dynamical spacetimes both in spherically and axially symmetric spacetimes. Using the appropriate null vectors, we have demonstrated the conditions under which a surface is trapped and marginally trapped. These investigations have revealed that a black hole region contains a trapped surface, a closed two-surface S with the property that for both ingoing and outgoing null vectors orthogonal to S, the expansion is negative everywhere on S. In the Schwarzschild spacetime with $r > 2M$, we have $\theta_l > 0$ and $\theta_n > 0$. This is a flat spacetime region. However, for $r < 2M$ (black hole region), both expansions of outgoing and ingoing null vectors are negative. Such surfaces are said to be trapped. For $r = 2M$ hypersurface, $\theta_l = 0$, $\theta_n < 0$. These surfaces are said to be marginally trapped. Thus, $r = 2M$ hypersurface is the boundary of the region in which the spherically symmetric trapped surfaces lie and are the signature of a black hole region. We also saw that, in a Schwarzschild black hole, the definitions of the surface of a black hole are the same. This implies the $r = 2M$ hypersurface is both the boundary of the trapped region and the event horizon. This means the event and apparent horizons of the Schwarzschild spacetime coincide. In dynamical situations, however, the apparent horizon lies within the black hole region. The dynamical black hole, example the Vaidya spacetime has been illustrated. We saw that there are trapped surfaces in the region $r < 2M(v)$ but outside the $r = 2M(v)$ surface, there are no trapped surfaces. The hypersurface $r = 2M(v)$ is foliated by round and marginally trapped spheres.

Investigating axisymmetric spacetimes in different coordinate systems have revealed that trapped surfaces exist in axisymmetric spacetimes where the Kerr in advanced Eddington-Finkelstein coordinates, Kerr in Doran coordinates, Kerr Vaidya coordinates and non extreme Kerr black hole ware chosen as typical examples. But in contrast, for the extreme Kerr black hole i.e. when $a = M$, there are no trapped surfaces. The three-dimensional boundary of the region of spacetime that contains trapped surfaces- the trapped region is the trapping horizon and its two-dimensional intersection with a spacelike hypersurface is called an apparent horizon. The apparent horizon is therefore a marginally trapped surface- a closed two-surface on which one of the congruences or null vectors vanishes. Trapped and marginally trapped surfaces play a very important role in the analysis of spacetime geometry. By the singularity theorems of Hawking and Penrose (Hawking and Ellis (1973); Page 266), a spacetime which satisfies suitable energy and causality conditions, and which in addition contains a trapped surface, must contain a black hole. Marginally trapped surfaces, serve as the quasi-local version of black hole boundary. In numerical general relativity, they are used as excision surfaces for the evolution of black hole initial data. The goal of these studies is to compute the covariant divergences and the fluxes of the appropriate null vectors in both spherically and axisymmetric spacetimes to determine the existence of trapped and marginally trapped surfaces as a new contribution to the existing knowledge. In doing so, a lot of other ideas were visited to add credence to the concept under study. In the end, it was confirmed that trapped and marginally trapped surfaces exist in both spherically and axiallysymmetric spacetimes. It is therefore desirable to use the notion of trapped surface and marginally trapped surface as a suitable complement in studying black holes. What can apparently be considered as a new result found is the application of the Gauss's divergence theorem as a new approach for studying black holes. Moreover, using trapped surfaces as local characterizations of black holes do not depend on global properties like the classical event horizon whose determination requires the knowledge of the entire future null infinity and also teleological (responds in advance to what will happen in the future).

7.2 Recommendation

The researcher highly recommends the use of local approach as a complementary means of studying black holes instead of the global concept which is teleological. The use of covariant divergence and the Gauss's divergence theorems are strongly recommended since they both apply to closed surfaces and can be used to define trapped and marginally trapped surfaces.

Finally, it is recommended that differential manifolds, Riemannian geometry, Lie groups and Lie derivatives could be taken as a course at the graduate and the post-graduate levels for applied Mathematics students.

REFERENCES

Andersson, L. and Metzger, J. (2008). the area of horizons and the trapped region, preprint arxiv: 0708.4252.

Ashtekar, A. and Corichi, A. (2000). Laws governing isolated horizons: Inclusion of dilaton coupling. class. *Quantum Grav*, 17:1317–1332.

Ashtekar, A., Corichi, A., and Sudarsky, D. (2001). Hairy black holes, horizon mass and solitons. class. *Quantum Grav.*, 18:919–940.

Ashtekar, A. and Galloway, G. J. (2005). Some uniqueness results for dynamical horizons; 2005 international press,. *Adv. Theor. Math. Phys*, 91:1–30.

Ashtekar, A. and Krishnan, B. (2004). Living rev. rel. 7 10 (www.livingreviews.org/lrr-2004-10).

Bardeen, J. M., Carter, B., and Hawking, S. W. (1973). commun. *Math. Phys.*, 3116.

Baumgarte, T. W. and Shapiro, S. L. (2010). *Numerical Relativity: Solving Einstein's Equations on the Computer*. Cambridge UP.

Ben-Dov, I. (2007). ”outer trapped surfaces in vaidya spacetimes” arxiv:grqc/0611057.

Bengtsson, I., Jacobsson, E., and Senovilla, J. M. M. (2013). Trapped surfaces in oppenheimer-snyder black holes; arxiv-1306.6486.

Booth, I. (2005). Black hole boundaries. *Can. J. phys.*, 831073.

Booth, I., Kunduri, H. K., and O'Grady, A. (2017). unstable marginally outer trapped surfaces in static spherically symmetric spacetimes. *Physics Reviewed D*, 96.

Borisenko, A. I. and Tarapov, I. E. (1979). *Vector analysis with applications*. Dover Publications, Inc. New York.

Carroll, S. M. (1997). *Lecture Notes on General relatitivty*. Institute for Theoretical Physics, University of California.

Carroll, S. M. (2004). Spacetime and geometry. an introduction to general relativity. *Addison Wesley, 1301 Sansome St. San Francisco*, CA94111.

Carter, B. (1973). *Black Holes*. Gordon and Breach, New York.

Chandrasekhar, S. (1983). *The mathematical theory of black holes;*. Oxford University press. New York.

Chrusciel, P. T., Costa, J. L., and Heusler, M. (2012). Stationary black holes: Uniqueness and beyond, living rev. rel. 157, arxiv: 1205.6112 [gr-qc].

Coley, A. A., McNutt, D. D., and Shoon, A. A. (2017). Geometric horizons. *Physics Letters B, Elsevier*, 771:131–135.

Demianski, M. and Lasota, J. P. (1973). Black holes in an expanding universe, nature. *Phys. Sci.*, 53:241.

DeWitt, C. and DeWitt, B. (1973). ”black holes les astres occlus” gordon and breach.

Doran, C. (2000). ”a new form of the kerr solution”, phys. rev. d61067503 [arxiv: gr-qc/9910099].

Eardley, D. M. (1997). Black hole boundary conditions and coordinate conditions, [arxiv; gr-qc/9703027v2].

Frolov, V. P. and Novikov, N. I. D. (1997). *Black hole Physics Basic concepts and new developments*. Oxford university press.

Frolov, V. P. and Zelnikov, A. (2011). *Introduction to black hole physics*. OUP, Oxford.

Gourgoulhon, E. (2017). *Geometry and physics of black holes, Lecture notes*.

Gourgoulhon, E. and Jaramillo, J. L. (2008). New theoretical approaches to black holes, arxiv.org.

Hamilton, A. J. S. (2014). *General relativity black holes and cosmology*.

Hartle, J. B. (2003). *An introduction to Einstein's General Relativity*. Addison Wesley, San Francisco.

Hawking, S. and Ellis, G. F. R. (1973). *The Large Scale Structure of Spacetime*. Cambridge University Press, Cambridge.

Hawking, S. and Hartle, J. B. (1972). Energy and angular momentum flow into a black hole. *Commun. Math. Phys*, 27:283.

Hawking, S. and Penrose, R. (2010). *The nature of space and time*. Princeton University Press,, 41 William Street, Princeton, New Jersey 08540.

Hawking, S. W. and Israel, W. (1979). *General relativity*. Cambridge University press London.

Hawking, S. W. and Penrose, R. (1970). The singularities of gravitational collapse and cosmology. *Proc. R. Soc. Lond*, 314(1519):529–548.

Hayward, S. A. (1994). "general laws of black hole dynamics" phys. rev. d49 6467 [arxiv; gr-qc/9303006].

Hayward, S. A. (2000). Black hole new horizons, world scientific singapore.

Helou, A. (2015). Dynamics of the four kinds of trapping horizons and existence of hawking radiation.

Hobson, M. P., Efstathiou, G., and Lasenby, A. N. (2006). *General Relativity; An introduction for physicists*. Cambridge University Press.

Jakobsson, E. (2017). *Shapes of Spacetimes - Collected tales of black holes*. Department of Physics, Stocholm University.

Jaramillo, J. L. (2011). An introduction to local black hole horizons in the 3+1 approach to general relativity, max-planck-institute fu¨r gravitations physik, albert einstein institut, am mu¨hlenberg 1, golm d-14476, germany.

Krishnan, B. (2012). The spacetime in the neighborhood of a general isolated black hole. *Class. Quant. Grav*, 29:205006.

Krishnan, B. (2013). Quasi-local black hole horizons, max planck institute for gravitational physics (albert einstein institute), callinstr. 38, 30167 hannover, germany. page 1.

Nielsen, A. B. (2009). Black holes and black hole thermodynamics without event horizons. *Gen. Relativ. Gravit*, 41:1539–1584.

Nielson, A. (2007). *Black hole horizon and black hole thermodynamics*. PhD thesis, University of Canterbury.

O'Neill, B. (1995). *The geometry of Kerr Black holes*. Hamilton printing company.

Penrose, R. (1965a). Gravitational collapse and space-time singularities. *Phys. Rev. Lett*, 14:57–59.

Penrose, R. (1965b). Zero rest mass fields including gravitation: asymptotic behaviour. *Proc. Roy. Soc. Lond*, A284159-203.

Penrose, R. (2004). *The road to reality. A complete guide to the laws of the universe*. Published by Jonathan Cape.

Poisson, E. (2002). *An advanced course in general relativity*. University of Guelph.

Poisson, E. (2004). *A Relativist's Toolkit (The Mathematics of Black-Hole Mechanics)*. Cambridge University Press: Cambridge, UK.

Pradhan, P. P. and Majumdar, P. (2011). Extremal limits and kerr spacetime, arxiv: 1108.2333v1 [gr-qc].

Raine, D. and Edwin, T. (2009). *Black holes an introduction*. Imperial college press, 57 Shelton street Covent Garden London WC2H9HE.

Schnetter, E., Krishnan, B., and Beyer, F. (2006). Introduction to dynamical horizons in numerical relativity. *Physical Review D (Particles, Fields, Gravitation, and Cosmology)*, 74(2):024028.

Schutz, B. (1980). *Geometrical methods of mathematical physics.* Cambridge University Press.

Schwarzschild, K. (1916). Sitzber. deut. akad. wiss. (berlin), 189.

Senovilla and Garfinkle (2015). The 1965 penrose singularity theorem. *Classical and Quantum Gravity*, 32.

Senovilla, J. M. M. (1998). Singularity theorems and their consequences. *(Review) General Relativity and Gravitation*, 30:701–848.

Senovilla, J. M. M. (2011). Trapped surfaces. In *International Journal of Modern Physics: Conference Series World Scientific Publishing Company*, arXiv1107.1344.

Suskind, L. and Lindesay, J. (2005). *Introduction to Black holes, information and the string theory revolution, the holographic Universe*. World Scientific publishing Co. Pte. Ltd. 5 Toh. Tuck Link, Singapore 596224.

Wald, R. (2001). The thermodynamics of black holes. living rev. relativ., 4, 6,.

Wald, R. M. (1984). *General Relativity,*. University of Chicago Press Chicago.

Wald, R. M. and Iyer, V. (1991). Phys. rev. d44 r3719.

Williams, C. (2007). Asymptotic behavior of spherically symmetric marginally trapped tubes; gr-qc/0702101.

Appendix A: The components of a curvature tensor or Riemann-Christoffel tensor

Given a vector field $\bar{U}, \bar{V}, \bar{W} \in T(M)$, we define a new tensor which operates on $\bar{U}, \bar{V}$ and $\bar{W}$ which leads to $\binom{1}{0}$ tensor $R(\bar{U}, \bar{V})\bar{W}$. This curvature tensor is of $\binom{1}{3}$ type. Let the components of this tensor in a coordinate basis be

$$\bar{U} = U^i \frac{\partial}{\partial x^i} = U^i \partial_i,\ \bar{V} = V^j \frac{\partial}{\partial x^j} = V^j \partial_j \text{ and } \bar{W} = W^k \frac{\partial}{\partial x^k = W^k \partial_k}$$

We have

$$R(\partial_i, \partial_j)\partial_k = \left(\nabla_i \nabla_j - \nabla_j \nabla_i - \nabla_{[\partial_i, \partial_j]}\right)\partial_k = \left(\nabla_{\partial_i} \nabla_{\partial_j} - \nabla_{\partial_j} \nabla_{\partial_i} - \nabla_{[\partial_i, \partial_j]}\right)\partial$$

Since $[\partial_i, \partial_j] = 0$

$$\begin{aligned}
R(\partial_i, \partial_j)\partial_k &= \nabla_{\partial_i}(\nabla_{\partial_j}\partial_k) - \nabla_{\partial_j}(\nabla_{\partial_i}\partial_k) = \nabla_{\partial_i}(\Gamma^l_{jk}\partial_l) - \nabla_{\partial_j}(\Gamma^l_{ik}\partial_l) \\
&= \left(\partial_i \Gamma^l_{jk}\right)\partial_l + \Gamma^l_{jk}\nabla_{\partial_i}\partial_l - (\partial_j \Gamma^l_{ik})\partial_l - \Gamma^l_{ik}\nabla_{\partial_j}\partial_l \\
&= \left(\partial_i \Gamma^l_{jk}\right)\partial_l + \Gamma^l_{jk}\Gamma^m_{il}\partial_m - (\partial_j \Gamma^l_{ik})\partial_l - \Gamma^l_{ik}\Gamma^m_{jl}\partial_m \\
&= \left(\partial_i \Gamma^l_{jk}\right)\partial_l + \Gamma^m_{jk}\Gamma^l_{im}\partial_l - (\partial_j \Gamma^l_{ik})\partial_l - \Gamma^m_{ik}\Gamma^l_{jm}\partial_l \\
&= \left(\partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^m_{jk}\Gamma^l_{im} - \Gamma^m_{ik}\Gamma^l_{jm}\right)\partial_l = R^l_{kij}\partial_l
\end{aligned}$$

Where

$$R^l_{kij} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^m_{jk}\Gamma^l_{im} - \Gamma^m_{ik}\Gamma^l_{jm} \qquad (7.1)$$

Equation (7.1) is called the curvature tensor or Riemann-Christoffel tensor of type $\binom{1}{\ }$. Hence, the components of the Riemann Christoffel tensor can be put in the

�� 3
for
m

$$R(\partial_i, \partial_j)\partial_k = [\partial_i, \partial_j]\partial_k - \nabla_{[\partial_i,\partial_j]}\partial_k = R^l_{kij}\partial_l \quad (7.2)$$

The curvature tensor $R_{kij}{}^l$ is skew-symmetric in i and j. From its definition, it is clear that

$$R(U,V) = -R(V,U)$$

That is

$$R_{kijl} = -R_{kjil} \quad (7.3)$$

In a non-coordinate basis define the commutation coefficients $C_{jk}{}^l$ by

$$[\bar{e}_j, \bar{e}_k] = C^l_{jk}$$

where $C_{jk}{}^l$ are differentiable functions which are called the structure coefficients. From equation ((7.2))

$$\begin{aligned} R^l_{kij}\bar{e}_l &= [\nabla_i, \nabla_j]\bar{e}_k - \nabla_{[\bar{e}_j,\bar{e}_k]}\bar{e}_k = [\nabla_i, \nabla_j]\bar{e}_k - \left((\nabla_{\bar{e}_i}\bar{e}_j) - (\nabla_{\bar{e}_j}\bar{e}_i)\right)\bar{e}_k \\ &= [\nabla_i, \nabla_j]\bar{e}_k - (\Gamma^m_{ij}\bar{e}_m - \Gamma^m_{ji}\bar{e}_m)\bar{e}_k = [\nabla_i, \nabla_j]\bar{e}_k - (\Gamma^m_{ij} - \Gamma^m_{ji})\bar{e}_k\bar{e}_m \\ &= [\nabla_i, \nabla_j]\bar{e}_k - C^m_{ij}T^l_{km} \end{aligned}$$

Substituting into equation (7.1)

$$R^l_{kij} = \partial_t\Gamma^l_{jk} - \partial_j\Gamma^l_{ik} + \Gamma^m_{jk}\Gamma^l_{im} - \Gamma^m_{ik}\Gamma^l_{jm} - C^m_{ik}T^l_{km}$$

Now, from equation (7.3), the curvature tensor is antisymmetric in i and j it implies

$$R^l_{k(ij)} = \frac{1}{2}\left(R^l_{kij} + R^l_{kji}\right) = 0 \quad (7.4)$$

In normal coordinates at P, $\Gamma^l{}_{jk}(P) = 0$ so that

$$R_{kijl}(P) = \Gamma_{ljk,i} - \Gamma_{lik,j}$$

This implies

$$3R^l_{[kij]} = \Gamma^l_{kj,i} - \Gamma^l_{ki,j} + \Gamma^l_{ik,j} - \Gamma^l_{jk,i} + \Gamma^l_{ji,k} - \Gamma^l_{ij,k} = 0$$

Therefore

$$R^l_{[kij]} = 0 \qquad (7.5)$$

Number of independent components $R_{kij}{}^l$

The four indices mean that we begin with n^4 components. Equation (7.4) is $n^2 \cdot \frac{1}{2}n(n+1)$ separate relations, since l and k are free, while there are $\frac{1}{2}n(n+1)$ symmetric pairs. This is the same as the number of independent components of a symmetric $n \times n$ matrix. Constraint (7.5) is entirely independent of (7.4) since it involves only $R_k{}^l{}_{[ij]}$. There are $\frac{1}{3!}n(n-1)(n-2)$ different antisymmetric triplets (kij) in this equation. The number of independent components of R^l_{kij} in an ndimensional manifold is given by

$$n^4 - n^2 \cdot \frac{1}{2}n(n+1) - \frac{1}{3!}n(n-1)(n-2) = \frac{1}{3}n^2(n^2-1) \qquad (7.6)$$

The Jacobi identity for covariant derivatives

The Jacobi identity for covariant derivatives is given by

$$[\nabla_i,[\nabla_j,\nabla_k]] + [\nabla_j,[\nabla_k,\nabla_i]] + [\nabla_k,[\nabla_i,\nabla_j]]$$

Proof

$$\begin{aligned}
[\nabla_i,[\nabla_j,\nabla_k]] &= \nabla_i[\nabla_j,\nabla_k] - [\nabla_j,\nabla_k]\nabla_i = -\nabla_i[\nabla_k,\nabla_j] - [\nabla_j,\nabla_k]\nabla_i \\
&= -\nabla_i(\nabla_k\nabla_j - \nabla_j\nabla_k) - (\nabla_j\nabla_k - \nabla_k\nabla_j)\nabla_i \\
&= -\nabla_i\nabla_k\nabla_j + \nabla_i\nabla_j\nabla_k - \nabla_j\nabla_k\nabla_i + \nabla_k\nabla_j\nabla_i
\end{aligned}$$

$$\begin{aligned}
[\nabla_j,[\nabla_k,\nabla_i]] &= \nabla_j[\nabla_k,\nabla_i] - [\nabla_k,\nabla_i]\nabla_j \\
&= \nabla_j(\nabla_k\nabla_i - \nabla_i\nabla_k) - (\nabla_k\nabla_i - \nabla_i\nabla_k)\nabla_j
\end{aligned}$$

$$= \quad \nabla_j\nabla_k\nabla_i - \nabla_j\nabla_i\nabla_k - \nabla_k\nabla_i\nabla_j + \nabla_i\nabla_k\nabla_j$$

$$\begin{aligned}
[\nabla_k,[\nabla_i,\nabla_j]] &= \nabla_k[\nabla_i,\nabla_j] - [\nabla_i,\nabla_j]\nabla_k \\
&= -\nabla_k(\nabla_j\nabla_i - \nabla_i\nabla_j) - (\nabla_i\nabla_j - \nabla_j\nabla_i)\nabla_k \\
&= -\nabla_k\nabla_j\nabla_i + \nabla_k\nabla_i\nabla_j - \nabla_i\nabla_j\nabla_k + \nabla_j\nabla_i\nabla_k
\end{aligned}$$

Putting the three terms together, we have

$$[\nabla_i,[\nabla_j,\nabla_k]] + [\nabla_j,[\nabla_k,\nabla_i]] + [\nabla_k,[\nabla_i,\nabla_j]] = 0 \qquad (7.7)$$

The Bianchi identities

Since in the normal coordinates $\Gamma^i{}_{jk} = 0$ but not necessarily their derivatives, equation (7.1) can be written as

$$R^l_{kij} = \partial_i\Gamma^l_{jk} - \partial_j\Gamma^l_{ik} \qquad (7.8)$$

Taking covering derivative with respect to m, we have

$$R_{kij,ml} = \partial_m\partial_i\Gamma_{ljk} - \partial_m\partial_j\Gamma_{lik} \qquad (7.9)$$

Permuting the indices in a i,j,m in a cyclic order we get two more equations as follows

$$R^l_{kjm,i} = \partial_i\partial_k\Gamma^l_{mk} - \partial_i\partial_m\Gamma^l_{jk} \qquad (7.10)$$

$$R_{kmi,jl} = \partial_j\partial_m\Gamma_{lik} - \partial_j\partial_i\Gamma_{lmk} \qquad (7.11)$$

Adding equations (7.9), (7.10) and (7.11) we get

$$R^l_{kij,m} + R^l_{kjm,i} + R^l_{kmi,j} = 0 \qquad (7.12)$$

This can be put in a compact form as

$$R_{kl\,[ij,m]} = 0 \qquad (7.13)$$

These are called the Bianchi identities.

Taking the inner product of (7.12), with g_{hl}, we obtain the covariant form of Bianchi identities as follows

$$
\begin{aligned}
g_{hl}(R_{kij,ml} + R_{kjm,il} + R_{kmi,jl}) &= 0 \\
R_{hkij,m} + R_{hkjm,i} + R_{hkmi,j} &= 0 \qquad (7.14)
\end{aligned}
$$

Now, consider equation (7.15)

$$R_{kijl} = \partial_i\Gamma_{ljk} - \partial_j\Gamma_{lik} + \Gamma_{mjk}\Gamma_{lim} - \Gamma_{ikm}\Gamma_{ljm} \qquad (7.15)$$

By permuting the indices k,i,j cyclically, we have two other relation as

$$R^l_{ijk} = \partial_j\Gamma^l_{ki} - \partial_k\Gamma^l_{ji} + \Gamma^m_{ki}\Gamma^l_{jm} - \Gamma^m_{ji}\Gamma^l_{im} \qquad (7.16)$$

$$R_{jkil} = \partial_k\Gamma_{lij} - \partial_i\Gamma_{lkj} + \Gamma_{mij}\Gamma_{lkm} - \Gamma_{mkj}\Gamma_{lim} \qquad (7.17)$$

Adding equations (7.15), (7.16) and (7.17)

$$R_{kijl} + R_{ijkl} + R_{jkil} = 0 \qquad (7.18)$$

Taking the inner product with g_{lh}

$$
\begin{aligned}
g_{lh}(R_{kijl} + R_{ijkl} + R_{jkil}) &= 0 \\
R_{hkij} + R_{hijk} + R_{hjki} &= 0 \qquad (7.19)
\end{aligned}
$$

The Ricci tensor

The inner product of g^{hi}.

$R_{kj} = g^{hi}R_{hkij} = g^{hi}R_{ijhk} = R_{jk}$ is called the Ricci tensor. $R_{kj} = R_{jk}$ is symmetric

$$e_{ijhk}R_{ihk} = 0$$

$$R = g_{hi}g_{kj}R_{hkij} = g_{hi}R_{hi}$$

R is known as the curvature scalar.

The Einstein's tensor

Consider the Bianchi identity equation (7.14), contract with g^{hi}, we have

$$\begin{aligned} g_{hi}R_{hkij,m} + g_{hi}R_{hkjm,i} + g_{hi}R_{hkmi,j} &= 0 \\ R_{kj,m} + g_{hi}R_{hkjm,i} - g_{hi}R_{hmki,j} &= 0 \\ R_{kj,m} + g_{hi}R_{hkjm,i} - R_{mk,j} &= 0 \end{aligned}$$

Contract again with g^{kj}

$$\begin{aligned} g^{kj}R_{kj,m} + g^{kj}g^{hi}R_{hkjm,i} - g^{kj}R_{mk,j} &= 0 \\ R_{,m} + g^{hi}R_{hm,i} - R_{m,j} &= 0 \\ R_{,m} + R^{i}_{m,i} - R^{j}_{m,j} &= 0 \\ R_{,m} - 2R^{i}_{m,i} &= 0 \qquad R^{i}_{m,i} = \frac{1}{2}R_{,m} \\ R^{i}_{m,i} - \frac{1}{2}\delta^{i}_{m}R_{,i} &= 0 \end{aligned}$$

This can also be written as

$$\left(R^{i}_{m} - \frac{1}{2}\delta^{i}_{m}R\right)_{,i} \tag{7.20}$$

$G^{k}{}_{m,i} = 0$ where

$$G^{k}_{m} = R^{i}_{m} - \frac{1}{2}\delta^{i}_{m}R \tag{7.21}$$

is called the Einstein's tensor.

Einstein's space

The space in which the Ricci tensor is proportional to the metric tensor is called the Einstein's space: i.e.

$$R_{lm} = \lambda g_{lm} \tag{7.22}$$

Contracting this equation with g^{lm}

$$g_{lm}R_{lm} = g_{lm}\lambda g_{lm} = \lambda N$$

$$R = \lambda N \text{ and } \lambda = \frac{R}{N} \tag{7.23}$$

Where N is the dimensionality of the space. Substituting (7.23) into (7.22), we have

$$R_{lm} = \frac{R}{N} g_{lm}$$

Multiplying through by g^{il}

$$g^{il} R_{lm} = g^{il} \frac{R}{N} g_{lm}$$

$$R^i_m = \frac{R}{N} \delta^i_m \qquad (7.24)$$

Equation (7.21) becomes

$$\left(\frac{R}{N} \delta^i_m - \frac{1}{2} \delta^i_m R \right)_i = \left(\frac{1}{N} - \frac{1}{2} \right) \delta^i_m R_{,i} = 0$$

$$R_{,i} = 0 \qquad (7.25)$$

Equation (7.25) means that, for Einstein's spaces the curvature scalar is a constant

Appendix B Geodesic Deviation

Another important aspect of the Riemann tensor involves geometric deviation, the fact that geodesics begun parallel do not stay parallel.

To measure this precisely, we consider a congruence of geodesics with tangent $\bar{U}$ ($\nabla_{\bar{U}} \bar{U} = 0$) and connecting vector $\bar{\xi}$ which is Lie dragged by the congruence $\pounds_{\bar{U}} \bar{\xi} = 0$ (see **figure (6.6)**) The manner in which $\bar{\xi}$ changes along $\bar{U}$ will be the measure of geodesic deviation. Its first derivative, $\nabla_{\bar{U}} \bar{\xi}$, depends upon initial conditions weather the geodesics are set up initially parallel or not. The geometry enters into the second derivative $\nabla_{\bar{U}} \nabla_{\bar{U}} \bar{\xi}$, which tells how the initial rate of separation of the geodesics changes.

Consider the congruence of geodesics C_U defined by $\nabla_{\bar{U}} \bar{U} = 0$

Let ξ be a vector field obtained by Lie dragging $\xi|_p$ along $\bar{U}$ i.e. $\pounds_{\bar{U}} \bar{\xi} = 0$.

$$\pounds_{\bar{U}} \bar{\xi} = U_i \bar{\xi}_{,i} - \xi_i \bar{U}_{,i} = U_i \bar{\xi}_{;i} - \xi_i \bar{U}_{;i} = \nabla_{\bar{U}} \bar{\xi} - \nabla_{\bar{\xi}} \bar{U}$$

We therefore have, since $\pounds_{\bar{U}} \bar{\xi} = 0$

$$= \nabla_{\xi}\nabla_{\bar{U}}\bar{U} + (\nabla_{\bar{U}}\nabla_{\xi})\bar{U} = [\nabla_{\bar{U}},\nabla_{\xi}]\bar{U}$$

since $\nabla_{\bar{U}}\bar{U} = 0$. This implies

$$\nabla_{\bar{U}}\nabla_{\bar{U}}\xi = R(\bar{U},\bar{\xi})\bar{U} \tag{7.26}$$

where $R(\bar{U},\bar{\xi}) = [\nabla_{\bar{U}},\nabla_{\xi}] - \nabla_{[\bar{U},\bar{\xi}]}$

$$[\bar{U},\bar{\xi}] = \pounds_{\bar{U}}\bar{\xi} = 0$$

In component form, we have

$$\nabla_{\bar{U}}\nabla_{\bar{U}}\xi = (U^i\nabla_i(U^j\nabla_j)\xi^k)e_k = (U^i(U^j\xi^k_{;j})_{;i})\bar{e}_k$$

$$= R(\bar{U},\bar{\xi}) = (R_{ijlk}U_iU_j\xi_l)\bar{e}_k$$

Or

$$U^i(U^j\xi^k_{;j})_{;i} = R^k_{ijl}U^iU^j\xi^l = U^iU^j_{;i}\xi^k_{;j} + U^iU^j\xi^k_{;ji} = U^iU^j\xi^k_{;ji}$$

Since $U^iU^j_{;i} = \nabla_{\bar{U}}U^j = 0$

$$U^iU^j\xi^k_{;ji} = R^k_{ijl}U^iU^j\xi^l \tag{7.27}$$

Equation (7.27) is called the equation of geodesic deviation.

Appendix C Lie derivatives

Proving some identities

(a) Proving of the Leibniz rule

$$\pounds_{\bar{V}}(A \otimes B) = (\pounds_{\bar{V}} A) \otimes B + A \otimes (\pounds_{\bar{V}} B) \qquad (7.28)$$

By definition

$$\pounds_{\bar{V}} f = \lim_{\triangle\lambda \to 0} \frac{f^*(\lambda_0) - f(\lambda_0)}{\triangle\lambda} = \lim_{\triangle\lambda \to 0} \frac{f(\lambda_0 + \triangle\triangle) - f(\lambda_0)}{\triangle\lambda} = \left[\frac{df}{d\lambda}\right]_{\lambda_0} = \frac{df}{d\lambda}$$

Proving from the left hand side

$$\begin{aligned}
\pounds_{\bar{V}}(A \otimes B) &= \lim_{\triangle\lambda \to 0} \frac{f^*(A \otimes B) - f(A \otimes B)}{\triangle\lambda} = \lim_{\triangle\lambda \to 0} \frac{f^*(A) \otimes f^*(B) - f(A) \otimes f(B)}{\triangle\lambda} \\
&= \lim_{\triangle\lambda \to 0} \frac{f^*(A) \otimes f^*(B) - f(A) \otimes f^*(B)}{\triangle\lambda} + \lim_{\triangle\lambda \to 0} \frac{f(A) \otimes f^*(B) - f(A) \otimes f(B)}{\triangle\lambda} \\
&= \lim_{\triangle\lambda \to 0} \frac{f^*(A) - f(A)}{\triangle\lambda} \otimes f^*(B) + \lim_{\triangle\lambda \to 0} f(A) \frac{f^*(B) - f(B)}{\triangle\lambda} \\
&= (\pounds_{\bar{V}} A) \otimes B + A \otimes (\pounds_{\bar{V}} B)
\end{aligned}$$

(b) Proving that, for any two twice-differentiable vector fields $\bar{V}$ and $\bar{W}$ on functions and fields

$$[\pounds_{\bar{V}}, \pounds_{\bar{W}}] = \pounds_{[\bar{V}, \bar{W}]} \qquad (7.29)$$

On functions say f, proving from the left hand side of equation

$$\begin{aligned}
[\pounds_{\bar{V}}, \pounds_{\bar{W}}]f &= \pounds_{\bar{V}} \pounds_{\bar{W}}(f) - \pounds_{\bar{W}} \pounds_{\bar{V}}(f) = \pounds_{\bar{V}}(\bar{W}(f)) - \pounds_{\bar{W}}(\bar{V}(f)) \\
&= \bar{V}(\bar{W}(f)) - \bar{W}(\bar{V}(f)) = (\bar{V}\bar{W} - \bar{W}\bar{V})f \\
&= [\bar{V}, \bar{W}]f = \pounds_{[\bar{V}, \bar{W}]} f
\end{aligned}$$

On vector fields $\bar{U}$ the left hand side of equation (7.29) becomes

$[\pounds_{\bar{V}}, \pounds_{\bar{W}}]\bar{U} = \pounds_{\bar{V}} \pounds_{\bar{W}} (\bar{U}) - \pounds_{\bar{W}} \pounds_{\bar{V}} (\bar{U}) = \pounds_{\bar{V}} [\bar{W}, \bar{U}] - \pounds_{\bar{W}} [\bar{V}, \bar{U}]$

$= [\bar{V}, [\bar{W}, \bar{U}]] - [\bar{W}, [\bar{V}, \bar{U}]]$

Consider the right hand side of (7.29), we have

$$\pounds_{[\bar{V}, \bar{W}]}\bar{U} = [[\bar{V}, \bar{W}]\bar{U}]$$

Equating both sides, we obtain

$$[\bar{V}, [\bar{W}, \bar{U}]] - [\bar{W}, [\bar{V}, \bar{U}]] - [[\bar{V}, \bar{W}]\bar{U}] = 0$$

Expanding, we have

$$\bar{V}[\bar{W}, \bar{U}] - [\bar{W}, \bar{U}]\bar{V} - \bar{W}[\bar{V}, \bar{U}] + [\bar{V}, \bar{U}]\bar{W} - [\bar{V}, \bar{W}]\bar{U} + \bar{U}[\bar{V}, \bar{W}]$$

$$= 0 \quad \bar{V}\bar{W}\bar{U} - \bar{V}\bar{U}\bar{W} - \bar{W}\bar{U}\bar{V} + \bar{U}\bar{W}\bar{V} - \bar{W}\bar{V}\bar{U} + \bar{W}\bar{U}\bar{V} +$$

$$\bar{V}\bar{U}\bar{W} - \bar{U}\bar{V}\bar{W} - \bar{V}\bar{W}\bar{U} + \bar{W}\bar{V}\bar{U} + \bar{U}\bar{V}\bar{W} - \bar{U}\bar{W}\bar{V} = 0$$

Hence on a vector field $\bar{U}$ we have $[\pounds_{\bar{V}}, \pounds_{\bar{W}}] = \pounds_{[\bar{V}, \bar{W}]}$

(c) Proving of the Jacobi identity for Lie derivatives on functions and vector fields: that is

where $\bar{X}, \bar{Y}$,

$[[\pounds_{\bar{X}}, \pounds_{\bar{Y}}], \pounds_{\bar{Z}}] + [[\pounds_{\bar{Y}}, \pounds_{\bar{Z}}], \pounds_{\bar{X}}] + [[\pounds_{\bar{Z}}, \pounds_{\bar{X}}], \pounds_{\bar{Y}}] = 0$ (7.30) $\bar{Z}$ are any three-

times-differentiable vector fields.

On functions f we prove the Jacobi identity as follows

$[\pounds_{[\bar{V}, \bar{W}]}\bar{U}]f = [[\bar{V}, \bar{W}]\bar{U}]f = [\bar{V}, \bar{W}]\bar{U}(f) - \bar{U}[\bar{V}, \bar{W}](f)$

$$= (\bar V\bar W - \bar W\bar V)\bar U(f) - \bar U(\bar V\bar W - \bar W\bar V)(f)$$

$$= (\bar V\bar W\bar U - \bar W\bar V\bar U - \bar U\bar V\bar W + \bar U\bar W\bar V)(f)$$

$$[£_{[\bar W,\bar U]}\bar V]f = [[\bar W,\bar U]\bar V]f = [\bar W,\bar U]\bar V(f) - \bar V[\bar W,\bar U](f)$$

$$= (\bar W\bar U - \bar U\bar W)\bar V(f) - \bar V(\bar W\bar U - \bar U\bar W)(f)$$

$$= (\bar W\bar U\bar V - \bar U\bar W\bar V - \bar V\bar W\bar U + \bar V\bar U\bar W)(f)$$

$$[£_{[\bar U,\bar V]}\bar W]f = [[\bar U,\bar V]\bar W]f = [\bar U,\bar V]\bar W(f) - \bar W[\bar U,\bar V](f)$$

$$= (\bar U\bar V - \bar V\bar U)\bar W(f) - \bar W(\bar U\bar V - \bar V\bar U)(f)$$

$$= (\bar U\bar V\bar V - \bar V\bar U\bar W - \bar W\bar U\bar V + \bar W\bar V\bar U)(f)$$

Adding the three terms, we have

$$[£_{[\bar V,\bar W]}\bar U]f + [£_{[\bar W,\bar U]}\bar V]f + [£_{[\bar U,\bar V]}\bar W]f = [[\bar W,\bar W]\bar U]f + [[\bar W,\bar U]\bar V]f + [[\bar U,\bar V]\bar W]f = 0$$

$\bar V = \bar X$, $\bar W = \bar Y$ and $\bar U = \bar Z$

$$[[£_{\bar X},£_{\bar Y}],£_{\bar Z}] + [[£_{\bar Y},£_{\bar Z}],£_{\bar X}] + [[£_{\bar Z},£_{\bar X}],£_{\bar Y}] = 0$$

For any three times-differentiable vector fields $\bar V, \bar W, \bar U$

$$[[\bar X,\bar Y]\bar Z] + [[\bar Y,\bar Z]\bar X] + [[\bar Z,\bar X]\bar Y] = 0$$

$$[[\bar{X},\bar{Y}]\bar{Z}] = [\pounds_{[\bar{X},\bar{Y}]},\bar{Z}] = [(\pounds_{\bar{X}}\pounds_{\bar{Y}} - \pounds_{\bar{Y}}\pounds_{\bar{X}}),\pounds_{\bar{Z}}]$$

$$= (\pounds_{\bar{X}}\pounds_{\bar{Y}} - \pounds_{\bar{Y}}\pounds_{\bar{X}})\pounds_{\bar{Z}} - \pounds_{\bar{Z}}(\pounds_{\bar{X}}\pounds_{\bar{Y}} - \pounds_{\bar{Y}}\pounds_{\bar{X}})$$

$$= \pounds_{\bar{X}}\pounds_{\bar{Y}}\pounds_{\bar{Z}} - \pounds_{\bar{Y}}\pounds_{\bar{X}}\pounds_{\bar{Z}} - \pounds_{\bar{Z}}\pounds_{\bar{X}}\pounds_{\bar{Y}} + \pounds_{\bar{Z}}\pounds_{\bar{Y}}\pounds_{\bar{X}}$$

$$[[\bar{Y},\bar{Z}]\bar{X}] = [\pounds_{[\bar{Y},\bar{Z}]},\bar{X}] = [(\pounds_{\bar{Y}}\pounds_{\bar{Z}} - \pounds_{\bar{Z}}\pounds_{\bar{Y}}),\pounds_{\bar{X}}]$$

$$= (\pounds_{\bar{Y}}\pounds_{\bar{Z}} - \pounds_{\bar{Z}}\pounds_{\bar{Y}})\pounds_{\bar{X}} - \pounds_{\bar{X}}(\pounds_{\bar{Y}}\pounds_{\bar{Z}} - \pounds_{\bar{Z}}\pounds_{\bar{Y}})$$

$$= \pounds_{\bar{Y}}\pounds_{\bar{Z}}\pounds_{\bar{X}} - \pounds_{\bar{Z}}\pounds_{\bar{Y}}\pounds_{\bar{X}} - \pounds_{\bar{X}}\pounds_{\bar{Y}}\pounds_{\bar{Z}} + \pounds_{\bar{X}}\pounds_{\bar{Z}}\pounds_{\bar{Y}}$$

$$[[\bar{Z},\bar{X}]\bar{Y}] = [\pounds_{[\bar{Z},\bar{X}]},\bar{Y}] = [(\pounds_{\bar{Z}}\pounds_{\bar{X}} - \pounds_{\bar{X}}\pounds_{\bar{Z}}),\pounds_{\bar{Y}}]$$

$$= (\pounds_{\bar{Z}}\pounds_{\bar{X}} - \pounds_{\bar{X}}\pounds_{\bar{Z}})\pounds_{\bar{Y}} - \pounds_{\bar{Y}}(\pounds_{\bar{Z}}\pounds_{\bar{X}} - \pounds_{\bar{X}}\pounds_{\bar{Z}})$$

$$= \pounds_{\bar{Z}}\pounds_{\bar{X}}\pounds_{\bar{Y}} - \pounds_{\bar{X}}\pounds_{\bar{Z}}\pounds_{\bar{Y}} - \pounds_{\bar{Y}}\pounds_{\bar{Z}}\pounds_{\bar{X}} + \pounds_{\bar{Y}}\pounds_{\bar{X}}\pounds_{\bar{Z}}$$

Adding the three terms, we obtain the Jacobi identity as follows

$$[[\pounds_{\bar{X}},\pounds_{\bar{Y}}],\pounds_{\bar{Z}}] + [[\pounds_{\bar{Y}},\pounds_{\bar{Z}}],\pounds_{\bar{X}}] + [[\pounds_{\bar{Z}},\pounds_{\bar{X}}],\pounds_{\bar{Y}}] = 0$$