KWAME NKRUMAH UNIVERSITY OF SCIENCE AND TECHNOLOGY, KUMASI

DETERMINING THE OPTIMAL TRADING STRATEGY OF A SECOND-TIER FUND MANAGER UNDER DIFFERENT COVARIANCE STRUCTURES

BY

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DECLARATION

I hereby declare that this submission is my own work towards the award of the M. Phil degree and that, to the best of my knowledge, it contains no material previously published by another person nor material which had been accepted for the award of any other degree of the university, except where due acknowledgement had been made in the text.

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DEDICATION

This work is dedicated to my husband and my son.

ABSTRACT

The 2^{nd}-tier fund managers have limited investment capital, making it impossible to invest in all securities on the stock market. However investing in fewer stocks may not present the needed financial base that could withstand the possible financial shocks and guarantee high returns. On the other hand, actuaries can employ actuarial tools such as covariance structure to forecast the prospects of a selected stocks which could complement each other to reduce the probable risk and improve investment output. Fund managers are therefore required to make an informed investment decisions adopting reliable actuarial tools that could minimize the risk that can be associated with investing in fewer stocks. The decision they make on their investments affect returns, and that of their clientèle eventually. The study looked at three different covariance structures: Toeplitz, Autoregressive(1) and Unstructured and the amount to optimally invest in each security under these structures. Correlated and uncorrelated stocks under the different covariance structures on simulation study were studied. Monthly data was taken from companies that deal in AAPL, TNET and BAX from 1^{st} January, 2008 to 31^{st} December, 2012. Kolmogorov - Smirnov and Anderson Darling tests were used to check the distributions of the market data. After a hypothesis testing,the observation was that AAPL, BAX and TNET were from the Normal, Weibull and Weibull distributions respectively with a negative correlation between BAX and TNET. Inferring from the results obtained in this study, investing more than half of the investment capital in the TNET security was considered optimal. In conclusion, this study has demonstrated that fund managers can rake in high returns if more than half of the amount available for investment was put in the TNET security. Additionally, the Toeplitz covariance structure proved efficient in predicting the suitable stocks to invest the 2 tier fund.

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LIST OF ABBREVIATION

CAP 30 .... Chapter 30 of the Pension Ordinance of 1946

NPRA .... National Pensions Regulatory Authority SSNIT

..........................Social Security and National Insurance Trust

PCP ....Presidential Commission on Pension PRIC

..........................Pensions Reform Implementattion Committee NHIS

.... National Health Insurance Scheme

ESS .... Employer Sponsored Scheme

MTS .... Master Trust Scheme

PAYG .... pay-as-you-go

MPT .... Modern Portfolio Theory CAPM

....Capital Asset Pricing Model

SEC ....Socurities and Exchange Commission

PDF ....Probability Density Function OTS

.... Optimal Trading Strategy

SDE .... Stochastic Differential Equation GOF

....Goodness of Fit

CDF ....Cumulative Distribution Function

GARCH ....Generalized Autoregressive Condition Heteroskedasticity

NWW .... combination of Nomal,Weibull and Weibull distributions NLL ....combination of Normal,Logistic and Logistic distributions

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CHAPTER 1

INTRODUCTION

In 1950, the Pensions Ordinance No. 42 established a pension scheme for public servants known as CAP. 30 scheme. However, after 15 years of operation coupled with increasing number of pensioners, the CAP.30 pension scheme could no longer cope with the financial burden and therefore was unsustainable. This necessitated the enactment of a social security Act 279 in 1965, to create a Social Security Fund for payment of superannuations, invalidity, survivors and other benefits for workers. To further strengthen the operations of the Social Security Fund, the National Insurance Trust was established in 1972 to administer a social security fund for all workers in the country. Since 1965, the country has been operating two major public pension schemes: CAP.30 and the SSNIT schemes. However, the disparity between the two schemes became more pronounced and therefore led to workers agitations which resulted in the replacement of the SSNIT scheme by the CAP.30, because CAP.30 was considered more favourable due to the lump sum factor (Agyeman, 2011). Going forward, the Government of Ghana in 2004 appointed a committee to initiate a major reform on the pension system that will ensure retirement income security for Ghanaian workers. This committee was established and commenced work immediately on November, 2006. The committee at the end of its tenure recommended the creation of a new three-tier contributing pension for Ghanaians comprising two mandatory schemes and a voluntary scheme and submitted proposals of National Pensions Reform Bill to Government in 2007 (Agyeman, 2011). The bill was subsequently passed by Parliament and received Presidential assent on December 2008. The new Pensions Law and National Pensions Act, 2008 was subsequently promulgated on 12th December, 2008 with the aim of improving pensions of workers with a more transparent pension scheme. The new Law specifically dealt with the establishment of a contributing three-tier Pension scheme with a Pension Regulatory Authority (Agyeman, 2011). The contributing three-tier schemes are:

1. firstly, the first- tier mandatory basic social security scheme which incorporate an improve system of SSNIT benefits and comes with no lump sum payment but only monthly pensions (Ashidan, 2011),

2. secondly, the second-tier which is the focus of this study is an occupational pension scheme mandatory for all employees but privately managed and designed to primarily give contributors higher lump sum benefits than CAP.30 or SSNIT Ashidan(2011), and last but not the least

3. the third-tier which is a voluntary provident fund and personal pension scheme designed to enhance workers pension benefits.

As stated above, the second tier pension scheme unlike the other two tiers does not only come with lump sum, it is also mandatory and managed by private manager. Therefore, the success of the second tier pension scheme will largely depend on the ability of the fund manager to invest in a promising stock market that will yield high dividend. It is therefore incumbent on fund managers to make competent informed decisions based on reliable stock market forecast employing suitable actuarial tools. Not long ago (about 2 years), there was an article on myjoyonline which stated emphatically that manager of the Pension fund, Social Security and National Insurance Trust were losing out on some of their investments (Agyeman, 2011). The author of the article reiterated that Social Security and National Insurance Trust had used part of employee's contribution in building a central car park which was not being patronised adequately and therefore yielded low returns. The big question that comes to mind is what informed this decision?

Managers of the second tier pension can perform better if issues on volatility becomes major concern as they invest monies of contributors on the stock market. The frequent changes in the stock market makes it difficult if not impossible for investors and fund managers to achieve high returns on their investments in these markets. This story is not peculiar but a common trend in the stock market. Considering the unpredictable nature the stocks has assume in recent times, fund manager will require accurate actuarial tools that could present reliable forecasts on trends in the stock market to be able to make high yielding investments in the interest of the clientèle. This is even more imperative when the investment is on a pension scheme such as the national tier 2 pension scheme which has zero tolerance for failure (Vives, 1999). Handling practical problems in modern finance require an understanding of the volatility and correlations of asset returns. Examples of such everyday problems include managing the risk of a multi-currency portfolio, optimal asset allocation and derivative pricing. Volatility or covariance matrix unfortunately for fund manager and investors cannot be observed directly, but must be estimated from data on daily returns. Many financial market participants, recognising the uncertainty in covariance matrix structure have given up on the hope of making informed decisions based on this type of data, however, we believe it cannot be avoided. Every decision, just as in life, requires that people make informed choices under conditions of uncertainty. The only issue at stake is whether they make the decision based on a more reliable information. For most financial decisions, the relevant information is the estimate of the covariance over a future horizon.

1.1 STATEMENT OF THE PROBLEM

Second tier fund manager is faced with the difficulty on how to optimally allocate the resource in securities on the stock market and therefore affects the investment prospects negatively. Taking into consideration the fact that the second tier pension scheme is mandatory for all employees but privately managed and designed primarily to give contributors higher lump sum than what is presently available under both CAP.30 and SSNIT pension schemes, it is incumbent on fund managers to use reliable actuarial tools that would guarantee higher returns on investment. However, more often than not, fund managers take investment decisions based on current trends and other stock market watches which are not robust enough to make reliable investment forecast on the stock market, leading to low returns on the investment, and culminating in lower lump sum pension in case of the tier two national pension scheme. Furthermore the problem is more compounded if the covariance structure between stocks are not considered in computing the optimal trading strategy. This study therefore seeks to present robust actuarial tools which could provide reliable forecast for successful investment by looking at the correlations and covariance structures between the stocks that would guarantee higher returns on investment on the tier two pension scheme by fund managers.

1.2 HYPOTHESIS

This study predicts that the optimal trading strategy formula being evaluated in this current study will yield optimal returns on the tier 2 pension if adopted by fund managers in the managers of this pension fund. Additionally, covariance structures will be convenient to operate and will present with high output.

1.3 OBJECTIVES

The study seeks to:

1. Investigate the optimal trading strategy formula for a fund manager under full information.

2. Compare the optimal trading strategy formula under two covariance structures that yield optimal returns.

CHAPTER 2

LITERATURE REVIEW

2.1 History on Pension Schemes in Ghana

Pension scheme in Ghana pre-dates independence. The first officially pension scheme in Ghana was instituted in 1946 and captioned CAP. 30 derived from Chapter 30 of the Pension Ordinance of 1946. The most intriguing part of the CAP.30 was that it was a non contributory scheme designed specifically for civil servants. Thus, the civil servants who were privileged to be under this scheme were not required to make contributions into the scheme because the scheme was being financed from general tax revenues. Initially the CAP. 30 was available to civil servants such as Armed forces, Prison and Police service (Kpessa, 2011). However similar scheme which was modelled on the original CAP.30 was later instituted for other public servants such as teachers. As the number of pensioners went up over the period, it became pertinent to find alternative scheme because the CAP.30 scheme was no longer sustainable. This led to the establishment of the Social Security and National Insurance Trust (SSNIT) pension scheme. It was established by the Social Security Act 279 in 1965 which set up a Social Security Fund and provided for the payment of lump sum through a Provident Fund Scheme. However, in 1991, the Social Security Law (PNDCL 247) converted the Provident Fund Scheme to a Pension Scheme. Unlike the CAP 30 pension scheme, workers under SSNIT were expected to contribute 17.5 percent of their monthly salaries made up of 5 percent by the employee and 12.5 percent by the employer. Additionally, the informal sector was encouraged to contribute but unlike the formal sector, they were expected to pay all the 17.5 percent themselves. In reality, CAP.30 was more lucrative than SSNIT. Under the CAP.30 scheme, contributors were entitled to a lump sum payment after working for a period of 10 years as against a period of 20 years under the SSNIT pension scheme. The lump sum payment under CAP 30 constituted 70 percent of the individual's final salary as against 50 percent of the average of three highest years' salaries under the SSNIT scheme. Additionally, CAP. 30 payments were indexed annually to current salary scales (Ofei-Kwapong, 2013). All these put together, the benefits under CAP.30 scheme were far better than those the SSNIT scheme presents. As expected, workers preferred the CAP 30 scheme to the SSNIT scheme. As was argued by Kpessa (2011), the CAP 30 scheme was more of a reward than a pension for members of the civil service especially the security services. The CAP 30 obviously was not sustainable financially, that was what necessitated the establishment of SSNIT to take off the financial burden from CAP 30 which is a non contributing scheme. As reported by Kumado & Gockel (2003), some of the inherent problems associated with the CAP 30 were its unfunded nature that leaves huge financial burden on the national coffers and the fact that it was not available to all Ghanaians.

This particular situation accounted for the incessant demonstrations that characterised the CAP 30 scheme, as those who were not part fought to get in to enjoy the privileges that go with it. Taking to account all these difficulties associated with CAP 30 coupled with managerial problems bedevilled the SSNIT scheme over the years which have led to low pensions received by workers, it was imperative to restructure the whole pension scheme to derive optimum output from it (Ashidan, 2011). Therefore, the Government of Ghana in July 2004 commenced a major reform in the pension system by constituting a commission and charged with the responsibility to examine the existing pension arrangements and to make appropriate recommendations for a sustainable national pension scheme that would ensure retirement income security for Ghanaian workers. Government accordingly accepted virtually all the recommendations by the commission and subsequently issued a white paper (W.P.No.1/2006) in July, 2006. The key recommendation of the commission was the creation of a new contributory three-tier pension system. The first-tier pension scheme was a mandatory basic national social security scheme which was suppose to incorporate an improved system of SSNIT benefits, mandatory for both private and public sector employees. Additionally, the second tier occupational pension scheme, mandatory for all employees and privately managed and primarily designed to give contributors higher lump sum benefits than what is presently available under the CAP 30 and SSNIT pension schemes. And last but not the least, the third tier voluntary provident fund and personal pension scheme supported by tax benefit incentives to provide additional funds for workers who want to make voluntary contributions to enhance their pension benefits and also for workers in the informal sector.

2.2 The Three- Tier Contributory Schemes

The pension reform was done in response to the incessant agitation from the Ghanaian workers and Pensioners on the inequalities in retirement benefits among workers under different pension schemes (CAP 30 and SSNIT) as well as inefficiencies in the SSNIT system. In July 2004, the NPP Government under the leadership of His Excellency President John Agyekum Kuffour instituted a presidential commission on Pension (PCP) headed by Mr. T. A. Bediako to address these concerns of organize labour. The mandate of the commission was to submit a proposal for pension reform that addresses the bottlenecks in the national pension scheme. The commission at the end of its tenure recommended the creation of a new contributory three tier pension scheme consisting of two mandatory schemes and a voluntary scheme. Furthermore, they recommended the abolishing of the CAP 30 pension scheme and called for the restructuring of the SSNIT pension scheme by revamping its administrative and management structures. Government of Ghana adopted almost all the recommendations and therefore Pensions Reform Implementation Committee (PRIC) was set up to see to the implementation of the recommendation. The work of the PRIC led to the promulgation of the new pensions law, the National Pensions Act,2008 (Act 766) on December 12, 2008 about the three-tier contributory scheme.

The three-tier contributory scheme which is a hybrid of the benefit and contribution schemes consists of tier one, tier two and tier three schemes. The tier one which is a mandatory contributory scheme with monthly contributions of 13.5 percent with 11 percent towards monthly pensions and 2.5 percent as contribution to NHIS on the basic salary of all employees (Kpessa, 2011). The tier one benefit and contributions are fully tax-exempt and are managed by SSNIT. This scheme is suppose to pay monthly benefits to employees upon retirement. Also, employees in the private sector may partake in this scheme although not mandatory. The tier two on the other hand is a mandatory contributory scheme with monthly contributions of 5 percent on the basic salary of employees. Just like the tier one, tier two contributions are fully tax-exempted, however, unlike the tier one, the tier two is privately managed by National Pension Regulatory Authority (NPRA), a licenced service provider. There are two forms of the tier two scheme, these are the Employer Sponsored Scheme (ESS) and Master Trust Scheme (MTS). The membership of ESS is limited to employees of a specific company whereas membership of the MTS is opened to employees of different companies. This scheme is suppose to pay out a lump sum benefit to employees upon retirement. And finally, the third tier which is an optional contribution scheme made up of a contribution of up to 16.5% of the employees' basic salary. The third tier scheme is also managed by NPRA licensed service provider and like tier one and tier two, is also tax- exempted. Contributors are suppose to be in the scheme for 10 years or more to receive all contributions made under the scheme in addition to all returns earned on their contributions at the time of exit. In the event of an exit prior to the contributors tenth anniversary, a marginal tax rate of 15% will be applied to the contributor's total redemption amount (Ofei-Kwapong, 2013).

2.3 Managing the Tier Three Pension Scheme

The management of the three tier pension scheme is left in the care of different service providers with varied responsibilities to ensure efficiency. Aside the NPRA which serves as a regulatory body, there are three main service providers who play key role in the management of the three tier pension scheme. These are the Trustee, the Pension Fund Manager and the Custodian (Aitken, 1994). The Trustee is an institution or an individual licensed by the NPRA and entrusted with the overall responsibility for the administration and prudent management of the tier two and tier three schemes. The Trustee as an independent thirdparty is separated from both the fund manager and the custodian. The Trustee is entrusted with the responsibility to ensure adherence of the investment objectives of the contributors. Trustees are mandated by law (Act 766) to appoint pension Fund managers, Custodians and other relevant service providers to assist in the management of the schemes. Among other things, Trustees are to ensure proper accounting and book keeping records. Additionally, they are to ensure that all engaged service providers comply with the regulations that govern the schemes. More importantly, Trustees do not have access to the pension funds and this is imperative to ensure transparency in the management of the pension scheme. The fund manager is a key figure in the management of the pension scheme. The fund manager is supposed to be licensed by SEC and registered by the NPRA before he or she can assume responsibility as a fund manager. The fund manager as part of his or her responsibility makes investment decisions in consultation with the NPRA and the Trustee. Also, the fund manager keeps records and account statements on transactions in relation to the pension funds and assets, and mandated to report monthly to the Trustee and quarterly to both the trustee and the NPRA. Like the Trustee, the fund manager does not have direct access to the pension funds and assets (Aitken, 1994). Like the fund manager, the Custodian is licensed by the Securities and Exchange Commission (SEC) and registered by the NPRA. The Custodians as the name implies receives contributions from employers and is responsible for the safe keeping of the pension funds and assets. Additionally, Custodian keeps records and statements of account on transactions related to the pension funds and assets, and expected to present monthly report to the Trustee and quarterly to both the Trustee and NPRA (Aitken, 1994).

2.4 The role of Actuaries in Pension fund management

Pension schemes by their nature require strategic investment of the pension fund to possibly maximise the pension benefits of Pensioners (Ivanova, 2010; Aitken, 1994). Fund managers will definitely require actuarial projections and analysis to ascertain the viability and sustainability of the pension scheme into the future (Marossy, 2001; Touahri, 2008). Actuaries have expertise in quantifying contingent risk and offering valuable advice to assist the fund manager and other stakeholders of pension schemes to understand and manage their risk appropriately. Additionally, actuaries have expertise in investment with special emphasis on strategic decision in relation to investments allocation with direct bearing on pension scheme benefits (Daykin, 2002). Proper management of the investments of a defined contribution pension scheme is important in pensions scheme management (Avrahampour, 2006). Therefore, fund managers of pension scheme are expected to make strategic investment decisions based on complete understanding of the trade - offs of risk, including a clear understanding of the risk profile (Camfield, 2000). The services of actuaries would therefore be needed in the strategic investment decision-making through asset- liability modelling and stochastic modelling of the investment portfolios. Another key part of the pension scheme management is the regular measurement and monitoring of the performance of the investment managers independent of the fund managers (Lewin, 2007; Dewotor, 2004). Actuaries have the best expertise to carry out this performance evaluation responsibilities. Reports on the performance evaluation of the pension scheme by the Actuaries are made available to the Fund managers, the Trustees and the Custodians. These reports could be used as part of the process of holding the fund managers to account and relying on it to make informed investment decisions (Ivanova, 2010). In a nutshell, pension schemes require active input from Actuaries.

2.5 Trends of Pension Schemes in Sub-Saharan Africa

According to a comprehensive study by Dorfman (2015) on pension patterns in Sub-Saharan Africa, there are currently four types of pension schemes being operated in Sub-Saharan Africa.These are non-contributory pension, mandatory contributory pension schemes, voluntary regulatory occupational or personal pension savings and insurance arrangements, and last but not the least, other informal voluntary savings arrangements and household assets. In that respect, public service pension schemes are classified as occupational pension scheme even though most of the schemes in Sub-Saharan Africa are non-contributory. Additionally, contributory pension schemes may be completely pay-as-you-go (PAYG) with the following categories: contributions financing benefits partially funded with reserve accumulations or fully funded with funds set aside for all pension liabilities. Interestingly, all the countries in Sub-Saharan Africa have public servant schemes (Dorfman, 2015; Ambachtsheer & Capelle, 2006). However, about a quarter of these countries are integrated into national contributory schemes. Countries like South Africa, Botswana, Namibia and Lesotho which do not have national contributory schemes have instituted their public service scheme as occupational schemes. The study also demonstrated that about three- fourth of the scheme are PAYG defined- benefit schemes, specifically 31 out of 44. Countries such as Burundi, Democratic Republic of Congo, Gambia and Kenya do not collect any contributions from public servants but pay benefits out of general national revenues (Catalan, 2004). Swaziland and South Africa on the other hand operate fully- funded defined benefit schemes. And whereas Nigeria, Botswana and Namibia have fully-funded defined contribution schemes (Stewart & Yermo, 2008), Ghana operates a hybrid scheme for both public servants and and private sector employee as demonstrated by the study Dorfman(2015), reiterated the fact that, 31 out of the 47 countries have national contributory PAYG defined benefit (DB) schemes, four have provident funds, Nigeria has a defined contribution (DC) scheme, Ghana has a hybrid of DB and DC schemes with four operating no national contributory schemes but have in place some form of non- contributory old age benefits. The remaining six countries either have no national scheme or the data for such schemes were not available.

2.6 Covariance Structure

Covariance is a measure of how changes in one variable are associated with changes in a second variable. Specifically, covariance measures the degree to which two variables are linearly associated or co- vary . Covariance Structures are just patterns in covariance matrices. Some of these patterns occur often enough in most statistical procedures that have been named. Some of these names are compound symmetry, variance components, unstructured, for example. They sound strange because they are often thrown about without any explanation. But they are just descriptions of patterns. For example, the compound symmetry structure means that all the variances are equal and all covariances are equal. Variance Components means that each variance is different and all covariances are zero. Unstructured means there is no pattern at all. Each covariance and each variance is completely different and has no relation to the other.

Models for covariance structures attempt to explain the relationships among a set of observed variables in terms of a generally smaller number of unobserved variables (Akbas, 2001). As the name of this technique implies, the relationships among the observed variables are characterised by the covariances among those variables, contained in the matrix P. This matrix is decomposed by a model that assumes that unobserved variables are generating the pattern or structure among the observed variables. Using a measurement model linking the observed variables to the unobserved variables, and a structural model relating the unobserved variables, an analysis of the covariance matrix is made to describe its structure (Akbas, 2001).

The term "analysis of covariance structures" was introduced by Akbas (2001) to describe what would now be called a confirmatory factor model. Since then, numerous authors have added to the complexity and generality of the model.The model has grown from the factor analytic model of Akbas (2001) to extremely general model in which the covariance matrix is considered to be any function of any set of parameters, with many intermediate forms of the model appearing along the way. Though progress has been made in the estimation and application of these extremely general forms of the model, emphasis had been on the more limit, albeit still quite general, form introduced by D. E. Wiley. In this more restrictive model, the covariances among the observed variables are decomposed in two conceptual distinct steps. Firstly, the observed variables are linked to the unobserved or latent variables through a factor analytic model, similar to that commonly found in psychometrics. Secondly, the casual relationships among these latent variables are specified through a structural equation model, similar to that found in Econometrics. The covariance structure model, in the form considered here, consists of the simultaneous specification of a factor model and a structural model, and as such represents a fruitful unification of psychometrics and econometrics. This synthesis was greatly facilitated by Kim (2006) programmatic article. In a test design including repeated measures, it is possible to get different features from variable structure test units with repeated measures made in different times for same features (Tabachnick, 2001). It is essential to properly identify the variance- covariance structure among the data in the analysis of the repeated measures (Akbas, 2001).

A primary aim of the covariance structure analysis is to specify enough restrictions, $\Sigma(\theta)$ so that substantively, it becomes a sufficiently simple and acceptable representation for the theoretical or interpretative issue being investigated (Akbas, 2001). Technically, also, the model should improve precision, that is reduce variance in the parameter estimator, at the expense of little or no bias in the estimator (Tabachnick, 2001).

2.7 Portfolio Management

Portfolio Management is an important area for long-term investors and fund managers. It is concerned with how to best diversify investments into different classes of assets in order to maximise the expected profit and to minimise the risk involve (Campbell, 2002; Tobin, 1958).

Things began with Harry Markowitz. His publication, M. 1952: Portfolio Selection in the Journal of Finance set the pace for Modern Portfolio Theory (MPT). The starting point of portfolio theory optimization is the work of Harry Markowitz on the mean-variance criteria to judge investment strategies in security markets (Darko, 2012). His innovation on Portfolio Selection awarded him a Noble Prize.

Tobin(1958) did more to the field of portfolio management by introducing the risk free asset. He brought to fore the concept of Capital Market Line and made a statement that portfolio comprises of risky portfolio and the risk free assets. William F. Sharpe also formalized the Capital Asset Pricing Model (CAPM) by introducing the beta.

The onset of computer allowed implementing software solutions to solve complex calculations. Sharpe(1994) reiterated that it is fundamentally assessment errors on performance expectations that have an impact on the setting up of portfolios. The assessment errors on variances and covariances clearly have less impact. Charles Stein Bayesian method also tackle on the improvement of the expected performance method (Sharpe, 1994; Pedersen, 2013). He talked about the fact that it is best to seek for the estimator that will minimise the impact of the assessment risk on the composition of an optimal portfolio. Parallel to the evolution of technical method, theoretical teaching methods also developed. Tyersky & Kahneman (1979) developed the Prospect Theory that will become the basis of the theory of behavioural finance.

Since then, this theory has allowed to systematically organize many illustrations showing the investor cognitive bias and the exceptions to the concept of rationality in traditional finance. Thaler & Bondt (1986) argued that investors irrationality may explain certain market inefficiencies. The introduction of the Sortino ratio of risk of the minimum asset return and other parameters give way to new avenues for refining the average variance (Munk, 2012; Tabachnick, 2001).

2.8 Stochastic Finance

Stochastic integration and the modelling of risky asset prices both began with Brownian motion. Mathematically, Brownian motion can be traced to three sources each knowing nothing about the other person's work (Chung & Williams, 1990). The first person being T. N. Thiele of Copenhagen who successfully proposed Brownian motion when studying time series in 1880. L. Bachelier of Paris, also created a model of Brownian motion while deriving the dynamic behaviour of the Paris stock market in 1900. A. Einstein, also proposed a model of the motion of small particles suspended in a liquid. Einstein's model became so prominent than the other two (Ameko & Baah, 2014).

The founder of modern Mathematical Finance is now attributed to Bachelier. While Bachelier was still ignorant of the works of the others, he attempted to initiate modelling the market noise of the Paris bourse. Exploiting the ideas of the Central Limit Theorem, he realised that increments of stock prices should be independent and normally distributed. He was able to define other processes related to Brownian motion.

Einstein's work assumed Brownian motion to be a stochastic process with continuous paths , independent increments, and stationary Guassian increments. However, Einstein was unable to show that the process he proposed actually existed as a mathematical object. The ideas of Borel and Lebesgue measure were proposed in 1905.

Daniell's approach to measure theory surfaced in 1913. N. Wiener used these ideas together with Fourier series in 1923 to construct Brownian motion. Wiener and others proved many properties of the paths of Brownian motion, an activity that still exists. One property of stochastic integration is that the paths of Brownian motion have infinite variation on compact time intervals, almost surely. If Einstein were to have assumed rectifiable paths, Wiener's construction would have proved the impossibility of the model. Wiener's construction of Brownian motion is often referred to as Wiener process. In 1951, K. Ito improved Wiener's ideas when trying to understand his papers.

Initially, the theory of stochastic integration from the non-finance perspective were intertwined with the theory of Markov processes in which A. N. Kolmogorov played a key role. In one of his papers, he develops a large part of his theory of Markov processes. In this paper, Kolmogorov showed that continuous Markov processes depend on two parameters: one for the speed of the drift and the other for the size of the purely random part. He was then able to relate the probability distributions of the process to the solutions of partial differential equations which were solved and now referred to as Kolmogorov's equations. Kolmogorov utilised the new concepts of martingales proposed by J.Ville in 1939 and understood the importance of studying sample paths.

The father of stoahastic is Kiyosi Ito. One of Ito's motivation for studying stochastic integrals was an attempt to establish a true stochastic differential to be used in the study of Markov processes. Wiener's integral did not permit stochatic processes as integrands, and it will be needed if one were to represent a diffusion as a solution of a stochastic differential equations. In 1944, Ito published his first paper on stochastic integration, and is in this same year, one Kakutani also published two brief notes connecting Brownian motion and harmonic functions. In 1948, E. Hille and K. Yosida independently gave the structure of semi- groups of strongly continuous operators clarifying the role of infinitesimal generators in Markov process theory. In Ito’s efforts and zeal to model Markov processes, he constructed a stochastic differential equation to have the form:

$$dX_t = \sigma(X_t)dW_t + \mu(X_t)dt \tag{2.1}$$

where W represents a standard Wiener process. His efforts resulted in publishing another paper in 1951, where he stated and proved what is now known as Ito’s formula:

$$f(X_t) = f^0(X_t)dX_t + 1/2f^{00}X_t d(X,X) \tag{2.2}$$

Ito’s formula is an extension of the change of variable formula in RiemannStieltjes integration. One of his key insights was to limit his space of integrands to those that were "non-anticipating". This meant that he allowed integrands that were adapted to the underlying filtration generated by the Brownian motion. It became necessary to see if Doob’s decomposition theorem could be extended to sub-martingales indexed by continuous time. However, the seminal paper of G. A. Hunt began to parallel the development of axiomatic potential theory. It took a decade for these papers to be fully appreciated. Some publications by P.

A. Meyer in one of his papers came to resolve this issue. As if it was to buttress the importance of probabilistic potential theory in the development of stochastic integral, Meyer’s first paper in the language of potential theory. He showed that is false but true if and only if one assumes that the sub-martingale has a uniform integrability property when indexed by stopping times. P. A. Meyer established the uniqueness of the Doob’s decomposition theorem. Also, Meyer’s second paper provides an analysis of the structure of L^2 martingales which will prove essential later in the development of the theory of stochastic integration. Ito and Watanabe in 1965 defined local martingales whilst studying multiplicative functionals of Markov processes. This allows Doob's conjecture to hold. This means that any sub-martingales X has a unique decomposition:

$$X_t = M_t + A_t \tag{2.3}$$

where M is a local martingale and A is a non decreasing, predictable process with $A_0 = 0$.

2.9 Probability distributions

The probability distribution (discrete random variable) is a list of probabilities associated with each of its possible values. Sometimes, it is called the probability function or the probability mass function. The probability distribution of a discrete random X is a function which gives the probability $p(x_i)$ that the random variable equals x_i, for each value x_i: $p(x_i) = P(X = x_i)$

And it satisfies the following conditions :

1. $0 \leq p(x_i) \leq 1$

2. $\sum p(x_i) = 1$

2.10 Cumulative distribution function

A discrete or continuous random variables have a cumulative distribution function and is a function giving the probability that the random variable X is less than or equal to x, for every value x.

Cumulative distribution function is giving by:

$F(x) = P(X \leq x)$ (Liptser & Shiryaev, 2001). The probability distribution function of a discrete random variable is obtained by adding all probabilities and that of a continuous random variable is the integral of its probability density function.

2.11 Independent Random Variables

W and Q random variables are said to be independent if and only if the value of one has no influence on the value of the other. The probability density functions of a continuous independent random variables are given by f(w,q)=g(w).h(q) where g(w) and h(q)are the marginal density functions. Probabilities of the discrete independent random variables are given by $P(W = w_i; Q = q_i) = P(W = w_i).P(Q = q_i)$ for each set (w_i,q_i).

2.12 Probability-Probability Plot (P-P Plot)

In order to check whether a given set of data follows a specified distribution, the P-P Plot could be used. If the distribution specified is correct, the P-P Plot should be approximately linear. The cumulative distribution function, F(x) of the said model is used to construct the P-P Plot. The cumulative distribution function, $F(x_i)$ plotted against (i-0.5)/n.

2.13 Quantile - Quantile Plot (Q-Q Plot)

The Q-Q Plot is also used to check whether a given data set follows a specified distribution. This plot will show an approximate linear graph when the data set follows that distribution. Here, x_i is plotted against $F^{-1}((i - 0.5)/n)$.

2.14 Normal Distribution

It is a continuous probability distribution with μ and σ as its parameters . These parameters μ and σ are real numbers with σ being positive. Its probability distribution function:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{(-\frac{1}{2}(\frac{x-\mu}{\sigma})^2)} . \tag{2.4}$$

Its cumulative distribution function:

$$F(x) = \frac{1}{2}[1 + erf(\frac{x-\mu}{\sigma\sqrt{2}})] \quad (2.5)$$

where erf is the error function.

2.15 Logistic Distribution

It is a continuous probability distribution with parameters μ and σ^2 as the location and scale parameters respectively. Its probability distribution function is given by:

$$f(x) = \frac{e^{-z}}{s(1+e^{-z})^2} \quad (2.6)$$

where $\sigma > 0$ and $-\infty < x < +\infty$. Its cumulative distribution is also given by:

$$F(x) = \frac{1}{1+e^{-z}} \quad (2.7)$$

where $z = \dfrac{x-\mu}{\sigma}$

2.16 Weibull Distribution

$f(x) = \dfrac{\alpha}{\beta}(\dfrac{x}{\beta})^{\alpha-1} e^{(-(\frac{x}{\beta})^{\alpha})}$. Its cumulative distribution function is

$$F(x) = 1 - e^{(-(\frac{x}{\beta})^{\alpha})} \quad (2.8) \qquad mean = \beta\, \Gamma(1 + \frac{1}{\alpha}) \quad (2.9)$$

$$variance = \beta^2 \Gamma(1 + \frac{2}{\alpha}) - \mu^2 \quad (2.10)$$

Where α>0 is the shape parameter and β>0 is the scale parameter of the distribution.

2.17 Cholesky Decomposition

A symmetric positive definite matrix A is given by:

$$A = L^T DL \quad (2.11)$$

where L is an upper triangular matrix and D is the diagonal matrix with positive diagonal elements (Haugh, 2004). The Cholesky Decomposition is one of the few numerically stable matrix algorithm used (Haugh, 2004). The variance-covariance matrix, Σ, is a symmetric positive definite matrix. Therefore,

$$\Sigma = L^T DL \quad (2.12)$$

$$\Sigma = (L^T\sqrt{D}L)(\sqrt{D}L) = (\sqrt{D}L)^T(\sqrt{D}L) \quad (2.13)$$

The matrix $Q = \sqrt{D}L$ therefore satisfies $Q^T Q = \Sigma$. It is called the Cholesky Decomposition of Σ.

CHAPTER 3

METHODOLOGY

3.1 Introduction

This section also considers the various mathematical concepts used in the paper. It provides a foundation for the structures that are built in the paper for a better appreciation of the mathematical formulas.

3.2 Probability Space

A probability space is a triplet consisting of these objects:

1. Ω,a non-empty set, called the sample space which contains all possible outcomes of a random experiment;

2. F, a σ-algebra of subsets of ω;

3. P, a probability measure on (ω,F) ie a function which assigns to each set A F a number Pϵ $[0,1]$, which represents the probability that the outcome of the random experiment lies in the set A (Ameko & Baah, 2014).

3.3 Sigma-Algebra and Filtration

3.3.1 Definition

Given that 2^{Ω} denotes the set of all subsets of Ω. We say that F is a σ-field (or a σ-Algebra), if

1. Ω F

2. If $A\epsilon$F then A^c ϵ F

3. If $A_i\epsilon$ F for *i* = 1,2,... then also $\bigcup A_i$ ϵ F

3.3.2 Definition

Let Ω be a nonempty finite set. A filtration is a sequence of σ-algebras $F_0,F_1,F_2,...,F_n$ such that each σ -algebra in the sequence contains all the sets contained by the previous σ-algebra (Haugh, 2004).

3.4 Radon Nikodym Theorem

3.4.1 Theorem

Let P and $\bar{P}$ be two probability measures on a space (Ω,F). Assume that for every A ∈ F satisfying P(A) = 0 , we also have $\bar{P}(A) = 0$. Then we say that $\bar{P}$ is absolutely continuous with respect to P. Under this assumption, there is a non-negative random variable Z such that

$$\bar{P} = \int_A Z dP, \qquad \forall A \in F.$$

3.5 Girsanov Theorem

3.5.1 Theorem

Let w_t, $0 < t < T$ be a Brownian motion on the probability space (Ω,F,P): let f_t, $0 < t < T$ be the accompanying filtration and let $\theta(t)$, $0 < t < T$ be a process adapted to the filtration. For $0 < t < T$, define

$$\bar{\mathbb{W}}(t) = \int_0^t \theta(u)du + W(t) \tag{3.1}$$

$$z(t) = exp - \int_0^t \theta(u)dW(u) - \frac{1}{2}\int_0^t \theta^2(u)du \tag{3.2}$$

and define a new probability measure by

$$\mathbb{P}(\bar{A}) = \int_A Z(T)dP, \forall A \inf \tag{3.3}$$

Under $\bar{P}$, the process $\bar{W}$,$0 < t < T$, is a Brownian motion. Given the following assumption about θ that

$$\mathbb{E}(e^{\frac{1}{2}\int_0^T \theta^2(u)du}) < \infty \tag{3.4}$$

Consequently, if X is a non negative random variable, then

$$E(\bar{x}) = E(XZ) \tag{3.5}$$

3.6 Stochastic Processes

3.6.1 Definition

Given Ω F a stochastic process X_t is a collection $X_t : t \epsilon I$ of random variables. When I is an interval in R, we say that X_t is a continuous time stochastic process, or a subset of 1,2,... then we say that X_t is a discrete time stochastic process.

The state space of a stochastic process is defined as the set of all possible values a random variable can assume. Then, we can say that a stochastic process is a family of random variables that describes the evolution through time of some processes.

3.7 Concept of Brownian Motion

3.7.1 Definition

A stochastic process $X_t, t > 0$ is said to be a Brownian motion if

1. $X_0 = 0$

2. X_t is a continuous function of t

3. X has independent normally distributed increments:

$If 0 = t_0 < t_1 < t_2 < ... < t_n$ and $Y_1 = X(t_1) - X(t_0), Y_2 = X(t_2) - X(t_1), ..., Y_n =$

$X(t_n) - X(t_{n-1})$ then

$Y_1, Y_2, \ldots, Y_n$

are independent $E(Y_i) = 0$, $\forall\, i$

$Var(Y_i) = t_i - t_{i-1}$, $\forall\, i$

Brownian motion is the most fundamental continuous time stochastic process. It is both a martingale and a Gaussian process. It has continuous sample path, independent increments and a strong Markov property. Also, Brownian motion is the cornerstone of the diffusion theory and stochastic integration.

3.8 Covariance and Correlation

Covariance and Correlation describe how two variables are related.

1. Variables are positively related if they move in the same direction

2. Variables are inversely or negatively related if they move in opposite direction

Both covariance and correlation indicate whether variables are positively or negatively related. Correlation also tells the degree to which the variables tend to move together linearly.

3.8.1 Covariance

Covariance indicates how two variables are related linearly. A positive covariance means the variables are positively related and a negative covariance means the variables are inversely related. The formula for calculating covariance of sample data is given by:

$$cov(x, y) = \frac{\Sigma_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{n-1} \qquad (3.6)$$

x=the independent variable y=the

dependent variable n=number of data

points in the sample $\bar{x}$ =the mean of x $\bar{y}$

=the mean of y

3.8.2 Correlation

Correlation is another way to determine how two variables are related. In addition to stating whether they are negatively or positively related, correlation also tells you the degree to which they tend to move together. The correlation measurement, called correlation coefficient, will always take on a value between -1 and 1:

1. If the correlation coefficient is one, the variables have a perfect correlation

2. If correlation coefficient is zero, no relationship exists between them:they are uncorrelated

3. If correlation coefficient is -1, the variables are perfectly negatively correlated and move in opposition to each other.

3.9 Unstructured Covariance

1. Different variances

2. No assumption of exponential decay

Unstructured

$$\begin{bmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \sigma_{23} & \cdots & \sigma_{2n} \\ \sigma_{31} & \sigma_{32} & \sigma_3^2 & \cdots & \sigma_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sigma_{n1} & \sigma_{n2} & \sigma_{n3} & \cdots & \sigma_n^2 \end{bmatrix}$$

3.10 Toeplitz Covariance Structure

1. Teoplitz: All correlations at the same distance have the same correlation

2. No assumption of exponential decay

Toeplitz

$$\sigma^2 \begin{bmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{n-1} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{n-2} \\ \rho_2 & \rho_1 & 1 & \cdots & \rho_{n-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \rho_{n-1} & \rho_{n-2} & \rho_{n-3} & \cdots & 1 \end{bmatrix}$$

$$1$$

$$\rho^{n-1}$$

3.11 Autoregressive (1) Covariance Structure, AR(1)

AR(1): Correlations decline exponentially with distance. eg. $\rho_{12} = \rho$, $\rho_{13} = \rho^2$, $\rho_{14} = \rho^3$ etc.

Autoregressive, AR (1)

$$\sigma^2 \begin{pmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{n-1} \\ \rho & 1 & \rho & \cdots & \rho^{n-2} \\ \rho^2 & \rho & 1 & \cdots & \rho^{n-3} \\ \cdots & \cdots & \cdots & \ddots & \\ \rho^{n-1} & \rho^{n-2} & \rho^{n-3} & \cdots & 1 \end{pmatrix}$$

3.12 Law of Large Numbers

3.12.1 Definition

Let $X_1, X_2, \ldots$ be a sequence of independent, identically distributed random variables, each with expected value μ and variance σ^2. We define a sequence of averages,

$$\frac{X_1 + X_2 + ... + X_n}{n}, n = 1, 2, ... \qquad (3.7)$$

Then we say that the sequence of averages converges to μ almost surely as n → ∞.

3.13 Kolmogorov - Smirnov Two Sample Test

3.13.1 Definition

Analyzes two different data samples for independence.

3.13.2 Assumptions

Data must meet these two assumptions:

1. Observations $X_1, X_2, ..., X_m$ are random samples from a continuous population 1, where the X- values are mutually independent and identically distributed.Likewise the other observation Y.

2. The two samples are independent.

3.14 Hypothesis Testing

3.14.1 Null and Research Hypotheses

H_0: [$F(t)$ = $G(t)$,$foreveryt$]

H_1: [$F(t)$ 6= $G(t)$, for at least one value of t]

3.15 Test Statistic:

where m and n are sample size of X and Y respectively.

Find the empirical distribution functions, $F_m(t)$ and $G_n(t)$ for the samples of X and Y respectively.Combine and rank order of both set of values.

$$F_m(t) = \frac{number\ of\ observed\ X's \leq t}{m}$$

$$G_n(t) = \frac{number\ of\ observed\ Y's \leq t}{n}$$

3.16 Finding Divergence Between Two Distribution Functions

Find each absolute value divergence D between the empirical distribution functions:

D = | $F_m(t)$ - $G_n(t)$ |

Use the largest divergence D_{max} with the formula below to calculate the K - S test statistic Z:

Z = $D_{max}\sqrt{\frac{mn}{m+n}}$

3.17 Decision Making

3.17.1 Critical Value

At a given level of significance, α, the critical value for the K - S test can be found on the K - S table N = m + n

3.17.2 Decision

We fail to reject H_0 if the test statistic Z is less than the critical value.

3.18 The Model

Let

$$(\Omega, F, P), F = F_t; 0 < t < T \tag{3.8}$$

be a complete filtered probability space with the N-dimensional price process

$$S = S_t = (S_1(t), S_2(t), ..., S_N(t)); t \epsilon [0, 1] \tag{3.9}$$

The dynamics of these processes are determined by a system of stochastic differential equations:

$$dS_i(t) = \mu_i(t) S_i(t) dt + \sum_{j=1}^{N} \sigma_{ij} dw_j^{(1)}(t) \tag{3.10}$$

In the above equation the drift

$$\mu = \mu_t = (\mu_1(t), \mu_2(t), ..., \mu_N(t))'; t \epsilon [0, 1] \tag{3.11}$$

is an adapted, measurable N-dimensional process such that

$$\int_0^T \|\mu_u\|^2 du < \infty \tag{3.12}$$

where $\|.\|$ is the Euclidean norm. The process

$$w^1 = (w_1^1(t), w_2^1(t), ..., w_N^1(t)); t \epsilon [0, 1] \tag{3.13}$$

is an N-dimensional Brownian motion and $\sigma = (\sigma_{i,j} = 1, N)$ is a non singular matrix of constants. Let r be a constant deterministic interest rate. Suppose the initial prices $S_i(0); i = 1, ..., N$ are deterministic positive constants.Let $F^s = F_t^s; t < T$ be the augmented filtration generated by the price process S. It shall be assumed that:

1. only F^s-adapted processes are observable

2. Agents in this market do observe the Brownian motion and the drift process μ
3. The constant interest rate r, the initial price vector S_0 and the volatility matrix σ are known to all agents acting in the market.

We define the positive local martingale by the equation $Z = Z_t; t < T$ by the equation

$$dZ_t = -(\mu_t - r1)^T (\sigma^T)^{-1} Z_t dw_t^1, Z_0 = 1 \tag{3.14}$$

where 1 is an N-dimensional vector with all entries equal to 1. The above equations have a unique solution

$$Z_t = exp - \int_0^t (\mu_t - r1) * (\sigma^*)^{-1} dw_u^1 - \frac{1}{2} \int_0^t \|\theta^{-1}(\mu_u - r1)\|)^2 du \tag{3.15}$$

From Assumption 2.1 of Lakner (1998), Z is a martingale.

Next, we shall define a trading strategy for an agent acting in this market. Let $\pi_i(t)$ be the amount of wealth invested in the ith security at time t. A trading strategy

$$\pi = \pi_t = (\pi_1(t), \pi_2(t), ..., \pi_N(t)) \tag{3.16}$$

is an N-dimensional measurable, F^s-adapted process such that:

$$\int_0^T \|U_t\|^2 dt < \infty \tag{3.17}$$

We emphasize that a trading strategy is required to be F^s-adapted, thus investors indeed observe the security prices only, not the drift or the Brownian motion. Let X_t be the wealth at time t of an agent who follows the trading strategy. The initial wealth $X_0 = x_0$ is a deterministic constant. The process $X = X_t;\ t\epsilon[0,T]$ is assumed to evolve according to the dynamics

$$dX_t = \pi_t^* \mu_t dt + \pi_t^* \sigma dw_t^{(1)} + (X_t - \pi^* 1) r dt \quad (3.18)$$

Ito's rule implies that the discounted wealth $e^{-rt}X_t$ has the form

$$d(e_{-rt}X_t) = e_{-rt}\pi_* \sigma dw_t \quad (3.19)$$

where

$$w_t = w_t^1 + \int_0^t \sigma^{-1}(\mu_u - r1)du \quad (3.20)$$

By Girsanov Theorem and Assumption 2.1 of Lakner (1998), the N-dimensional process

$$\bar{W} = \bar{W}_{\approx} = (\bar{W}_1(t),...,\bar{W}_N(t)); 0 < t < T \quad (3.21)$$

is a Brownian motion under the probability measure $\bar{P}$ where

$$\frac{d\bar{\mathbb{P}}}{d\mathbb{P}} = Z_T \quad (3.22)$$

We denote by $\bar{E}$ the expectation operator corresponding to the measure $\bar{P}$. A trading strategy is called admissible if $X_i > 0$, a.s. $t\epsilon[0,T]$.

A function U: $[0,\infty)$ 7−→ R ∪ $\{-\infty\}$ is called a utility function if it is continuous, strictly increasing, strictly concave on it domain, continuously differentiable on $(0,\infty)$ with derivative function $U^0(.)$ satisfying the relation

$$lim U^0(0) = 0, x \to \infty \quad (3.23)$$

Our optimization problem is to maximize the expected utility from terminal wealth, i.e., $maxE[U(X_T)]$ over all admissible trading strategies. We define the N-dimensional return process:

$$R = R_t = (R_1(t), R_2(t), ..., R_N(t)); t\epsilon[0,T] \quad (3.24)$$

by

$$dS_i(t) = S_i(t)dR_i(t), i = 1,...,N \tag{3.25}$$

So we have the following decompositions for the return process

$$dR_t = \mu_t dt + \sigma dw_t^{(1)} \tag{3.26}$$

and

$$dR_t = r1dt + \sigma d\bar{\mathbb{W}}_t^{(1)} \tag{3.27}$$

Equation (3.24) and (3.25) imply that S, R and $\bar{W}$ each generate the same filtration. Thus, F^s is continuous (Karatzas & Shreve, 1988), corollary 2.7.8).

$$\zeta = \zeta_t; t\epsilon[0,T] \tag{3.28}$$

be the optional projection of the P-martingale Z to F^s, so that

$$\zeta_t = E[Z_t|F]a.s., t\epsilon[0,T] \tag{3.29}$$

We note that ς is a martingale with respect to (P, F^s) and for every F_t^s measurable random variable W with $0 < t < u < T$

$$\bar{E}V = E\varsigma_t V \tag{3.30}$$

$$\bar{\mathbb{E}}[Y|F_t^s] = \frac{1}{\zeta_t}E[Z_u Y|F_t^s] \tag{3.31}$$

and

$$\bar{\mathbb{E}}[W|F_t^s] = \frac{1}{\zeta_t}E[Z_u Y|\zeta_t^s] \tag{3.32}$$

The last identity implies that $\frac{1}{\zeta}$ is a $(\bar{P}, F^s)$- martingale since F^s is generated by $\bar{W}$, so $\frac{1}{\zeta}$ and also ζ, must be continuous.Let the function $I : (0,\infty) 7\to [0.\infty)$ be the pseudo inverse function of the strictly decreasing derivative of the utility function:

$$I(y) = infx > 0 : U^0 < y \tag{3.33}$$

Equation(3.32) defined function I actually becomes the inverse function of U_0 if $limU^0(x) = \infty.\ x \to 0$

However, we did not make this assumption. We recall the following theorem from (Lakner, 1998).

3.18.1 Theorem

Suppose that for every constant $x\epsilon(0,\infty)$ $\mathbb{E}[I(x\zeta_T)] < \infty$ then the optimal level of terminal wealth is

$$X_T = I(ye^{-rT}\zeta_T)] = x_0 \tag{3.34}$$

The optimal wealth process X and the trading strategy π is implicitly determined by:

$$e^{-rt}X_t = \bar{\mathbb{E}}[e^{-rt}I(ye^{-rt}\varsigma)_T|F_t^s] = x_0 + \int_0^T e^{-rt} \tag{3.35}$$

3.19 Explicit Representation of the Optimal Terminal Wealth Level

We assume that the drift process μ_t for the various securities follows the following process:

$$d\mu_t = \alpha(\delta - \mu_t)dt + \beta \ \ d\omega_t^{(2)} \tag{3.36}$$

where $\omega_t^{(2)}$ is an N-dimensional Brownian motion with respect to (F,P), independent of $\omega_t^{(1)}$ under P,α and β are known $N * N$ matrices of real numbers, and δ is a known N-dimensional vector of real numbers. We shall assume that β is invertible and that μ_0 follows an N-dimensional normal distribution with mean vector m_0 and covariance matrix γ_0. The vector m_0 and the matrix γ_0 are assumed to be known to all agents in the market. We note that if α is a diagonal matrix with positive entries in the diagonal, then μ_t will be an N-dimensional Orstein-Uhlenbeck process with mean-reverting drift. We shall also assume that $tr(\gamma_0)$ and k β k are "small".

Now we are ready to state the main theorem of the paper:

3.19.1 Theorem

Suppose that U is twice continuously differentiable on $(0,\infty)$ and $I(x) < K_2(1+ x^{-5})$ $-I^0(x) < K_2(1 + x^{-2})$ for some $K_2 > 0$. Then the optimal trading strategy is

$$\pi_t = H_t \frac{1}{\zeta} t E[I\prime(ye^{-rT}\zeta_T)\zeta_T^2 - \gamma(t)(\phi)^*(t))^{-1} \int_t^T \phi^*(u)(\sigma^*)^{-1} dW_u - m_t + r1 | F_t^s] \tag{3.37}$$

where $H(t) = e^{r(t-2T)}y(\sigma)\sigma_*)^{-1}$. The constant y is uniquely determined by 2.22 of Lakner (1998) and ζ and m in 3.5 and 4.9 of the same paper. We will consider a fund manager with a logarithmic utility function say: $U(x) = log(x)$. Considering the case of full information, we will obtain the feedback form of the optimal trading strategy to be

$$(\sigma\sigma^T)^{-1}(\mu_t - r1)X_t \tag{3.38}$$

which is proven by example 4.3 of (Ocone & Karatzas, 1990).

CHAPTER 4

ANALYSIS

4.1 Introduction

This chapter presents, discusses and interprets results obtained in this chapter. Furthermore, it gives a practical illustration of the methodology discussed in chapter three of the study. Computations involved in this study will be outlined numerically and graphically where needed. The study looks at simulation study scenarios and fitting real data. The scenario study considers five (5) cases and the real data looks at three companies that deal in AAPL, BAX and TNET from 1^{st} January, 2008 to 31^{st} December, 2012.

4.2 Scenario Study

This section looks at the optimal trading strategy in five(5) scenarios:

1. Uncorrelated stocks: each following different distribution

2. Uncorrelated stocks: All Normal,or all Logistic or all Weibull

3. Correlated stocks: NWW with stocks 2 and 3 being negatively correlated (Toeplitz and AR(1))

4. Correlated stocks: All Normal, Logistic or Weibull distributed stocks
5. Correlated stocks: Positive Toeplitz and positive AR(1), NLW combination

All the scenarios were performed at different level of observations (500,1000,2000,10000). The law of large numbers was resorted to and that only output under the 10000 was considered. The three distributions Normal(N), Logistic(L) and Weibull(W) were considered in the study. With each of them, we looked at the three covariance structure (AR(1) and Toeplitz) and considered the unstructured covariance. Investment made in three different stocks at the same time is associated with minimum risk as against putting all in one stock. We therefore looked at the best way to invest to obtain the optimum returns employing different scenarios.

For the study, a constant risk-free interest rate of 0.23 (23%) was assumed and also, the order of the combinations considered did not matter. The possible combination of the distributions are discussed.

The first scenario was uncorrelated stocks with each following one of the three distribution. Whereas stock one had Normal distribution, the stock two and three had Logistic and Weibull distributions respectively.

Scenario two had all the stocks being Normal, Logistic or Weibull. This meant that all three stocks might follow the Normal, Logistic or Weibull distributions. The third scenario looked at correlated stocks. Thus, stocks one and two were correlated, stocks one and three were correlated, with stocks two and three demonstrating the same trend.

In the case of scenario four, two stocks out of the three exhibited the same distribution, however stocks two and three had negative relation, with the rest demonstrating positive relation.

Under scenario five, each stock presented different distribution. Thus, stock one had Normal distribution whereas stock two and three demonstrated Logistic and Weibull distributions respectively.

As shown in chapter 3 of the work, the optimal trading strategy formula is giving by:

$\pi_t = (\sigma\sigma^T)^{-1}(\mu_t - r1)X_t$ where

1. π_t represents the investment strategy

2. σ represents the volatility matrix of N securities

3. r represents the market risk - free rate

4. X_t represents the wealth at time t

5. μ_t represents the N dimensional vector of drift of the various securities

6. 1 represents N-dimensional vector of entries 1

7. T represents transposition

We applied this model to the simulation study and observed the dynamics of the trading strategy with:

1. X_t=1000 (Ghana cedis)

2. r_t=0.23

4.3 Uncorrelated Stocks

4.3.1 Scenario 1

Table 1: Each following a different distribution, NLW (Toeplitz and AR(1))

Stocks	500	1000	2000	10000

1	464.7592	461.6985	461.6141	472.4474
2	122.1314	126.6158	126.9035	127.5162
3	413.1094	411.4824	411.4824	400.0364
Stocks	500	1000	2000	10000
1	257.5717	248.2272	258.0490	256.7341
2	254.7679	266.7788	258.7846	261.3797
3	487.6604	484.9939	483.1664	481.8863

Table 1 presents the average wealth allocation of uncorrelated stock returns of a 2^{nd}-tier fund manager with a monthly wealth of 1000 (in currency) with a fixed interest rate of 0.23 (23%) under the combinations of the distributions. The table depicted the trading strategy for all the stocks under that possible combinations. The simulation was performed for N different observations (N=500,1000,2000 and 10000) because of the law of large numbers. As the observations got larger, the more it approached the real value. Here, the fund manager invested more in stock one (472.4474), followed by the third stock (400.0364) and the second with 400.0364 in that order under the Toeplitz structure. However, under the AR(1), the fund manager allocated 481.8863, 261.3797 and 256.7341 to stock three, two and one respectively.

4.3.2 Scenario 2

Table 2: All Normal,all Logistic or all Weibull stocks (AR(1))

Stocks	500	1000	2000	10000
1	344.5437	356.2183	339.6972	344.6787
2	323.9374	319.3188	326.9432	325.5433
3	331.5189	324.4629	333.3597	329.7780
Stocks	500	1000	2000	10000
1	242.0926	237.7636	249.7979	238.7860
2	299.8119	304.7037	299.4258	301.7887
3	458.0955	457.5326	460.7763	459.4253
Stocks	500	1000	2000	10000
1	244.2758	230.9347	241.0286	242.4579
2	300.9561	302.9600	300.0655	300.0655

3	454.7682	466.1053	458.9059	456.8547

Table 2 showed uncorrelated stocks under the AR(1) structure when all stocks were from the Normal, Logistics or Weibull distributions respectively. The fund manager allocated more into stock one (344.6787), followed by stock three (325.5433) and then stock two (325.5433) under the Normal distribution. However, under both Logistic and Weibull distribution, more funds were allocated to stock three followed by stock two and then stock one.

4.4 Correlated Stocks

4.4.1 Scenario 3

Table 3: All Normal or All Logistic or All Weibull respectively (Negative AR(1))

Stocks	500	1000	2000	10000
1	314.6433	321.6259	315.8193	305.7662
2	446.8821	434.3632	436.9575	450.9393
3	238.4747	244.0109	247.2232	243.2945
Stocks	500	1000	2000	10000
1	339.6569	336.1094	330.8138	331.1678
2	183.8857	163.2464	178.3383	180.4170
3	476.4574	500.6443	490.8479	488.4152
Stocks	500	1000	2000	10000
1	122.3566	118.3025	120.9128	120.4130
2	297.2838	287.3646	293.9658	290.1142
3	580.3596	594.3330	585.1213	589.4728

It can be observed from Table 3 that when all the stocks are Normal under a correlated AR(1), the fund manager invested more in stock 2 (450.7662), followed by stock 1 (305.7662) and then stock three 243.2945. Under the same preamble but with stocks from the Logistic and Normal distribution, the order was stock 3 (488.4152), followed by stock 1 (331.1678) and then stock 2 (180.4170), and stock 3 (589.4728),stock 2 (290.1142) and then stock 1 (120.4130) respectively.

4.4.2 Scenario 4

Table 4a: NWW with stocks 2 and 3 being negatively correlated (Toeplitz and AR(1))

Stocks	500	1000	2000	10000
1	230.1817	246.4945	246.6373	245.4066
2	306.2204	281.6433	289.7680	290.7323
3	463.5979	471.8623	463.5947	463.8610
Stocks	500	1000	2000	10000
1	559.6553	565.5496	566.6194	571.1754
2	246.4831	241.8581	243.5183	238.4312
3	193.8616	192.5923	189.8623	190.39935

As demonstrated in Table 4a, when two stocks were negatively correlated under NWW, Toeplitz structure allowed the fund manager to invest 24.5% in the Normal (stock 1), followed by 29.1% stock 2 (Weibull) and then 46.4% in stock 3 (Weibull) in that order. However, under the AR(1) for the same preamble, the fund manager invested 57.1% of the investment capital in the Normal (stock 1), followed by stock 2 (238.4312) and then stock 3 (190.3935).

Table 4b: NWW with stocks 1 and 2, 1 and 3 being negatively correlated (Toeplitz and AR(1))

Stocks	500	1000	2000	10000
1	394.0344	391.2662	396.7341	396.4353
2	144.9887	148.6085	147.4586	147.7462
3	460.9770	460.1253	455.8072	455.8185
Stocks	500	1000	2000	10000
1	428.3639	425.0678	432.1498	420.0365
2	107.0797	100.7806	107.6024	106.2649
3	464.5564	474.1516	460.2477	473.6986

As demonstrated in Table 4b, when stocks were NWW, with ρ_{12}= ρ_{13}= negative correlation, Toeplitz structure allowed the fund manager to invest 39.6 % in the Normal stocks and 60% in the Weibulls (14.8% in stock 2 and 45.6% in stock 3).

However, under the AR(1) for the same preamble,the fund manager invested 42% of the wealth in stock 1 (Normal), followed by 10.6% in stock 2 (Weibull)then 47.4 in stock 3 (Weibull).

4.4.3 Scenario 5

Table 5: Positive Toeplitz and Positive AR(1), NLW combination

Stocks	500	1000	2000	10000
1	526.1456	527.2522	521.1962	524.4594
2	161.1236	159.0362	160.7839	160.9833
3	312.7308	313.7116	318.0199	314.5574
Stocks	500	1000	2000	10000
1	569.1710	577.9316	574.7257	577.2156
2	188.6386	186.7201	190.1530	190.9417
3	242.1903	235.3482	235.1213	231.8426
Stocks	500	1000	2000	10000
1	268.4619	267.2001	259.5498	260.2275
2	410.7273	401.1949	416.2475	412.1607
3	320.8108	331.6051	324.2027	327.6118

Table 5 also showed Normal, Logistic and Weibull (NLW) combination of correlated stocks under Toeplitz and AR(1) respectively with the third or last table being an unstructured Normall, Weibull and Weibull stocks. The table presented a similar order in investing in these stocks though they were of different structures. The first two tables depicted a positive correlation amongst the stocks with $\rho_{12} = 0.3449474$, $\rho_{13} = 0.6435245$, $\rho_{23} = 0.8217723$) under the Toeplitz and

($\rho_{12} = 0.1678110$, $\rho_{13} = 0.1514601$, $\rho_{23} = 0.9574626$) under AR(1) structure. Much of the investment capital (52.4%) was put in the Normal stock with 16.1% in the Logistic stock and 31.5% in the Weibull stock under the Toeplitz structure.

4.5 conclusion on the simulation study

Critically examining the tables showed that when the distribution of the stocks were a combination of the Normal, Weibull and Weibull distributions (NWW), the optimal trading formula performed 500, 1000, 2000 and 10000 times and the averages found for each of the stocks, more is invested in stock 3 (one of the Weibull distribution with 455.8185), followed by stock 1 (396.4353) and the stock 2 (147.7462) under the Toeplitz structure. It prompts a fund manager as to what should constitute his portfolio.

4.6 Real Data Fitting

This section seeks to fit real market data to the model and analyze how it helps the second-tier fund manager to make informed decision about his investments. The data comprise three (3) stocks sourced from yahoofinance (www.yahoofinance.com) for a period of 5 years from 1^{st} January, 2008 to 31^{st} December, 2012. The stocks are companies that operates in AAPL, BAX and TNET.

4.6.1 Statement of Hypothesis

The following hypothesis was used as the criterion of selection for each of the distributions:

H_0: The statistical distribution provides an accurate statistical model for the data

H_1: The statistical distribution does not provide an accurate statistical model for the data

4.7 Maximum Likelihood Estimates

This part sought to come out with a statistical distribution that best fit the stock prices collected. It is pertinent to define the parameters of each of the distribution.Since we are only attempting to liken the data to a known statistical distribution, we can only estimate such parameters. We literally defined the distribution upon estimating those parameters. We used the maximum likelihood estimator to define these parameters in this study. The maximum likelihood estimates of the selected distribution were organized in table D. Curve fitting software and R software were relied upon for the curve fitting procedure and the maximum likelihood estimates.

Table 6: Fitted Parametric Distribution of AAPL

Distribution	Scale	Location	Shape
Normal	σ=172.24	μ=388.34	
Logistic	σ=94.962	μ=388.34	
Weibull	β=446.38		α=1.8765

From EasyFit

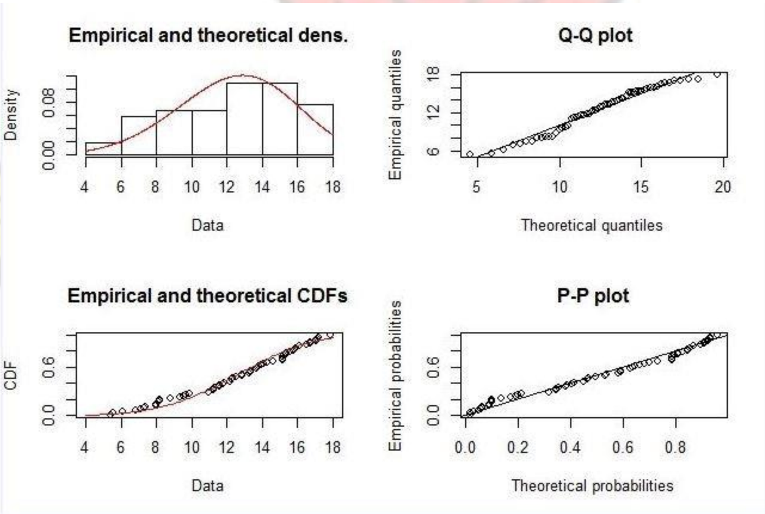

Figure 4.1: Q-Q PLOT (TNET, WEIBULL)

Figure 4.1 showed the PDF, CDF, Q-Q plot and P-P plot of TNET obtained using the R software. In addition to the Q-Q plot, Kolmogorov- Smirnov and Anderson- Darling tests were used for the analysis in this section. It could be seen from figure 4.1 that comparatively, Weibull best describes TNET prices as most of the points fell on or were closer to the reference line. The Normal and Logistic had greater deviations in the body as shown in the Appendix than the Weibull. The Q-Q plots for the Normal and Logistic are provided in the Appendix.

Weibull [#3]					
Kolmogorov-Smirnov					
Sample Size Statistic P-Value Rank	60 0.08655 0.727 1				
α	0.2	0.1	0.05	0.02	0.01
Critical Value	0.13573	0.15511	0.17231	0.19267	0.20673
Reject?	No	No	No	No	No
Anderson-Darling					
Sample Size Statistic Rank	60 0.66693 2				
α	0.2	0.1	0.05	0.02	0.01
Critical Value	1.3749	1.9286	2.5018	3.2892	3.9074
Reject?	No	No	No	No	No

Figure 4.2: GOF Weibull

From Figure 4.2, the statistical distributions were subjected to the various algebraic distributions. The rejection or acceptance decision was determined at various levels of significance. For the Kolmogorov-Smirnov and Anderson-Darling tests the null hypothesis that the data comes from the said distribution is rejected if the test statistic obtained was greater than the critical value obtained at the specified level of significance. For the Kolmogorov-Smirnov test, the rejection or otherwise of the null hypothesis could also be decided by considering the Pvalue. If the P-value is lesser than the level of significance, the null hypothesis is rejected. Otherwise we fail to reject it. Based on the Kolmogorov-Smirnov test, all the distributions (Normal, Logistic, Weibull) passed the test at a significance level of 0.01, that is, a 99 percent confidence level. They had test statistic values of 0.10287, 0.12495 and 0.08655 respectively. These values were all lesser than the critical value of 0.13573 obtained at a 99 percent confidence level. The AndersonDarling test recorded a similar results as obtained by the Kolmogorov-Smirnov test. Considering the two tests, we could deduce that Weibull distribution best fit the TNET prices as it is the one with the least test statistic. In a nutshell, it could be concluded that TNET follows the Weibull distribution, AAPL follows the Normal distribution whereas BAX follows the Weibull distribution. Coming out with these distributions would contribute immensely to the investment decision of the 2nd-tier fund manager.

4.8 Reduction of Risk

Portfolios had some advantages over a single security. The return of one security might tend to move in the same direction as the return of another security, probably in the opposite direction of the third security's return. Because of this, for a given expected return, when securities are grouped into a portfolio, the variance of that return can be reduced. The joint tendencies between the returns can be measured by covariances.

4.8.1 Correlation Matrix

Correlation tells the degree to which variables move together. Both correlation and covariance showed whether variables are positively or negatively correlated. Table 7: correlation matrix

Stocks	AAPL	BAX	TNET
AAPL	1.00000000	0.06516486	0.04304308
BAX	0.06516486	1.00000000	-0.01782627
TNET	0.04304308	-0.01782627	1.000000

Table 7 showed that there is a weak positive correlation between AAPL and BAX, AAPL and TNET. However there was a weak negative relation between BAX and TNET. This automatically reduced the portfolio risk.

Table 8: Variance - Covariance Matrix

Stocks	AAPL	BAX	TNET
AAPL	0.068227754	0.0014520680	0.0011011020
BAX	0.001452068	0.0072775621	-0.0001489351

TNET	0.001101102	-0.0001489351	0.0095915190

Table 8 indicated that AAPL and BAX, AAPL and TNET showed a positive covariance, indicating the degree to which those stocks move together. But that of BAX and TNET indicated a negative correlation. This showed the degree to which those stocks move inversely.

4.9 Optimal Trading Strategy (OTS) Of The Real Data

It has been noticed from the previous section that, the stocks AAPL, BAX and TNET followed Normal, Weibull and Weibull respectively and from the correlation matrix, BAX and TNET were the only stocks that showed a weak negative relation. The other combinations were all positive relations. We applied the optimal trading strategy model to the data under the distributions obtained with:

1. X_t=1000(Ghana cedis)

2. r_t=0.23 as fixed risk free rate and observe the dynamics.

Table 9: OTS of Normal, Weibull, Weibull correlated stock prices

Stocks	Amount to invest
AAPL(Normal)	24.80
BAX(Weibull)	221.41
TNET(Weibull)	753.79

It can be observed from the Table 9 that, the OTS of the said combination of distribution (Normal, Weibull,Weibull (NWW)) for AAPL, BAX and TNET are 24.80,221.41 and 753.79 respectively. A weak negative correlation was observed between stocks BAX and TNET. Similar trend was observed in the simulation study in scenario 4 as it also reported a negative correlation between stocks 2 and

3.

Additionally, the Appendices showed several combinations of the distributions under the covariance structures that yielded optimal returns. A weak negative correlation with a Toeplitz structure with NWW combination gave similar observation.It again showed the GOF tests for each of the stocks and their corresponding figures.

CHAPTER 5

CONCLUSION

The 2^{nd}-tier fund manager can not invests more than what has been giving to him. But there are millions of ways he can invests this money. The study therefore opted to investigate and compare the optimal trading strategy formula under three covariance structures and observed the dynamics. In order to find solution to the problem statement of this study, investment in stocks with fixed interest rate for each of the stock evaluated. Short selling and borrowing were not allowed. The results brought to fore a weak negative relation between two of the securities, and that two stocks followed the same distribution. The fund manager had to invest 2.5% (Gh¢24.79783) of the investment capital in the Normal stock (AAPL), 22% (Gh¢221.41690) in the Weibull (BAX) and 75.4% (Gh¢753.78527) in the other Weibull (TNET) to obtain a higher return. The simulation study depicted a combination where one of the securities followed Normal and the other two followed the Weibull distribution. Therefore, fund managers could rely on information obtained from these structures to make informed decisions to obtain optimal returns on their investments.

5.1 Recommendations

According to George E.P.Box "Essentially all models are wrong but some are useful". The optimal trading strategy for both the market data and the simulated data assumed fixed interest rate for the N-securities. In spite of this, the optimal trading strategy model can be recommended for the 2^{nd}-tier fund managers to help in making informed decision in their asset allocation process.

5.2 Further Studies

Further study can be carried on Optimal trading strategy of a 2^{nd}-tier fund manager, the case of power utility function, where interest rate will not be held constant. Additionally, "An Application of stochastic differential equations in determining the optimal trading strategy of a 2^{nd}-tier fund manager in Ghana using models like HESTON and GARCH to solve problem of constant volatility" could be investigated in future studies.

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APPENDIX A

5.3 Optimal Trading Strategy Tables(Simulation Study)

5.3.1 Uncorrelated Stocks

5.3.2 Scenario 2

Table 5.1: Logistic and Weibull respectively

Stocks	500	1000	2000	10000
1	249.0716	244.7568	244.7181	245.9049
2	377.1707	394.4207	383.4892	391.5335
3	373.7577	360.8225	371.7927	362.5615
Stocks	500	1000	2000	10000
1	445.3976	446.1637	448.5174	446.8120
2	285.2196	280.9476	280.2246	282.4458
3	269.3828	272.8887	271.2580	270.7421

5.4 CORRELATED STOCKS

5.4.1 Scenario 3

Table 5.2: All Normal or All Logistic or All Weibull(Weak posiive AR(1))

Stocks	500	1000	2000	10000
1	186.5205	184.5471	186.9436	186.5304
2	364.1931	367.8507	366.7510	366.9686
3	449.2864	447.6022	446.3055	446.5010
Stocks	500	1000	2000	10000
1	152.9246	141.9023	137.8841	142.0636
2	236.6783	218.3182	225.1401	221.8471
3	610.3971	639.7795	636.9758	636.0892
Stocks	500	1000	2000	10000
1	330.3305	323.1419	316.6905	316.7193
2	430.8069	437.9583	440.7302	439.2070
3	238.8626	238.8998	242.5793	244.0737

Table 5.3: All Normal or All Logistic or All Weibull(Weak posiive Toeplitz)

Stocks	500	1000	2000	10000
1	120.1922	129.7878	127.2657	123.6519
2	377.0481	372.0508	379..9179	384.8767
3	502.7598	498.1614	492.8164	491.4714
Stocks	500	1000	2000	10000
1	108.5830	120.8015	118.0022	118.6438
2	225.4399	209.1891	212.1736	215.7235
3	665.9771	670.0094	669.8242	665.6327
Stocks	500	1000	2000	10000
1	312.7548	311.2577	319.4403	313.4380
2	442.1416	443.5732	438.3553	441.9566
3	245.1036	245.1691	242.2044	244.6054

Table 5.4: All Normal or All Logistic or All Weibull(Weak negative Toeplitz)

Stocks	500	1000	2000	10000

1	657.5969	663.5239	661.9698	658.4407
2	210.8263	210.1143	209.5703	211.6835
3	131.5769	126.3618	128.4599	129.8758
Stocks	500	1000	2000	10000
1	325.8165	317.6317	316.6160	317.3202
2	437.2539	438.7504	438.5382	439.8797
3	236.9296	243.6179	244.8458	242.8001
Stocks	500	1000	2000	10000
1	323.2473	318.4028	319.4056	317.2401
2	439.8948	441.1827	438.7100	440.8677
3	236.8579	240.4145	241.8844	241.8923

5.4.2 Scenario 5

Table 5.5: Positive Toeplitz(NLL,NNL,WWL respectively)

Stocks	500	1000	2000	10000
1	682.5132	689.1732	682.4062	690.2861
2	151.6531	150.3268	153.3804	149.2881
3	165.8337	160.4999	164.2133	160.0658
Stocks	500	1000	2000	10000
1	496.08074	483.77344	492.377338	490.40789
2	411.98212	423.73990	421.43433	421.67046
3	91.93715	92.48667	86.19228	87.92165
Stocks	500	1000	2000	10000
1	738.0351	737.26355	730.33787	731.24753
2	93.4153	91.86788	95.28131	96.52226
3	168.5496	170.868858	174.38081	172.23022

Table 5.6: Negative Toeplitz(NLL,NNL,WWL)

Stocks	500	1000	2000	10000
1	764.4113	756.9383	760.1066	765.9002
2	100.0456	105.2846	102.9500	99.1908
3	135.5432	137.7771	136.9433	134.9090
Stocks	500	1000	2000	10000

1	546.8097	547.4252	547.3183	551.8915
2	217.9106	218.6690	217.5351	216.1274
3	235.2797	233.9058	235.1466	231.9811
Stocks	500	1000	2000	10000
1	464.2017	466.9373	458.5343	459.6018
2	362.7019	357.1919	353.8689	354.1516
3	173.0964	175.8708	187.5968	186.2466

5.4.3 Scenario 6

Table 5.7: Unstructured(NNN,WWW,LLL,NWL respectively)

Stocks	500	1000	2000	10000
1	259.4607	249.5902	254.3738	261.4602
2	455.7246	470.3485	459.6763	462.1433
3	284.8147	280.0613	285.9500	276.3965
Stocks	500	1000	2000	10000
1	185.7912	190.8554	193.1418	190.0761
2	534.4179	517.4821	525.5742	525.4749
3	279.7909	291.6626	281.2840	284.4490
Stocks	500	1000	2000	10000
1	191.0828	179.4065	183.7003	187.2113
2	405.9772	411.5794	418.5778	412.5160
3	402.9400	409.0141	397.7219	400.2727
Stocks	500	1000	2000	10000
1	445.9088	440.95865	439.06628	434.92270
2	478.7712	484.12876	484.85704	489.08457
3	75.3200	74.91259	76.07667	75.99273

APPENDIX B

Table 5.8: Fitted Parametric Distribution of BAX

Distribution	Scale	Location	Shape
Normal	α=3.4289	μ=12.42	

Logistic	α=1.8904	μ=12.42	
Weibull	β=13.668		α=3.7614

Table 5.9: Fitted Parametric Distribution of TNET

Distribution	Scale	Location	Shape
Normal	σ=3.4289	μ=12.42	
Logistic	σ=1.8904	μ=12.42	
Weibull	β=13.668		α=3.7614

APPENDIX C

Normal [#2]					
Kolmogorov-Smirnov					
Sample Size Statistic P-Value Rank	60 0.10287 0.51616 3				
α	0.2	0.1	0.05	0.02	0.01
Critical Value	0.13573	0.15511	0.17231	0.19267	0.20673
Reject?	No	No	No	No	No
Anderson-Darling					
Sample Size Statistic Rank	60 0.84979 3				
α	0.2	0.1	0.05	0.02	0.01
Critical Value	1.3749	1.9286	2.5018	3.2892	3.9074
Reject?	No	No	No	No	No

Figure 5.1: GOODNESS OF FIT TABLE(TNET, NORMAL)

Logistic [#1]					
Kolmogorov-Smirnov					
Sample Size Statistic P-Value Rank	60 0.12495 0.28191 4				
α	0.2	0.1	0.05	0.02	0.01
Critical Value	0.13573	0.15511	0.17231	0.19267	0.20673
Reject?	No	No	No	No	No
Anderson-Darling					
Sample Size Statistic Rank	60 1.2366 4				
α	0.2	0.1	0.05	0.02	0.01
Critical Value	1.3749	1.9286	2.5018	3.2892	3.9074
Reject?	No	No	No	No	No

Figure 5.2: GOODNESS OF FIT TABLE(TNET, LOGISTICS)

Normal [#2]					
Kolmogorov-Smirnov					
Sample Size Statistic P-Value Rank	60 0.16634 0.06411 3				
α	0.2	0.1	0.05	0.02	0.01
Critical Value	0.13573	0.15511	0.17231	0.19267	0.20673
Reject?	Yes	Yes	No	No	No
Anderson-Darling					
Sample Size Statistic Rank	60 1.647 3				
α	0.2	0.1	0.05	0.02	0.01
Critical Value	1.3749	1.9286	2.5018	3.2892	3.9074
Reject?	Yes	No	No	No	No

Figure 5.3: GOODNESS OF FIT TABLE(BAX, NORMAL)

Weibull [#3]					
Kolmogorov-Smirnov					
Sample Size Statistic P-Value Rank	60 0.13702 0.19143 2				
α	0.2	0.1	0.05	0.02	0.01
Critical Value	0.13573	0.15511	0.17231	0.19267	0.20673
Reject?	Yes	No	No	No	No
Anderson-Darling					
Sample Size Statistic Rank	60 1.2008 1				
α	0.2	0.1	0.05	0.02	0.01
Critical Value	1.3749	1.9286	2.5018	3.2892	3.9074
Reject?	No	No	No	No	No

Figure 5.4: GOODNESS OF FIT TABLE(BAX, WEIBULL)

Normal [#2]					
Kolmogorov-Smirnov					
Sample Size Statistic P-Value Rank	60 0.13125 0.23145 3				
α	0.2	0.1	0.05	0.02	0.01
Critical Value	0.13573	0.15511	0.17231	0.19267	0.20673
Reject?	No	No	No	No	No
Anderson-Darling					
Sample Size Statistic Rank	60 1.1951 2				
α	0.2	0.1	0.05	0.02	0.01
Critical Value	1.3749	1.9286	2.5018	3.2892	3.9074
Reject?	No	No	No	No	No

Figure 5.5: GOODNESS OF FIT TABLE(AAPL, NORMAL)

Weibull [#3]					
Kolmogorov-Smirnov					
Sample Size Statistic P-Value Rank	60 0.12663 0.26773 2				
α	0.2	0.1	0.05	0.02	0.01
Critical Value	0.13573	0.15511	0.17231	0.19267	0.20673
Reject?	No	No	No	No	No
Anderson-Darling					
Sample Size Statistic Rank	60 1.9833 4				
α	0.2	0.1	0.05	0.02	0.01
Critical Value	1.3749	1.9286	2.5018	3.2892	3.9074
Reject?	Yes	Yes	No	No	No

Figure 5.6: GOODNESS OF FIT TABLE(APPL WEIBULL)

APPENDIX D

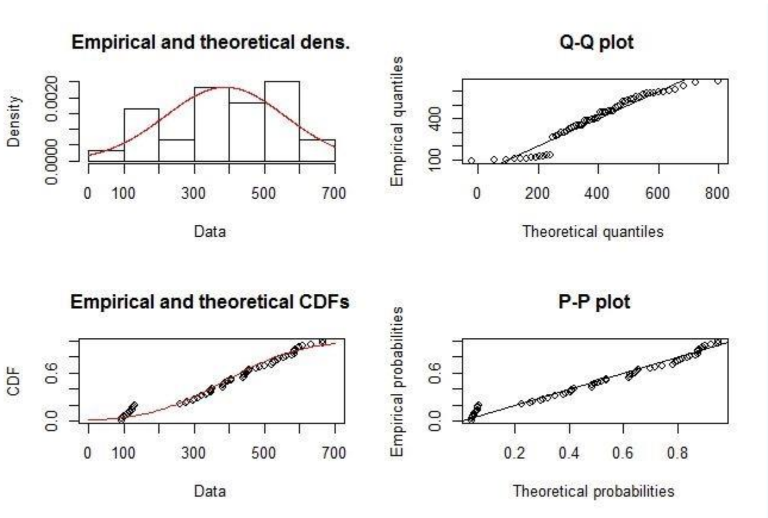

Figure 5.7: Q-Q PLOT (APPL, NORMAL)

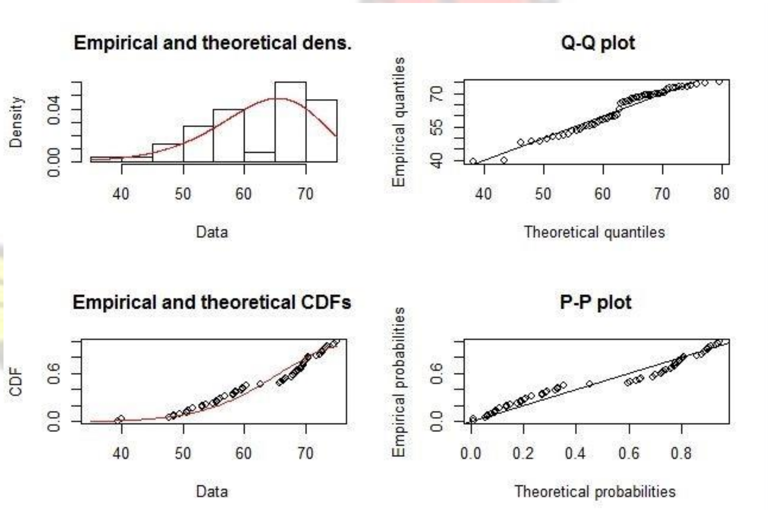

Figure 5.8: Q-Q PLOT (BAX, WEIBULL)

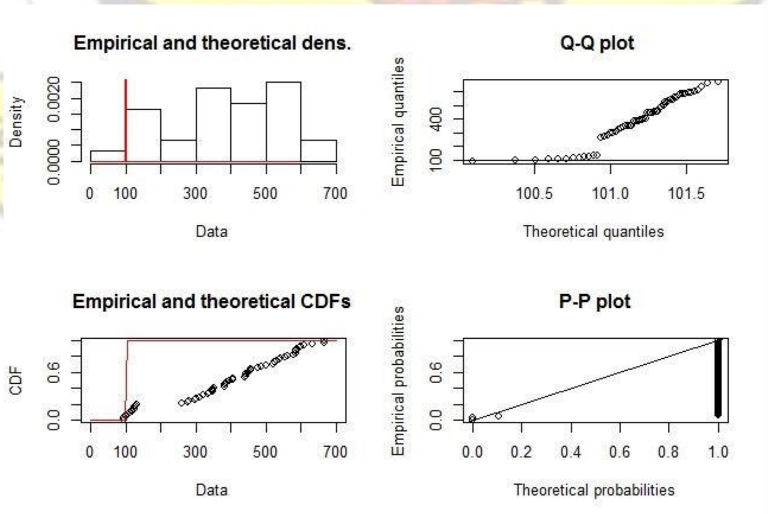

Figure 5.9: Q-Q PLOT(AAPL,WEIBULL)

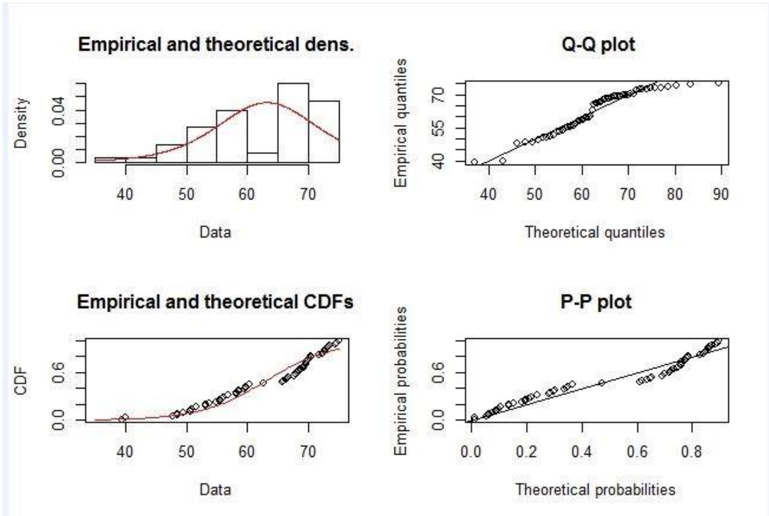

Figure 5.10: Q-Q PLOT(BAX, LOGISTICS)

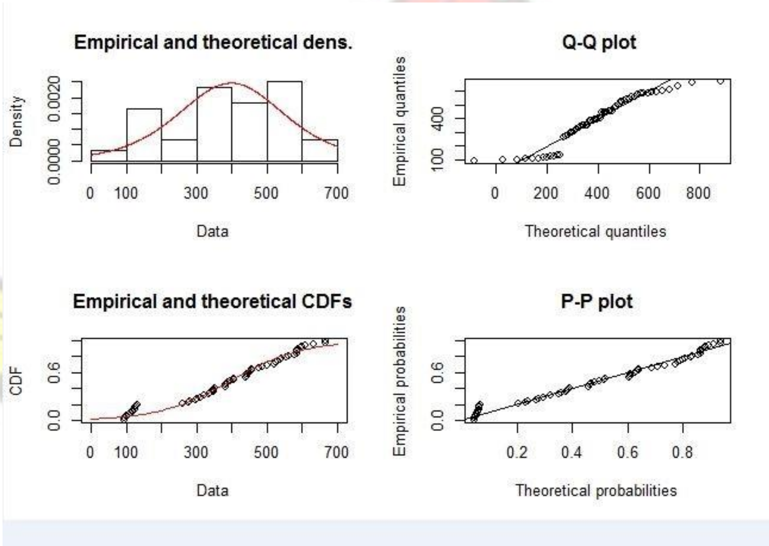

Figure 5.11: Q-Q PLOT(APPL, LOGISTIC)

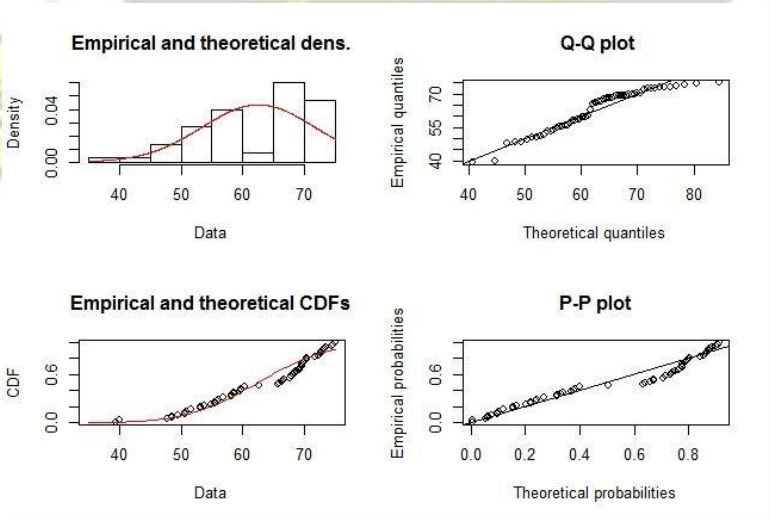

Figure 5.12: Q-Q PLOT(BAX, NORMAL)

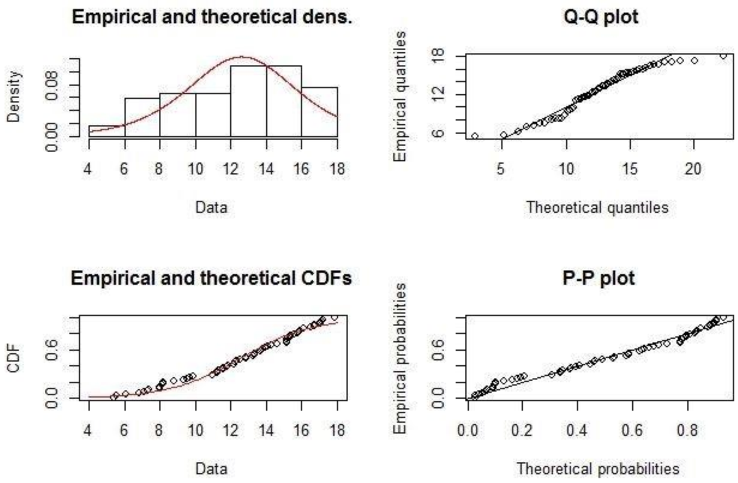

Figure 5.13: Q-Q PLOT(TNET, LOGISTICS)

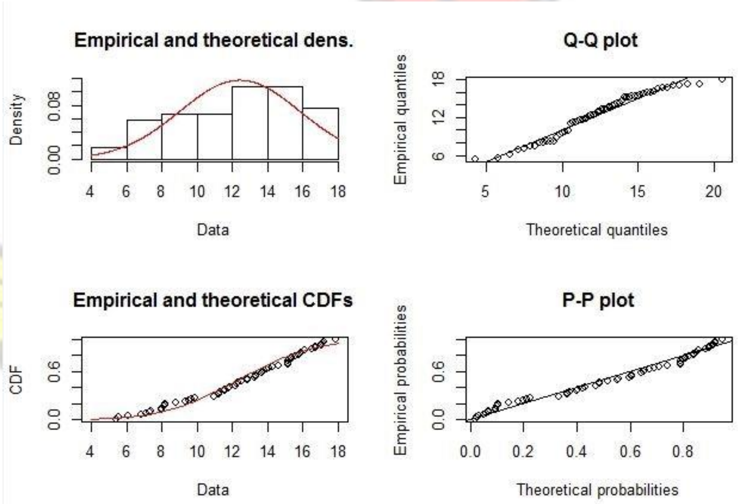

Figure 5.14: Q-Q PLOT(TNET, NORMAL)

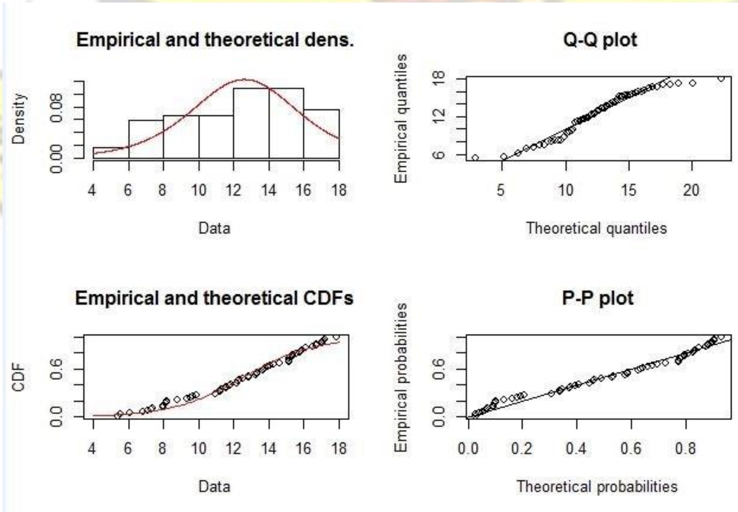

Figure 5.15: Q-Q PLOT(TNET,LOGISTIC)

APPENDIX E

5.5 This chunk of R codes is to generate independent Cholesky Decomposition of stock prices

```
#N is the number of observations
#n is the number of stocks being worked with
#sigma is the monthly volatility of the historical stock prices
#mu is the expected return of the historical stock prices
#cs is the covariance structure
#m is the correlation matrix
#z is the distribution
#x is the simulated stock prices

UNCORRCOMBAR<- function(N,n,mu,sigma,identitycs){
z=matrix(NA, nrow=n, ncol=N) for(i in 1:n){ if(i=1){
z[i,]=rweibull(N, mu, sigma) }else if(i=2){ z[i,]=rnorm(N, mu,
sigma)
}else z[i,]=rnorm(N, mu, sigma)
} identitycs<-function(n){ m<-
diag(n)
cs=1.5*m cs }
c=chol(identitycs(n)) c z x=t(c)*z
x t(x) } TRY
sp=UNCORRCOMBAR(60,3,10,1)
sp
#we compute the returns, mean(muR) and the volatility matrix(volmat)
returns=diff(log(sp)) muR= colMeans(returns, na.rm=TRUE) muR volmat=var(returns,
na.rm=TRUE) volmat cor(returns)
```

```
#Trading Strategy function
TradStra<-function(W,n,R)
{ beta<-matrix(runif(n*n, 0, 1), nrow=n) alpha<-diagg(runif(n, 0, 1)) delta<-
matrix(runif(n, 0, 1), nrow=n) mu=muR+alpha*(delta-muR)+beta*matrix(rnorm(n,
mean=0, sd=1), nrow=n) TS=solve(volmat*t(volmat))*(mu-R*matrix(rep(1,n),
nrow=n)) prop=1/sum(TS)
TS=prop*TS
TS=TS*W
sum=0 for(i in
1:n) {
value=TS[i];
if(value>0) {
sum=sum+value
} } if (sum<=W) { return(TS) }else {
return(matrix(c(mat.or.vec(n,1))))
}
}
##Simulation of Trading Strategy for large numbers
simulatedTS<-function(T, W, n, R){ totalRequired<-T; counter<-
1
PI=matrix(data=NA, nrow=n, ncol=totalRequired)
while(counter<-totalRequired){ strategy=TradStra(W, n,
R) zerovector=matrix(c(mat. or. vec(n,1)))
if(all(strategy=zerovector)){#Bad strategy next
}else{#Good strategy PI[,counter]=strategy
counter<-counter+1
}
}
```

PI<-matrix(c(rowMeans(PI)),nrow=n,ncol=1)

return(PI) } ##TRY THIS

simulatedTS(500,1000,3,r=0.23)

simulatedTS(1000,1000,3,r=0.23)

simulatedTS(2000,1000,3,r=0.23)

simulatedTS(10000,1000,3,r=0.23)

5.6 This chunk of codes is to generate correlated combination of the distribution and Cholesky Decomposition of stock prices

#N is the number of observations

#n is the number of stocks being worked with

#sigma is the monthly volatility of the historical stock prices

#mu is the expected return of the historical stock prices

#cs is the covariance structure

#m is the correlation matrix

#z is the distribution

#x is the simulated stock prices

#CORRCOMBTOEP is correlated combination(Toeplitz structure) CORRCOMBTOEP<-function(N,n,mu,sigma,TOEPL){ z=matrix(NA, nrow=n, ncol=N) for(i in 1:n){ if(i=1){

z[i,]=rweibull(N, mu, sigma) }else if(i=2){ z[i,]=rnorm(N, mu, sigma)

}else z[i,]=rlogis(N, mu, sigma)

}

TOEPL<-function(n){ m<-matrix(c(1,-0.2,0.15,-0.2,1,0.12,0.15,0.12,1)) cs=1.5*m cs } c=chol(TOEPL(n)) c z

x=t(c)*z x t(x) } TRY sp=CORRCOMBTOEP(60,3,10,1) sp

```
#we compute the returns, mean(muR) and the volatility matrix(volmat)
returns=diff(log(sp)) muR= colMeans(returns, na.rm=TRUE) muR
volmat=var(returns, na.rm=TRUE)
volmat cor(returns)
#Trading Strategy function
TradStra<-function(W,n,R)
{ beta<-matrix(runif(n*n, 0, 1), nrow=n) alpha<-diagg(runif(n, 0, 1)) delta<-
matrix(runif(n, 0, 1), nrow=n) mu=muR+alpha*(delta-muR)+beta*matrix(rnorm(n,
mean=0, sd=1), nrow=n) TS=solve(volmat*t(volmat))*(mu-R*matrix(rep(1,n),
nrow=n)) prop=1/sum(TS)
TS=prop*TS
TS=TS*W
sum=0 for(i in
1:n)
{ value=TS[i];
if(value>0) {
sum=sum+value
} } if (sum<=W) { return(TS) }else {
return(matrix(c(mat.or.vec(n,1)))) }
}
##Simulation of Trading Strategy for large numbers
simulatedTS<-function(T, W, n, R){ totalRequired<-T; counter<-
1
PI=matrix(data=NA, nrow=n, ncol=totalRequired)
while(counter<-totalRequired){ strategy=TradStra(W, n,
R) zerovector=matrix(c(mat. or. vec(n,1)))
if(all(strategy=zerovector)){#Bad strategy next
}else{#Good strategy PI[,counter]=strategy
counter<-counter+1
```

```
}
}
PI<-matrix(c(rowMeans(PI)),nrow=n,ncol=1)
return(PI) } ##TRY THIS
simulatedTS(500,1000,3,r=0.23)
simulatedTS(1000,1000,3,r=0.23)
simulatedTS(2000,1000,3,r=0.23)
simulatedTS(10000,1000,3,r=0.23)
```