A MATHEMATICAL MODEL OF A SUSPENSION BRIDGE
CASE STUDY: ADOMI BRIDGE

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BY

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Institute of Distance Learning.

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DECLARATION

I hereby declare that this submission is my own work towards the MSc and that, to the best of my knowledge, it contains no material(s) previously published by another person(s) nor material(s), which have been accepted for the award of any other degree of the University, except where the acknowledgement has been made in the text.

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ABSTRACT

The Lazer – McKenna mathematical model of a suspension bridge applied to the Adomi Bridge in Ghana is presented.

Numerical methods accessible in commercially available Computer Algebraic System "MATLAB" are used to analyse the second order non-linear ordinary differential equation.

Simulations are performed using an efficient SIMULINK scheme, the bridge responses are investigated by varying the various parameters of the bridge

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DEDICATION

This thesis is dedicated to my lovely, supportive and tolerant wife Nana Menyin Kwofie and our daughter; princess and pride of my life Nana Adjoa Kwofie.

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CHAPTER ONE

INTRODUCTION

The collapse of the Tacoma Suspension Bridge in 1940 stimulated interest in mathematical modeling of suspension bridges. The reason of collapse was originally attributed to resonance and this was generally accepted for fifty years until it was challenged by mathematicians Lazer and McKenna (Lazer and McKenna, 1990). Using a system of uncoupled non-linear ordinary differential, these mathematicians explained the collapse of the Bridge. Their model with appropriate engineering constants will be used to determine the response of the Adomi Bridge subjected to large induced initial oscillations.

1.1 BACKGROUND

1.1.1 Overview of Suspension bridges

Suspension bridges are of utmost importance in bridge engineering and dates as far back as early 19^{th} century. Suspension bridge technology enables the longest span of any design. Normally two enormous cables stretch their length. Shorter cables dangle down to hold the roadway. This connection of cables to other cables makes suspension bridge very flexible and susceptible to noticeable oscillations.

Suspension bridges have the longest free span of all the different type of bridges constructed, currently the first fifteen bridges with the longest free span in the world are of the suspension bridge type. (http://en.wikipedia.org/wiki/List_-of_longest_-suspension_bridge_spans) The bridge with the longest free span in the world is The Akashi-Kaikyō Bridge.on the Kobe-Awaji Route in Japan with a free maximum span of almost 2000m (figure 1.1).

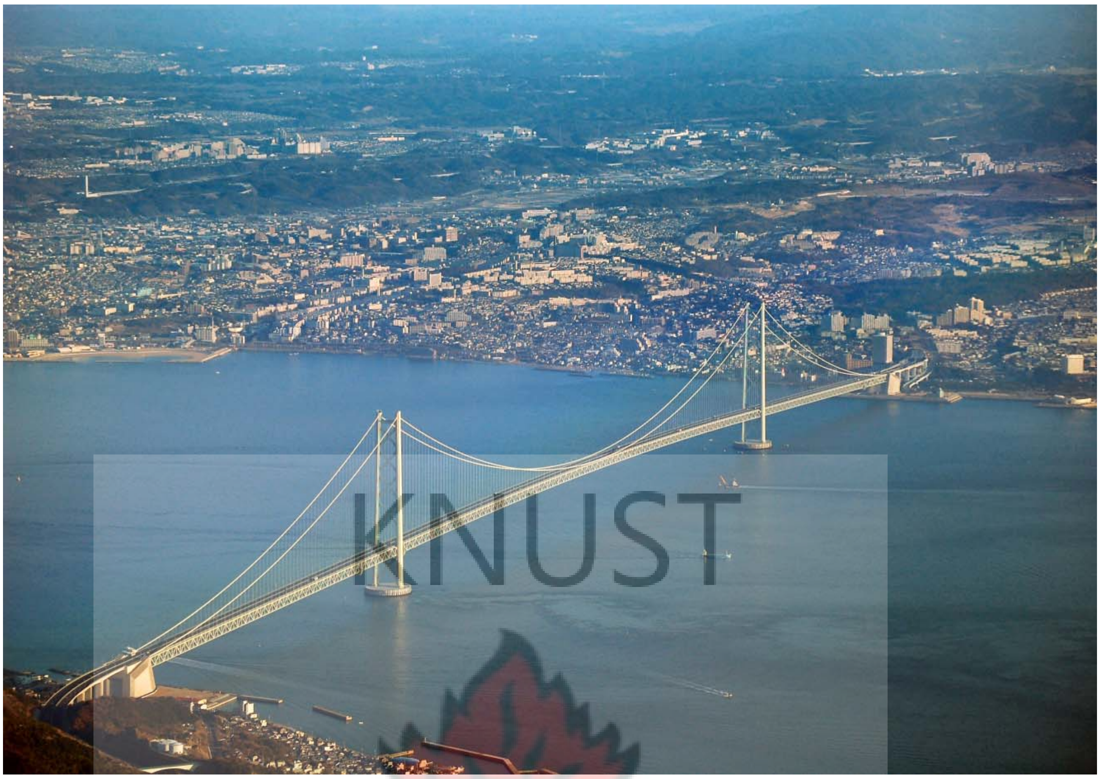

Figure 1.1: Akashi-Kaikyō Bridge.on the Kobe-Awaji Route in Japan (source: http://upload.wikimedia.org/wikipedia/commons/f/f1/Akashi_Bridge.jpg)

As stated above, suspension bridge technology currently provides maximum free span of bridges than any other type of bridge, this makes researching technology of suspension bridges of special interest to both engineers and mathematicians. An event which triggered renewed interest in the modeling and behaviour of suspension bridges by the scientific community is the collapse of the Tacoma suspension bridge in 1940.

1.1.2 The collapse of the Tacoma Suspension Bridge

On July 1, 1940, the Tacoma Narrows Bridge in the state of Washington was completed and opened to traffic. From the day of its opening the bridge began to undergo vertical oscillations, and it was soon nicknamed "Galloping Gertie". As a result of its novel behaviour, traffic on the bridge increased tremendously. People came from hundreds of miles to enjoy riding over a galloping, rolling bridge. For four months, everything was all right, and the authorities in charge became more and more confident of the safety of the bridge that they were even planning to cancel the insurance policy on the bridge (Tajcová, 1997).

The collapse of the bridge as described in (Tajcová, 1997) and (Menkveld and Pence, 2001) is paraphrased follows….

"*At about 7:00 a.m. of November 7, 1940, the bridge began to undulate persistently for three hours. Segments of the span were heaving periodically up and down as much as three feet. At about 10:00 a.m., the bridge started suddenly oscillating more wildly and concerned officials closed the bridge. Shortly after the bridge was closed, the character of the motion changed from vertical oscillation to two-wave torsional motion. The torsional motion caused the roadbed to tilt as much as 45 degrees from horizontal. At one moment, one edge of the roadway was twenty eight feet higher than the other; the next moment it was twenty-eight feet lower than the other edge. The centre span, remarkably, endured the vertical and torsional oscillation for about a half hour, but then a centre span floor panel broke off and dropped into the water below At 10:30 a.m. the bridge began cracking, and finally, at 11:00 a.m. the entire structure fell down into the river"*.

The collapse of the Tacoma suspension bridge is of particular interest and thus referenced in many papers addressing mathematical modeling of suspension bridges. The aftermath of this collapse is that it generated a lot of interest in the mathematical modeling of suspension bridges within the mathematics community. Initially the Tacoma Narrows bridge failure was considered as a classic example of the resonance effects on structures, in this case under the action of time-periodic forcing caused by a von kármán street of staggered vortices due to impinging wind on the bridge structure (Amann et al, 1941). In this manner acknowledging that the ultimate source of problem is the interaction between the periodicities of the bridge oscillations and the vortices that are created in the von Kármán street, later it became clearer that the standard textbook explanation of the collapse, based on linear resonance arguments was erroneous (Billah and Scanlan, 1991). Linear resonance is a rather narrow phenomenon and very difficult to occur in an irregularly changing environment (Lazer and McKenna , 1990)

Surprisingly it took more than half a century for mathematicians (Lazer and McKenna, 1990) to suggest that the generally accepted cause of the Tacoma bridge collapse being resonance was flawed and could not have been the reason of collapse. Even so, up until now the science of the dynamics of suspension bridges has many unexplained gaps as was made abundantly clear by the opening and rapid closing of the Millennium Bridge in London in 2000 (McKenna and Moore, 2002)

One of the most challenging and not fully explained areas of mathematical modeling involves nonlinear dynamical systems, in particular systems with so called jumping nonlinearity. It can be seen that its presence brings into the whole problem unanticipated difficulties and very often it is a cause of several solutions. The suspension bridge is an example of such a dynamical system. The nonlinearity is caused by the presence of the vertical supporting cable stays which restrain the movement of the centre span of the bridge in a downward direction, but have no influence on its behaviour in the upwards direction.

After the collapse of the Tacoma Narrows Bridge, it became important to establish what factors caused this disastrous failure so that these factors would be taken into consideration for the design of future suspension bridges. Although questions still persists about the exact cause for the Tacoma Narrows Bridge failure, mathematical models have been developed to illustrate how the bridge behaved during its final moments. There are models that illustrate both the vertical motion, as well as the torsional motion exhibited by the bridge. (McKenna, 1999)

1.1.3 The Adomi Bridge

One of Ghana's most treasured landmarks and national heritage is the Adomi Bridge (originally opened as the Volta Bridge) which is the main link between the Eastern and Volta regions of Ghana. It bridges the Volta River at Atimpoku which is near Akosombo dam (the site of Ghana's hydroelectric power plant). Figure 1.2 Shows the Adomi Bridge which is rightly described as arched suspension bridge. The Adomi Bridge is an arch suspension type whereby the roadway is suspended off two giant arches via cables.

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According to a 1958 article in the Structural Engineer, the bridge has a span of 805 feet and the rise to the crown of the arches is 219 feet. There have been debates in the past as to whether Adomi Bridge can be described to be suspension bridge or not. It suffice to say that so far as the roadbed are suspended by means of a vertical cable stays (hangers) connected to the steel truss arches, the bridge can be considered as a suspension type but undeniably in conventional suspension bridges the vertical cable stays are connected to a main cables which are strung between two supporting towers at the ends of the span as shown in figure 1.1.

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Figure 1.2 Adomi Bridge.
(source: https://www.myc4.com/Images/Users/21333/Adomi%20Bridge%20006.jpg)

According to Paper No. 6290 – The Volta Bridge (Scott and Adams, 1958), the Adomi Bridge is a steel arch structure having a clear span of 805 feet. The deck is suspended from the arch at 35 feet intervals by 2¼ inch high tensile steel cables (hangers), and is of composite reinforced concrete and steel construction. The carriageway has a width of 22 feet surfaced with a coat of mastic asphalt 1 inch thick. On each side are cantilevered footways of 4 feet 9inches wide. The arch is of crescent form bearing on concrete abutments founded on rock on each bank of the river. The arch itself is 40 feet wide overall. The rise of the lower chord is 158 feet 6 inches above the hinges, and the overall depth of the truss is 32 feet at the centre.

Adomi Bridge was designed by Sir William Halcrow & Partners and Freeman Fox & Partners and constructed by Dorman Long Bridge and Engineering Limited. The Bridge was built from March 1955 to November 1956. It was unveiled by the Honourable Dr. Kwame Nkrumah– the then Prime Minister of the Gold Coast (now

Ghana), on 25th January, 1957 to commemorate the opening of the bridge by H.E Sir Charles Noble Arden-Clarke Governor of The Gold Coast

Adomi Bridge is a major landmark and a national heritage and remains so even after fifty four years of exploitation. According to Ghana Highway Authorities (GHA), the Bridge is the main means by which an average of 120,000 workers, traders and tourists cross the Volta River daily to and from the eastern corridor and northern regions of the country. An average of 3,000 vehicles uses the Bridge daily.

1.2 PROBLEM STATEMENT

Suspension bridges are generally susceptible to visible oscillations, which if not controlled can lead to failure of the bridge. An uncoupled system of non-linear differential equation first derived in (McKenna, 1990) was used to explain the ultimate failure of the Tacoma Bridge. We apply this model to the Adomi Bridge with few modifications and appropriate engineering constants to predict the response of the Bridge to large oscillatory motions.

As far as we are aware, there are no studies published previously on the response of the Adomi Bridge to possible oscillations (vertical and torsional). In this thesis, we analyse the problem of stability of the Adomi Bridge when it is subjected to large initial vertical displacement or large torsional rotation. We determine whether small or large amplitude oscillations once started on the Bridge, will eventually diminish or rather continue oscillatory motion unceasingly until the Bridge collapses.

1.3 OBJECTIVES OF THESIS

- Use appropriate software program to create a simulation of the mathematical model of suspension bridge proposed in (McKenna, 1999) with some modifications.
- To determine using numerical experiments the response of Adomi Bridge when subjected to large initial vertical displacement or large torsional rotation.
- Investigate the stability of the Adomi Bridge under various initial conditions and varying engineering constants
- To establish if in spite of the apparent rigidity of steel arched- suspension bridge, they are as susceptible to large oscillation as in the conventional type.
- To make an input to the general stock of knowledge available to determine the safe and economical parameters for design and construction of steel arched- suspension bridges

1.4 JUSTIFICATION

As stated previously, the Adomi Bridge is a major landmark and a national heritage and remains so even after fifty four years of exploitation. Any unforeseen damage to the bridge will be very costly to the country and bring about economic hardship to many people as well as loss of revenue to the country. The estimated more than one hundred and twenty thousand human traffic(workers, tourists and traders) and three thousand vehicles which use the bridge daily will have to find longer alternate route and as such loss of valuable man-hours and business and trade opportunities. In this view any research paper which provides any suggestions or some form of scientific knowledge on the continued exploitation and maintenance of the Bridge is justifiable

1.5 METHODOLOGY

Recent advances in computer science have given us the necessary tools to evaluate the more sophisticated non-linear models rather than simplify the models using method such as linearization scheme. By means of available numerical tools, one can provide accurate numerical solutions in affordable computing times.

In this dissertation, the non-linear differential equation which models the oscillations of the Bridge is evaluated by employing numerical methods specifically Runge-Kutta algorithm. This is done indirectly by the use of "Matlab simulink" a computer software program. Different simulations are performed for different initial conditions and also varying some of the engineering constants associated with the Bridge. The data needed for the model are the physical constants of the materials from which the Bridge was built, the physical dimensions of the Bridge, the spacing and the physical constants of cables stay (hangers).

1.6 SCOPE OF THESIS

The scope of this dissertation is limited to the application of a modified McKenna's mathematical model of a suspension bridge to the Adomi Bridge. The mathematical model derived in (McKenna, 1999) is modified by adding other forcing term (periodic impulse and periodic random forces) to create a more realistic model.

1.7 THESIS ORGANISATION

The dissertation comprises the following five chapters. Chapter one is introduction which covers background, problem statement, objectives, justification, methodology, scope of thesis and thesis organisation.

Chapter two reviews related researches starting with a paper by Lazer and McKenna (Lazer and McKenna, 1990), which sparked interest in mathematical modeling of suspension bridges. Thereafter several other papers on mathematical modeling of suspension bridges and alternative models are assessed.

Chapter three focuses on the methodology for the analysis of non-linear differential equations derived for oscillation of the suspension bridge. The derivation is from first principle using applied mechanics and Euler-Langrage principle. A brief overview of "Matlab simulink software" which is used for the analysis is presented and reasons for choosing a specific algorithm in this software are explained. The numerical method applied in the "Matlab simulink software" specifically the fourth order Runge-Kunte method for solving system of ordinary differential equation is outlined and it specific features which makes it the most suitable numerical method as compared to others are discussed.

Chapter four is data collection and analysis; here by using available engineering data of the Adomi Bridge, we choose the appropriate values of the physical constants in the system of differential equations which models the oscillation of a suspension bridge. We perform multiple numerical experiments by specifying different initial conditions and assuming different physical constants. We use the Runge-Kutta method to solve the initial value problem over long period of time. The results of the numerical experiments are displacement - time plot for vertical oscillations and angle of rotation - time plot for torsional oscillations.

Chapter five concludes the thesis. In this chapter we summarise the results of numerical experiments and draw conclusions on the results obtained. Recommendations based on the numerical experiments will be articulated. The inadequacies of the mathematical model and the accuracy of results obtained from the model will be discussed.

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CHAPTER TWO

REVIEW OF RELATED STUDIES

2.1 PIONEERING STUDIES

The pioneering paper on mathematical modeling of suspension bridge (Lazer and McKenna, 1990) was published fifty years after the collapse of the Tacoma Suspension Bridge. Their research directly contradicted the long-standing view that resonance phenomena caused the collapse of the Tacoma Narrows Bridge.

They suggested several alternative types of differential equations that govern the motion of such suspension bridges. In their paper the authors made a strong case against the popular notion that the collapse of the Tacoma Bridge was due to resonance. They contended that a complete mathematical explanation for the Tacoma Narrows disaster must isolate the factors that make suspension bridges prone to large-scale oscillations; show how a bridge could go into large oscillations as the result of a single gust and at other times remain motionless even in high winds; and demonstrate how large vertical oscillations could rapidly change to a twisting motion. One significant detail, they asserted, lies in the behaviour of the cable stays (hangers), connecting the roadbed to a bridge's main cable.

Civil engineers usually assume that the stays always remain in tension under a bridge's weight, in effect acting as stiff springs. That allows them to use relatively simple, linear differential equations to model the bridge's behaviour. When a bridge starts to oscillate, however, the stays begin alternately loosening and tightening. That produces a nonlinear effect, changing the nature of the force acting on the bridge.

When the hangers are loose, they exert no force, and only gravity acts on the roadbed. When the hangers are tight, they pull on the bridge, countering the effect of gravity. Solutions of the nonlinear differential equations that correspond to such an asymmetric situation suggest that, for a wide range of initial conditions, a given push can produce either small or large oscillations. Lazer and McKenna went on to argue that the alternate slackening and tightening of cables might also explain the large twisting oscillations experienced by a suspension bridge.

In (Lazer and McKenna, 1987) the authors proposed a nonlinear beam equation as a model for vertical oscillations in suspension bridges. They modelled the restoring force from the cable as a piecewise linear function of the displacement in order to capture the fact that the suspension cables resist elongation, but do not resist compression. Later investigations of the qualitative and quantitative properties of solutions to this type of asymmetric system suggest that this is a convincing model for nonlinearly suspended structures.

The results on existence, uniqueness, multiplicity, bifurcation, and stability of periodic solutions are consistent with the nonlinear behaviour of some suspension bridges; see (Chen and McKenna, 1999), (Doole and Hogan, 2000), (Humphreys and McKenna, 1999) (Lazer and McKenna, 1990) and (McKenna and Walter, 1987) for example. In (McKenna, 1999) and (Moore, 2002), McKenna and Moore extended the models of Lazer and McKenna to the coupled vertical and torsional motions of suspension bridges. Though they were able to replicate the phenomena observed on the Tacoma Narrows Bridge on the day of its famous collapse, the model had several shortcomings. First, the treatment of the restoring force from the cables was oversimplified; the nonlinear terms in the model describe cables that behave perfectly linearly when in tension (regardless of the size of the oscillation) and that can lose tension completely. Moreover, the parameter values for which they could induce the desired phenomena were physically unreasonable.

In (McKenna and O'Tuama, 2001), the authors proposed a modified nonlinearity that addressed the shortcomings described above. In (McKenna and O'Tuama, 2001) and (McKenna and Moore, 2002) McKenna, O'Tuama, and Moore found that smoothing the nonlinearity yields a significant qualitative change in the structure of the set of periodic solutions to the nonlinearly coupled vertical-torsional system

2.2 THE McKenna's MATHEMATICAL MODEL

In (McKenna, 1999), the author considered a horizontal cross section of the centre span of a suspension bridge and proposed an ordinary differential equation model for the torsional motion of the cross section. Using physical constants from the engineers' reports of the Tacoma Narrows collapse, he investigated this model numerically. In the paper, the author formulated a mechanical model for a beam oscillating torsionally about equilibrium, and suspended at both or ends by cables. He showed how the "small-angle" linearization can remove a large class of large-amplitude non-linear solutions that can be sustained by extremely small periodic forcing terms

To model the motion of a suspension bridge, McKenna considered the horizontal cross section of the suspension bridge as a beam (rod) of

length $2l$ and

mass m suspended by non-linear cables,

$y(t)$ denote the downward distance of the centre of gravity of the rod from the unloaded state and

$\theta(t)$ denote the angle of the rod from horizontal at time t

The uncoupled differential equation derived by the author in (McKenna, 1999) for the torsional and vertical motion of a beam assuming that the vertical cables never lose tension was given as.

$$\ddot{\theta} = -\frac{6K}{m}\cos\theta\sin\theta - \delta_1\dot{\theta} + f(t) \qquad 2.1$$

$$\ddot{y} = -\frac{2K}{m}y - \delta_2\dot{y} + g \qquad 2.2$$

where δ_1 and δ_2 are damping constants,

g is the force due to gravity, and

$f(t)$ is the external force at time t,

K is the spring constant of the nonlinear cable-like springs

By specifying the initial position and velocity of the cross section and using the Runge-Kutta method to solve the initial value problem over long time, McKenna demonstrated that under the same small periodic forcing term, small or large amplitude periodic motion may result; the ultimate outcome depends on the initial conditions.

2.2.1 Modifications to McKenna's model

In (McKenna and Moore, 2000), the authors contended that the methodology in the (McKenna, 1999) was somewhat primitive. In (McKenna, 1999), different initial conditions were prescribed randomly and the eventual behaviour of the solution of the initial value problem was observed. For such a method sometimes the motion converged to a large amplitude solution and sometimes to the small near-equilibrium solution. In their paper, McKenna and Moore presented a more systematic approach to the study of the equation for the torsional motion of a cross section of the center span. They used Leray-Schauder degree theory to prove that, under certain physical assumptions, the undamped equation has multiple periodic solutions. They demonstrated numerically that for small forcing, multiple periodic solutions exist and that whether large or small amplitude motion results depends only on the initial conditions. Finally, they used a more sophisticated approach to compute periodic solutions of the nonlinear differential equation. Using continuation methods, they examined the bifurcation properties of periodic solutions as the amplitude of the forcing term varies and they demonstrated that bifurcation from single to multiple periodic solutions occurs for small forcing.

In Ben-Gal and Moore, 2006, the authors study the nature of periodic solutions of two non-linear spring-mass equations; their non-linear terms were similar to earlier models of motion in suspension bridges. Firstly they considered the known piecewise linear model and then proposed a smoothed non-linear cable model. For the piecewise linear cable force Ben-Gal and Moore were motivated by the original Lazer-McKenna model (Lazer and McKenna, 1987) and thus arrive at equations similar to ones in equations 2.1 and 2.2. The authors then proposed a smoothed non-linear cable force, here they contended that there are some inherent flaws in the piecewise model. For example, they stated that the fact that the expression for the cable force cannot be differentiated makes it an unlikely candidate to describe a physical system. Relying on physical intuition they indicated that the transition of an object from resisting to no resisting of a noticeable force is not sudden, but must be smooth. They argued that the cable stay do in fact exert some resistance when compressed and exert super-linear force when overstretched. Motivated by (McKenna and O'Tuama, 2001) and (McKenna and Moore, 2002), Ben-Gal and Moore proposed the equation of the smoothed non-linear cable force *F*. as

$$F = mg(e^{\frac{Ky}{mg}} - 1) \qquad 2.3$$

The corresponding differential equation is given as

$$y'' = -g(e^{\frac{Ky}{mg}} - 1) - \left(\frac{\delta}{m}\right) + \left(\frac{\lambda}{m}\right)\sin(\mu t) \qquad 2.4$$

Where y is downwards displacement of the mass from the equilibrium point,

g is acceleration due to gravity,

δ is damping constant,

λ and μ are the amplitude and frequency of forcing term and

K is the spring constant of the nonlinear cable-like springs

They contrast the multiplicity, bifurcation, and stability of periodic solutions for a piecewise linear and smooth non-linear restoring force. The authors conclude that while many of the qualitative properties are the same for the two models, the nature of the secondary bifurcations (period-doubling and quadrupling) differs significantly.

2.3 AN ALTERNATIVE MATHEMATICAL MODEL

A more complex model as compared to the model suggested by McKenna (McKenna, 1999) is found in (Tajcová, 1997). In his paper, the author proposed two mathematical models describing a dynamical behaviour of suspension bridges such as Tacoma Narrows Bridge. The author's attention was concentrated on their analysis concerning especially the existence of a unique solution.

In the first and simpler model proposed by Tajcová, the construction holding the cable stays was taken as a solid and immovable object. Then he described the behaviour of the suspension bridge by a vibrating beam with simply supported ends. The suspension bridge is subjected to the gravitation force, to the external periodic force (e.g. due to the wind) and in an opposite direction to the restoring force of the cable stays hanging on the solid construction. The model illustrated in figure. 2.1 shows the bending beam with simply supported ends, held by nonlinear cables, which are fixed on an immovable construction

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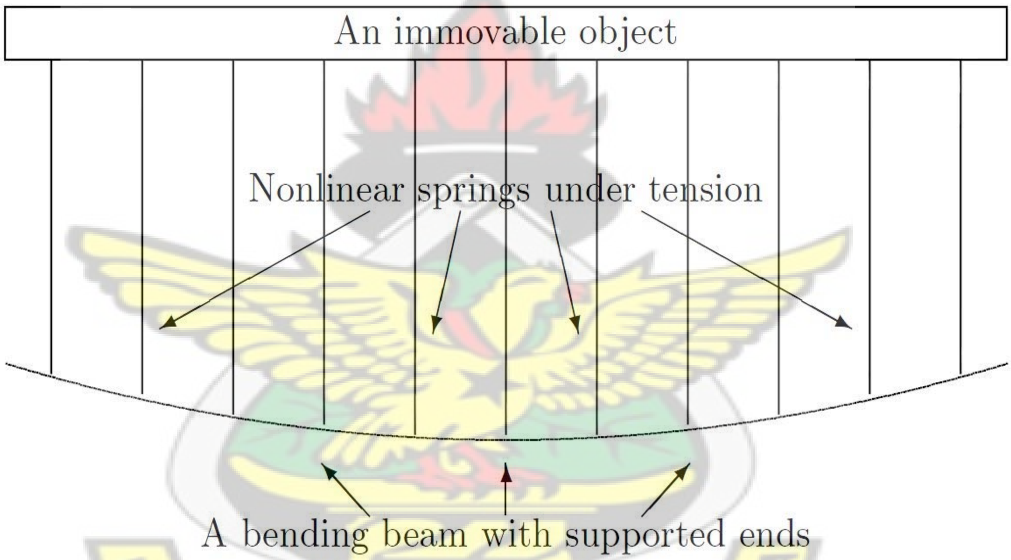

Figure 2.1 A simple model of a suspension bridge proposed by Tajcová

In this model the displacement $u(x,t)$ of this beam was described by non-linear partial differential equation:

$$m\frac{\partial^2 u(x,t)}{\partial t^2}+EI\frac{\partial^4 u(x,t)}{\partial x^4}+b\frac{\partial u(x,t)}{\partial t}=-ku^+(x,t)+W(x)+\varepsilon f(x,t) \qquad 2.5$$

With the boundary conditions

$$\begin{aligned}&u(0,t)=u(L,t)=\frac{\partial^2 u(0,t)}{\partial x^2}=\frac{\partial^2 u(L,t)}{\partial x^2}=0\\&u(x,t+2\pi)=u(x,t), \qquad -\infty<t<\infty, \quad x\in(0,L)\end{aligned} \qquad 2.6$$

In the other and more complicated model proposed by Tajcová, the construction holding the cable stays was not taken as a solid and immovable object but rather as a vibrating string, coupled with the beam of the roadbed by non-linear cable stays as shown in Fig 2.2

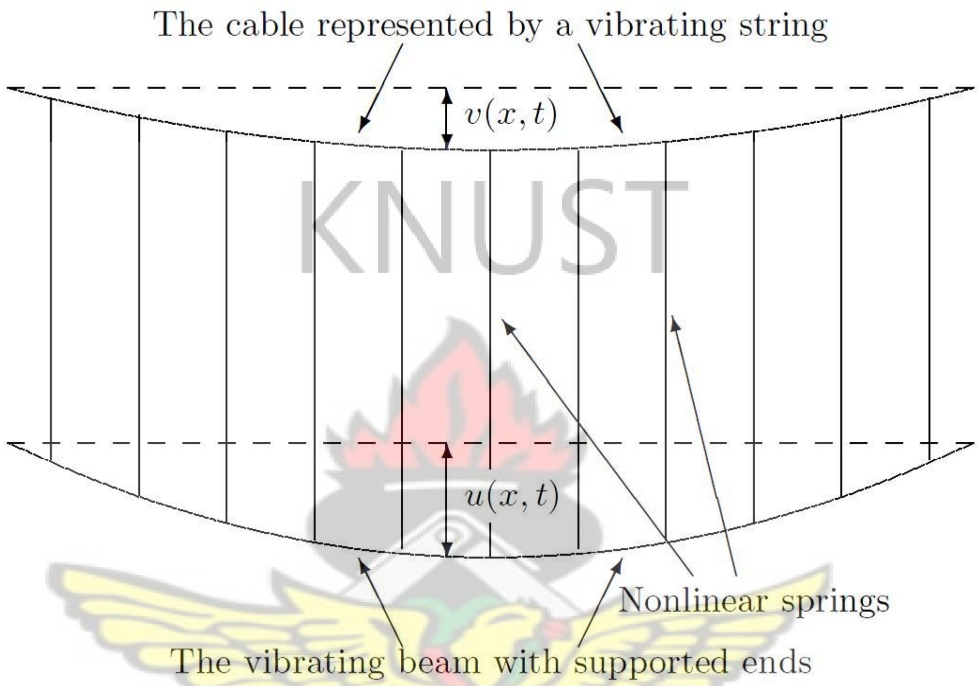

Figure 2.2 A more complicated model proposed by Tajcová

For this model the displacement $u(x,t)$ of the beam and $v(x,t)$ of that of vibrating string was given by the author as a coupled non-linear partial differential equation:

$$\begin{aligned} &m\frac{\partial^2 u(x,t)}{\partial t^2} + EI\frac{\partial^4 u(x,t)}{\partial x^4} + b\frac{\partial u(x,t)}{\partial t} + k(u-v)^+ = W(x) + \varepsilon f(x,t) \\ &m_1\frac{\partial^2 v(x,t)}{\partial t^2} + T\frac{\partial^2 v(x,t)}{\partial x^2} + b_1\frac{\partial v(x,t)}{\partial t} - k(u-v)^+ = W_1(x) + \varepsilon f_1(x,t) \end{aligned} \qquad 2.7$$

With the boundary conditions

$$u(0,t) = u(L,t) = \frac{\partial^2 u(0,t)}{\partial x^2} = \frac{\partial^2 u(L,t)}{\partial x^2} = v(0,t) = v(L,t) = 0 \qquad 2.8$$

In equations 2.5, 2.6, 2.7 and 2.8

m and m_1	mass per unit length of bridge and main cable respectively,
E	Young's modulus,
I	Moment of inertia of cross section,
b and b_1	damping coefficient of bridge deck and main cable respectively,
k	stiffness of cables (spring constant),
W and W_1	weight per unit length of the bridge and main cable respectively,
L	length of the centre-span of the bridge,
T	inner tension of main cable,
εf and εf_1	external time-periodic forcing term (due to wind) on bridge and main cable respectively.

In the paper (Tajcová, 1997), the author used the same non-linear springs assumption for the cable stays (hangers) as proposed in (Lazer and McKenna, 1990). That is the cable stays are considered as one-sided springs, obeying Hooke's law, with a restoring force proportional to displacement when stretched and with no restoring force when compressed. Thus if an unloaded cable is expanded downward by a distance u from the unloaded state, the cable should have a resisting force ku^+ in other words, ku if u is positive, and 0 if u is negative.

Finally Tajcová presented his own results concerning existence and uniqueness of time-periodic solutions of two chosen models. He used two different approaches; the first one was based on the Banach contraction theorem which needs some restrictions on the bridge parameters. The second approach works in relatively greater generality but with an additional assumption of sufficiently small external forces. One conclusion the author arrives at, consistent with the conclusion of other researches was that strengthening the cable stays (hangers), which means increasing the spring constant *k*, can paradoxically lead to the destruction of the bridge. That is in some range of k values the more flexible the cable stays are, the better the bridge response to oscillations (large amplitude oscillations settle down more quickly)

Research in the area of mathematical modeling of suspension bridges started by Lazer and McKenna is still continuing with researchers constantly providing interesting and useful results.

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CHAPTER THREE

METHODOLOGY

This chapter will start with the derivation of the system of second order differential equation governing the vertical and torsional oscillations of a suspension bridge. The equations with the necessary engineering constants were used in (McKenna, 1999) to explain the probable cause of collapse of the Tacoma Narrows Suspension Bridge. Herein, this differential equations is applied to model the vertical and torsional oscillations of the Adomi Bridge. Numerical methods specifically the fourth order Runge-Kutta method is employed to solve the equations, hence the formulation of this method (Runge-Kutta) is presented. This is done indirectly by the use of "Matlab simulink" which is imbedded in "Matlab" a computer software program. The chapters end with an overview of the capabilities of "Matlab" and "Matlab simulink".

3.1 THE MODEL OF CROSS SECTION OF BRIDGE'S SPAN

We first develop the differential equation governing the vertical and torsional oscillations of the horizontal cross section of the centre span of a suspension bridge.

We treat the centre span of the bridge as a beam of length L and width $2l$ suspended by cables (see figure 3.1).

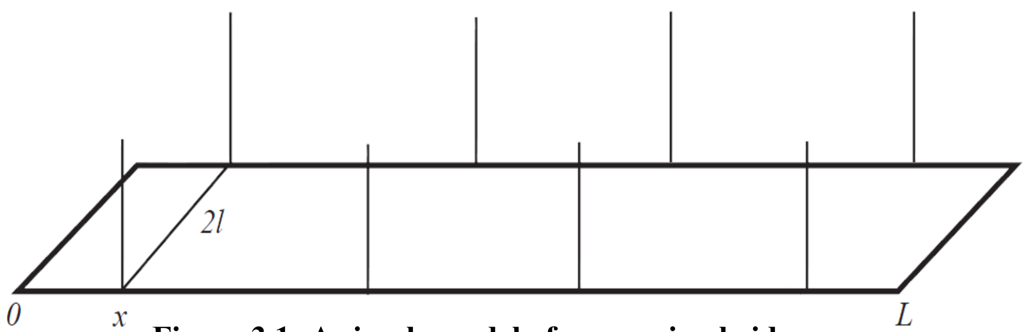

Figure 3.1: A simple model of suspension bridge

To model the motion of a horizontal cross section of the beam, we treat it as a rod of length $2l$ and mass m suspended by cables. Let

$y(t)$ denote the downward distance of the centre of gravity of the rod from the unloaded state

$\theta(t)$ denote the angle of the rod from horizontal at time t (see figure 3 2).

We will assume that the cables do not resist compression, but resist elongation according to Hooke's Law with spring constant K; i.e., the force exerted by the cable is proportional to the elongation in the cable with proportionality constant K. In Figure 3.2 we see that the extension in the right hand cable is $(y - l\sin\theta)$ hence the force exerted by the right hand cable is

$$-K(y - l\sin\theta)^{+} = \begin{cases} -K(y - l\sin\theta), & y - l\sin\theta \geq 0 \\ 0, & y - l\sin\theta \leq 0 \end{cases} \qquad 3.1$$

Where $v^{+} = \max(v, 0)$

Similarly the force exerted by the left hand cable is

$$-K(y + l\sin\theta)^{+} = \begin{cases} -K(y + l\sin\theta), & y + l\sin\theta \geq 0 \\ 0, & y + l\sin\theta \leq 0 \end{cases} \qquad 3.2$$

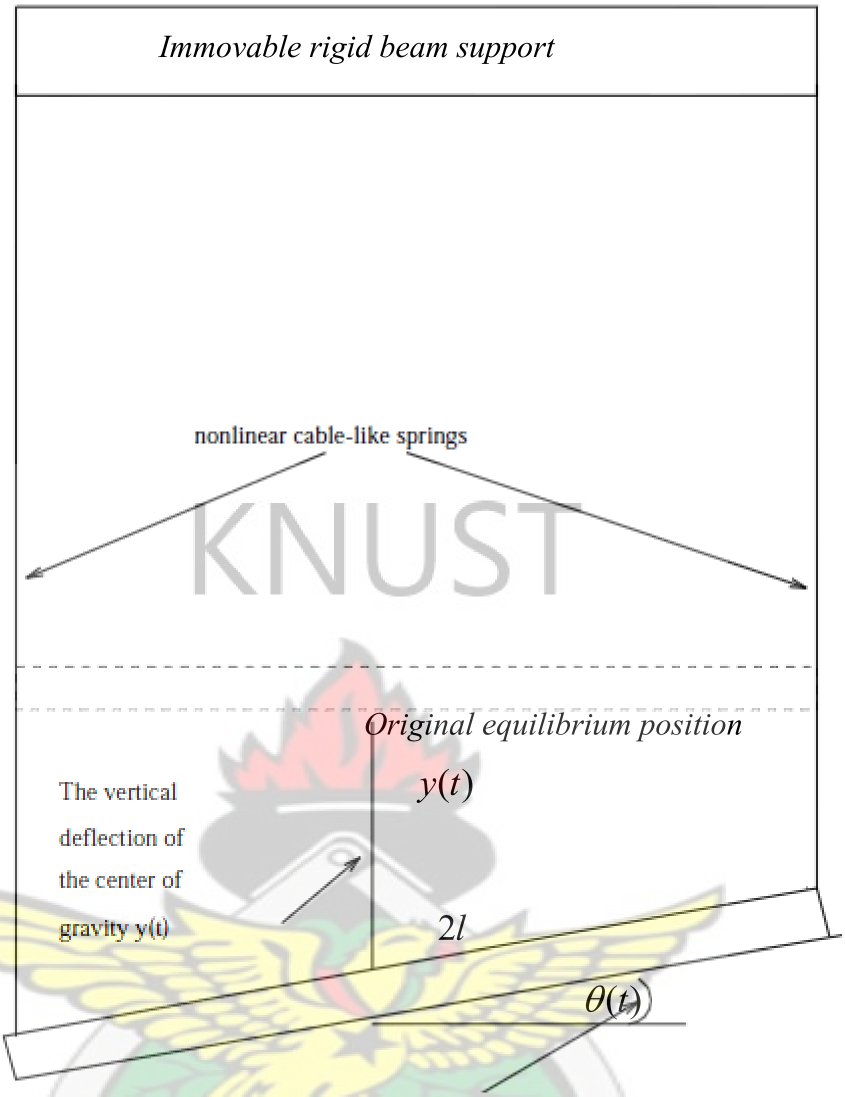

Figure 3.2: A horizontal cross section of suspension bridge

The derivation is as follows; the potential energy $(P.E)$ of a spring with spring constant k stretched a distance x from equilibrium position is given by

$$P.E = \int kxdx = \frac{1}{2}kx^2 \qquad 3.3$$

Thus total potential energy $(P.E_T)$ of right and left hand cable (figure 3.2) will be given by

$$P.E_T = \frac{1}{2}K\left(\left((y - l\sin\theta)^+\right)^2 + \left((y + l\sin\theta)^+\right)^2\right) \qquad 3.4$$

The potential energy $P.E_R$ due to weight of rod with mass m displaced downwards from equilibrium by distance y is given by

$$P.E_R = -mgy$$

where g is acceleration due to gravity

Therefore total potential energy of model $P.E_M$ is given by

$$P.E_M = \frac{K}{2}\left(\left[(y - l\sin\theta)^+\right]^2 + \left[(y + l\sin\theta)^+\right]^2\right) - mgy \qquad 3.5$$

Now we proceed to find the total kinetic energy $K.E_M$ of model. For the vertical oscillatory motion the kinetic energy $K.E_R$ of the centre of mass of the rod is given by

$$K.E_R = \frac{1}{2}m\dot{y}^2$$

where $\dot{y}$ is the velocity of the centre of mass of rod.

The formula for finding the kinetic energy $K.E_T$ about the centroid of the rod due to the torsional oscillatory (rotational) motion is derived from first principles as;

$$K.E_T = \frac{1}{6}ml^2\dot{\theta}^2$$

where $\dot{\theta}$ is the angular velocity

To prove the formula for $K.E_T$ consider an infinitesimal part of the rod with mass dm at a distance r from the centre of rod as shown in figure 3.3.

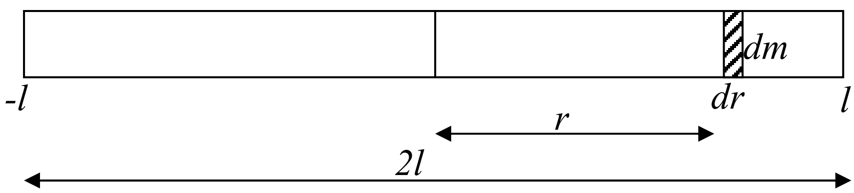

Figure 3.3 Rod representing cross section of bridge

The kinetic energy $K.E_{dm}$ of mass *dm* is given by

$$K.E_{dm} = \frac{1}{2} dm(r\dot{\theta})^2,$$

$r\dot{\theta}$ is linear velocity *v* of infinitesimal part *dm*. The mass of rod is *m* and length *2l,* thus

$$dm = \frac{m}{2l} dr$$

Substituting this in $K.E_{dm}$ and integrating over limit $[-l,l]$ we have

$$K.E_T = \frac{m\dot{\theta}^2}{4l} \int_{-l}^{l} r^2 dr = \frac{1}{6} ml^2 \dot{\theta}^2$$

Thus total Kinetic energy of system will be given by

$$K.E_M = K.E_R + K.E_T = \frac{1}{2} m\dot{y}^2 + \frac{1}{6} ml^2 \dot{\theta}^2 \qquad 3.6$$

Now we form the Lagrangian *L*

$$L = K.E_M - P.E_M$$

$$L = \frac{1}{2} m\dot{y}^2 + \frac{1}{6} ml^2\dot{\theta}^2 - \frac{K}{2}\left(\left[(y - l\sin\theta)^+\right]^2 + \left[(y + l\sin\theta)^+\right]^2\right) + mgy \qquad 3.7$$

According to the principle of least action, the motion of the beam obeys the Euler-Lagrange equations,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \qquad \text{and} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0 \qquad 3.8$$

We proceed by evaluating the required derivatives needed in the Euler-Lagrange equations,

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{ml^2\dot{\theta}}{3}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{ml^2\ddot{\theta}}{3}$$

$$\frac{\partial L}{\partial \theta} = Kl\cos\theta\left[(y - l\sin\theta)^{+} - (y + l\sin\theta)^{+}\right]$$

Thus $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$ becomes

$$\frac{ml^2\ddot{\theta}}{3} = Kl\cos\theta\left[(y - l\sin\theta)^{+} - (y + l\sin\theta)^{+}\right] \qquad 3.9$$

Similarly we evaluate

$$\frac{\partial L}{\partial \dot{y}} = m\dot{y}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = m\ddot{y}$$

$$\frac{\partial L}{\partial y} = -K\left[(y - l\sin\theta)^{+} + (y + l\sin\theta)^{+}\right] + mg$$

Thus $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0$ becomes

$$m\ddot{y} = -K\left[(y - l\sin\theta)^{+} + (y + l\sin\theta)^{+}\right] + mg \qquad 3.10$$

Simplifying and adding damping terms $\delta_1\dot{\theta}$ and $\delta_2\dot{y}$ to equations (3.9) and (3.10) respectively, as well as external forcing function $f(t)$ to equation (3.9) we get the following system of coupled second order differential equations

$$\left\{\begin{aligned} \ddot{\theta} &= \frac{3K}{ml}\cos\theta\left[(y - l\sin\theta)^{+} - (y + l\sin\theta)^{+}\right] - \delta_1\dot{\theta} + f(t) \\ \ddot{y} &= -\frac{K}{m}\left[(y - l\sin\theta)^{+} + (y + l\sin\theta)^{+}\right] - \delta_2\dot{y} + g \end{aligned}\right\} \qquad 3.11$$

Assuming that the cables never lose tension,

we have $y \pm l\sin\theta \geq 0$ and hence

$(y \pm l\sin\theta)^{+} = y \pm l\sin\theta$.

Thus, the equations (3.11) become uncoupled and the torsional and vertical motion satisfy

$$\ddot{\theta} = -\frac{6K}{m}\cos\theta\sin\theta - \delta_1\dot{\theta} + f(t) \qquad 3.12$$

$$\ddot{y} = -\frac{2Ky}{m} - \delta_2\dot{y} + g \qquad 3.13$$

Equations 3.12 and 3.13 were used in (McKenna, 1999) to explain the cause of collapse of the Tacoma Narrows suspension bridge.

Equation 3.13 model the vertical oscillatory motion and is simply the equation for a damped, forced, linear harmonic oscillator and the behaviour of its solutions is well known (Blanchard, Devaney and Hall, 2006). The equation for the torsional motion is a damped, forced, pendulum equation, which is known to possess chaotic solutions (Blanchard, Devaney and Hall, 2006). McKenna approximated periodic solutions of (3.12) in (McKenna, 1999). In this dissertation we investigate numerically the bifurcation properties of these periodic solutions.

3.2 FOURTH (4^{th}) ORDER RUNGE-KUTTA METHOD

The fourth order Runge-Kutta method (RK4) is the most widely used numerical method for solving ordinary differential equation (ODE). RK4 belongs to the family of explicit Runge-Kutta method.

Let an initial value problem (IVP) be specified as follows

$$y' = f(t, y), \qquad y(t_0) = y_0 \qquad 3.14$$

The explicit Runge-Kutta method is then given by

$$y_{n+1} = y_n + h\sum_{i=1}^{s} b_i k_i \qquad 3.15$$

Where

$$k_1 = f(t_n, y_n),$$
$$k_2 = f(t_n + c_2 h, y_n + a_{21} h k_1),$$
$$k_3 = f(t_n + c_3 h, y_n + a_{31} h k_1 + a_{32} h k_2), \qquad 3.16$$
$$\vdots$$
$$k_s = f(t_n + c_s h, y_n + a_{s1} h k_1 + a_{s2} h k_2 + \cdots + a_{s,s-1} h k_{s-1}),$$

s - The number of stages.

h - The step size

$a_{ij} (1 \le j < i \le s)$, $b_i (i = 1, 2, \cdots, s)$ and $c_i (i = 2, 3, \cdots, s)$ are coefficients to be specified for a particular Runge – Kunta (RK) order and chosen so as to minimize local truncation error (LTE). The LTE, of an RK method is defined to be the difference between the exact and the numerical solution of the IVP at time $t = t_{n+1}$. These data are conveniently displayed in a tableau known as the Butcher array shown in table 3.1.

Table 3.1: The Butcher array for an explicit RK method

$$\begin{array}{c|ccccc}
0 & 0 & 0 & \cdots & & 0 \\
c_2 & a_{21} & 0 & \cdots & & 0 \\
c_3 & a_{31} & a_{3,2} & \cdots & & \vdots \\
\vdots & \vdots & & \ddots & 0 & 0 \\
c_s & a_{s,1} & a_{s,2} & \cdots & a_{s,s-1} & 0 \\
\hline
 & b_1 & b_2 & \cdots & b_{s-1} & b_s
\end{array}$$

For explicit RK method the following conditions are imposed in specifying the coefficients.

$a_{ij} = 0$ for all $j \ge i$

$c_i = \sum_{j=1}^{s} a_{ij}, \qquad i = 2 : s \qquad$ and

$$\sum_{j=1}^{s} b_j = 1$$

RK4 is a four-stage RK method. The most popular of all RK methods (of any stage number) is the four-stage, fourth order method with corresponding Butcher array shown in table 3.2.

Table 3.2: The Butcher array for the classic four-stage, fourth-order method

$$
\begin{array}{c|cccc}
0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
\hline
 & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6}
\end{array}
$$

Thus equivalently, the RK4 method for the IVP specified in equation 3.14 will be given by the following equations:

$$
\begin{aligned}
y_{n+1} &= y_n + \tfrac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4) \\
t_{n+1} &= t_n + h
\end{aligned}
\qquad 3.17
$$

Where y_{n+1} is the RK4 approximation of $y(t_{n+1})$ and

$$
\begin{aligned}
k_1 &= f(t_n, y_n) \\
k_2 &= f(t_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hk_1) \\
k_3 &= f(t_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hk_2) \\
k_4 &= f(t_n + h, y_n + hk_3)
\end{aligned}
\qquad 3.18
$$

Thus, the subsequent value (y_{n+1}) is determined by the current value (y_n) plus the product of the size of the interval (h) and an estimated slope. The slope is a weighted average of slopes:

- k_1 is the slope at the start of the interval;
- k_2 is the slope at the midpoint of the interval, using slope k_1 to determine the value of y at the point $t_n + h / 2$ using Euler's method;
- k_3 is again the slope at the midpoint, but now using the slope k_2 to determine the y-value;
- k_4 is the slope at the end of the interval, with its y-value determined using k_3.

In averaging the four slopes, greater weight is given to the slopes at the midpoint:

$$slope = \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

The RK4 method is a fourth-order method, meaning that the error per step is on the order of h^5, while the total accumulated error has order h^4. The above formulae are valid for both scalar- and vector-valued functions (i.e., y can be a vector and f an operator). The fourth-order Runge–Kutta scheme requires four function evaluations per time step. However, it also has superior stability as well as excellent accuracy properties. These characteristics, together with its ease of programming, have made the fourth-order RK one of the most popular schemes for the solution of ordinary and partial differential equations. A straightforward implementation of RK4 method applied to a system of ODE is as follows:

We wish to solve the system of differential equations

$$\frac{dy_1}{dx} = f_1(x, y_1, y_2, \cdots, y_n)$$
$$\frac{dy_2}{dx} = f_2(x, y_1, y_2, \cdots, y_n)$$
$$\ldots$$
$$\frac{dy_n}{dx} = f_n(x, y_1, y_2, \cdots, y_n)$$

The pseudo-code is given by:

x **- scalar;** ***y, k1, k2, k3, k4, slope*** **are vectors;** ***n*** **number of equations;** ***h*** **is step size.**

On exit, both ***x*** **and** ***y*** **are updated for the next station in marching.**

```
SUB RK4 ( x , y , n , h )
CALL Derivs ( x , y , k1 )
DO i = 1 , n
ym ( i ) = y ( i ) + k1 ( i ) * h / 2
END DO
xm = x + h / 2
CALL Derivs ( xm , ym , k2 )
DO i = 1 , n
ym ( i ) = y ( i ) + k2 ( i ) * h / 2
```

```
END DO
CALL Derivs ( xm , ym , k3 )
DO i = 1 , n
ym ( i ) = y ( i ) + k3 ( i ) * h
END DO
xm = x + h
CALL Derivs ( xm , ym , k4 )
DO i = 1 , n
slope ( i ) = ( k1 ( i ) + 2 * ( k2 ( i) + k3 ( i ) ) + k4 ( i ) ) / 6
y ( i ) = y ( i ) + slope ( i ) * h
END DO
x = xm
END
SUB Derivs ( x , y , f )
f ( 1 ) = function-1 ( x , y )
f ( 2 ) = function-2 ( x , y )
f ( n ) = function-n ( x , y )
END
```

A more complex implementation of RK4 method is available in commercial mathematical packages such as MATLAB, MAPLE, MATHCAD etc. The implementation in such packages will routinely consider adaptive steps techniques. This method adapts the step size *h* during the course of the iteration in attempt to keep the LTE within some specified bound. An example of such method is the Runge-Kutta-Fehlberg Variable-Step method (RKF45)

3.3 COMPUTER ALGEBRAIC SYSTEM "MATLAB"

MATLAB (an abbreviation of MATrix LABoratory) is a computer algebraic package, registered trademark of computer software, now at version 7.10 (release R2010a) developed by the Math Works Inc. The software is widely used in many of science and engineering fields. MATLAB is an interactive program for numerical computation and data visualization. MATLAB is supported on Unix, Macintosh and

Windows environments. MATLAB integrates mathematical computing, visualization, and a powerful language to provide a flexible environment for technical computing. The open architecture makes it easy to use MATLAB and its companion products to explore data, create algorithms and create custom tools, that provide early insights and competitive advantages. Known for its highly optimized matrix and vector calculations, MATLAB offers an intuitive language for expressing problems and their solutions both mathematically and visually. Typical uses include:

- Numeric computation and algorithm development.
- Symbolic computation (with the built-in Symbolic Math functions).
- Modeling, simulation and prototyping.
- Data analysis and signal processing.
- Engineering graphics and scientific visualization.

MATLAB offers engineers, scientists, and mathematicians an intuitive language for expressing problems and their solutions mathematically and graphically. It integrates computation, visualization, and programming in a flexible, open environment. Complex numeric and symbolic problems can be solved in a fraction of the time required with other languages such as C, Fortran, or Java.

Simulink (Simulation and Link) is an extension of MATLAB by Mathworks Inc. It works with MATLAB to offer modeling, simulating, and analyzing of dynamical systems under a graphical user interface (GUI) environment. The construction of a model is simplified with click-and-drag mouse operations. Simulink includes a comprehensive block library of toolboxes for both linear and nonlinear analysis. Models are hierarchical, which allow using both top-down and bottom-up approaches. As Simulink is an integral part of MATLAB, it is easy to switch back and forth during the analysis process and thus, the user may take full advantage of features offered in both environments.

- A graphical, interactive software tool for modeling, simulating, and analyzing dynamic systems
- Enables rapid construction of "virtual prototypes" to explore design concepts at any level of detail with minimal effort
- Ideally suited to linear, nonlinear, continuous-time and discrete-time systems
- Commonly used in control system design, DSP design, communication system design, and other simulation applications
- A graphical plug-in for MATLAB®, offering additional access to a range of non-graphical analysis and design tools

Traditional approaches to system design typically include building a prototype followed by extensive testing and revision. This method can be both time-consuming and expensive. As an effective and widely accepted alternative, simulation is now the preferred approach to engineering design. Simulink is a powerful simulation software tool that enables you to quickly build and test virtual prototypes so that you can explore design concepts at any level of detail with minimal effort. By using Simulink to iterate and refine designs before building the first prototype, engineers can benefit from a faster, more efficient design process.

Simulink provides an interactive, block-diagram environment for modeling and simulating dynamic systems. It includes an extensive library of predefined blocks that you can use to build graphical models of your systems using drag-and-drop operations. Supported model types include linear, nonlinear, continuous-time, discrete-time, multirate, conditionally executed, and hybrid systems. Models can be grouped into hierarchies to create a simplified view of components or subsystems. High-level information is presented clearly and concisely, while detailed information is easily hidden in subsystems within the model hierarchy.

As shown in table 3.3 (a composite of a couple of tables in the MathWorks MATLAB Function Reference, Vol. 1), currently in MATLAB there are seven ODE solvers with varying speed of execution, accuracy and suitability for particular type of ODE. The ODE solvers in MATLAB fall, broadly, into two types: solvers for non-stiff and stiff problems. An (ODE) problem is said to be "stiff" when stability requirements force the solver to take a lot of small time steps, this happens when there is a system of coupled differential equations that have two or more very different scales of the independent variable over which integrating is done.

The solver **ode45** is based on an explicit Runge-Kutta (4,5) formula of the Dormand-Prince pair (Dormand and Prince, 1980). That means the numerical solver **ode45** combines a fourth order method and a fifth order method, both of which are similar to the classical fourth order Runge-Kutta (RK4) method discussed above. The modified RK varies the step size, choosing the step size at each step in an attempt to achieve the desired accuracy. Therefore, the solver **ode45** is suitable for a wide variety of initial value problems in practical applications. In general, **ode45** is the best function to apply as a "first try" for most problems.

Table 3.3: ODE solver types in MATLAB

Solver	Problem Type	Order of Accuracy	Mathematical Method	When to use
ode45	non-stiff	medium	Explicit Runge-Kutta (4,5) formulation.	Most of the time. **This should be the first solver to try.**
ode23	non-stiff	low	Explicit Runge-Kutta (2,3) formulation.	If using crude error tolerances or solving moderately stiff problems. It *may* be more efficient than ode45 at crude tolerances.
odell3	non-stiff	low to high	Variable order Adams-Bashforth-Moulton predictor-evaluate-corrector-evaluate (PECE) method.	If using stringent error tolerances or solving a computationally intensive ODE file. This one *may* be more efficient than ode45 at strict tolerances.
odel5s	stiff	low to medium	Variable order solver based on numerical differentiation formulas (NDF).	If ode45 is slow (stiff systems). Try this one if ode45 failed.
ode23s	stiff	low	Modified Rosenbrock formula of order 2.	If using crude error tolerances to solve stiff systems. This one may work at crude tolerance when ode 15s fails.
ode23t	moderately stiff	low	The trapezoidal rule using a "free" interpolant.	If the problem is only moderately stiff and you need a solution without numerical damping.
ode23tb	stiff	low	An implicit Runge-Kutta formula with first stage trapezoidal rule and second stage backward differentiation formula of order 2 (TR-BDF2).	If using crude error tolerances to solve stiff systems. This one may work when ode 15s failed at crude tolerances.

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The Dormand-Prince method (DOPRI) is a member of the Runge–Kutta family of ODE solvers. More specifically, it uses six function evaluations to calculate fourth- and fifth-order accurate solutions. The difference between these solutions is then taken to be the error of the (fourth-order) solution. This error estimate is very convenient for adaptive stepsize integration algorithms.

Other similar integration methods are Fehlberg (RKF) and Cash–Karp (RKCK). The Dormand–Prince method has seven stages, but it uses only six function evaluations per step because it has the FSAL (First Same As Last) property: the last stage is evaluated at the same point as the first stage of the next step.

Dormand and Prince choose the coefficients of their method to minimize the error of the fifth-order solution. This is the main difference with the Fehlberg method, which was constructed so that the fourth-order solution has a small error. For this reason, the Dormand–Prince method is more suitable when the higher-order solution is used to continue the integration, a practice known as local extrapolation (Shampine 1986; Hairer, Nørsett & Wanner 2008, pp.178–179).

Dormand–Prince is currently the default method in MATLAB and GNU Octave's ode45 solver and is the default choice for the Simulink's model explorer solver.

A Fortran free software implementation of the algorithm called DOPRI5 is also available at http://www.unige.ch/~hairer/prog/nonstiff/dopri5.f (http://en.wikip-edia.org/wiki/Dormand%E2%80%93Prince_method, accessed April 10th 2011).

In this thesis we attempt to apply the existing mathematical models of suspension bridges to the Adomi Bridge in Ghana. By the recommendations presented in Table 3.3 above, we perform numerical experiments and simulation on the mathematical model of the Adomi Bridge in the next chapter.

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CHAPTER FOUR

DATA COLLECTION AND ANALYSIS

4.1 MODEL PARAMETERS FOR ADOMI BRIDGE

The data required for modeling the oscillations of the Adomi Bridge is the physical properties of the materials which were used in constructing the Bridge. Also necessary is the detailed geometric configuration of the Bridge. The following information was gathered from a comprehensive engineering report (Scott and Adams, 1958) on the Adomi Bridge.

- The bridge is a two hinged latticed steel arched structure of 805 feet clear span, bearing on concrete abutments founded on rock on each bank of the river. The arch is of crescent form.
- The deck is suspended from the arch at 35 feet intervals by 2¼ inch high tensile steel cables, the cables consist of 127 wires, 0.164 inches diameter with a breaking stress of 100-110 tons/square inch before galvanizing
- The deck is of composite reinforced concrete and steel construction. The deck slab is made up of 23 reinforced concrete panels each 35 feet giving a total length of 805 feet.
- The carriageway has a width of 22 feet surfaced with a coat of mastic asphalt 1 inch thick. On each side are cantilevered footways of 4 feet 9inches wide. The footways have natural concrete finish with wooden floats, and protected by galvanized steel handrails with teak capping.
- The arch itself is 40 feet wide overall. The rise of the lower chord is 158 feet 6 inches above the hinges, and the overall depth of the truss is 32 feet at the centre.

- Weight of steel in main span is 880 tons, this is made up of 580 tons for the arch steel work and 300 tons for the deck steelwork. The total volume of concrete of the entire deck is 520 cubic yards ($\approx 400\ m^3$)

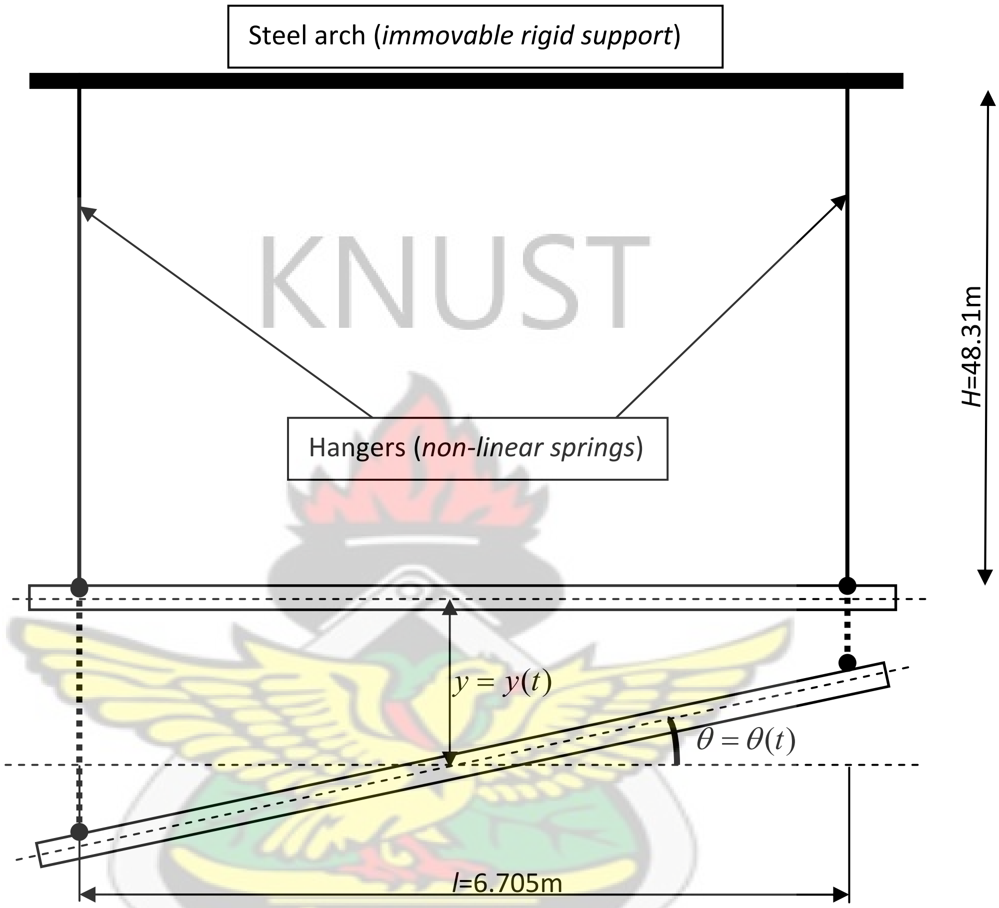

Figure 4.1: Mathematical model of cross section of Adomi Bridge

Figure 4.1 shows the mathematical model of the vertical and torsional motion of a cross section of the Adomi Bridge. The differential equations modeling the torsional and vertical motion of a suspension bridge was proposed by McKenna (1999) and derived in Chapter 3 as;

$$\ddot{\theta} = -\frac{6K}{m}\cos\theta\sin\theta - \delta_1\dot{\theta} + f(t) \qquad 4.1$$

$$\ddot{y} = -\frac{2Ky}{m} - \delta_2\dot{y} + g \qquad 4.2$$

The parameters needed are *m,* mass per unit length of the bridge deck. For the Adomi Bridge, this is evaluated as (from Engineering details of Bridge above).

$$m = (300 + 400 * 2.5 + 6.705 * 0.025 * 245.36 * 2.5 + 2 * 1.45 * 0.02 * 245.36 * 2.5) / 245.36$$
$$m = 5.862\, tons / m = 5862\, kg / m \approx 6,000 kg / m$$

This value of *m* fairly compares with bigger suspension bridges as listed in (Tajcová, 1997); Tacoma – 8,500 *kg/m*, Golden Gate – 31,000 *kg/m*, Bronx-Whitestone – 16,000 *kg/m*.

The real value of the stiffness of the cable stays *k* in our mathematical model cannot be easily determined. Based on observations during the collapse of the Tacoma Bridge, the value of *K* for the Bridge was approximated as 1,000 kg/s^2 per foot (0.3m).Thus stiffness $K = 3,333\, kgm^{-1}s^{-2}$ for the Tacoma Bridge. In this thesis we will investigate the mathematical model with vastly varying value of the stiffness *K* (between 1,000 and 300,000 $kgm^{-1}s^{-2}$).

The damping coefficients δ_1 and δ_2 also are not easily determined, again for the Tacoma Bridge a value of 0.01 was used in (McKenna, 1999), we also use same value of 0.01

In modeling the collapse of the Tacoma Bridge, the forcing function $f(t)$ was assumed to be sinusoidal with constant amplitude λ of form $f(t) = \lambda \sin \mu t$, the value of μ was chosen between 1.2 to 1.6, this was based on the fact that the frequency of motion of the bridge before the collapse was about 12 to 14 cycles per minute. The value of λ specified between 0.02 - 0.06 was so chosen, in order to induce oscillations of three degrees near equilibrium in the linear model (McKenna, 1999). In this thesis we use similar values for the forcing term as used for modeling the Tacoma Bridge. We also investigate the Adomi Bridge responses to different forcing term like periodic impulsive force, periodic random forces and the combination of these.

4.2 NUMERICAL EXPERIMENTS

4.2.1 Vertical Motion

Firstly we consider the vertical motion of the bridge which is the familiar forced harmonic oscillator.

$$\ddot{y} = -\frac{2Ky}{m} - \delta_2 \dot{y} + g \Leftrightarrow \ddot{y} + \delta_2 \dot{y} + \frac{2Ky}{m} = g \qquad 4.3$$

This is standard second order linear ordinary differential equation; with a known analytical solution:

$$y = e^{-\delta_2 t/2}\left(A\cos\left(\tfrac{1}{2}\sqrt{\tfrac{8k}{m} - \delta_2^2}\right)t + B\sin\left(\tfrac{1}{2}\sqrt{\tfrac{8k}{m} - \delta_2^2}\right)t\right) + \frac{mg}{2K} \qquad 4.4$$

The constants *A* and *B* are determined by the initial conditions (initial displacement and initial velocity of the mass). Due to the presence of damping (i.e., because of the $e^{-\delta_2 t/2}$ term), we point out that

$y(t) \to \frac{mg}{2K}$ as $t \to \infty$.

Therefore the long term response of this system is independent of the initial conditions and is driven entirely by the external forcing.

As we know the damping coefficient δ_2 is usually small (in our model we have settled on a value of 0.01) so the square of it can be neglected as compared to the value of $8K/m$ hence equation 4.4 simplifies to

$$y = e^{-\delta_2 t/2}\left(A\cos\left(\sqrt{\tfrac{2k}{m}}\right)t + B\sin\left(\sqrt{\tfrac{2k}{m}}\right)t\right) + \frac{mg}{2K} \qquad 4.5$$

Given $\delta_2 = 0.01,\ m = 6000,\ g = 10$ assuming that *K*=3000

$$y = e^{-0.005t}\left(A\cos t + B\sin t\right) + 10 \qquad 4.6$$

Considering initial condition of $y(0) = 14,\ \dot{y}(0) = 0$, we have $A = 4,\ B = 0.02$. Final solution is thus:

$$y(t) = e^{-0.005t}\left(4\cos t + 0.02\sin t\right) + 10$$

An initial condition of $y(0) = 10,\ \dot{y}(0) = 0$, yields $A = 0,\ B = 0$ which corresponds to equilibrium position of the bridge deck under its own weight. In this case equation is simply:

$$y(t) = 10 .$$

The original differential equation for the vertical motion after substituting parameters in equation 4.3 becomes

$$\ddot{y} + 0.01\dot{y} + y = 10 \qquad 4.7$$

Further on we use the MATLAB SIMULINK to simulate the numerical solution of the differential equation and compare it with the analytical solution to determine the accuracy of the numerical method.

Figure 4.2 shows the SIMULINK scheme and the numerical solution of the differential equation in form a graph of *y* (vertical displacement) against *t* (time), for *t* up to 1500 seconds. In table 4.1, we present the values of the solution of the differential equation analytically (in closed form) and numerically by SIMULINK over time ranging from *t*=0 to *t* =6000 at varying intervals. A comparison of the values shows very little error which confirms the accuracy of the chosen algorithm in the SIMULINK scheme as well as the scheme itself for the solution of the equation.

The solver used in the SIMULINK program is the variable step fourth and fifth order solver RK45, which as stated in Chapter 3 is based on the Dormand-Prince method (DOPRI), a member of the Runge–Kutta family of ODE solvers. The solver options in the program was set as follows; Type – Variable-step, solver – ode45 (Dormand-Prince), Max step size – auto, Min step size – auto, Initial step size

– auto, Relative tolerance – 1e-6, Absolute tolerance – auto, Shape preservation – Disable all.

The time taken to solve the equation on 64 bit core 2 duo laptop with 4gb of memory for a time up to t=6000 was 10 seconds, which for all purpose can be deemed to be fast enough.

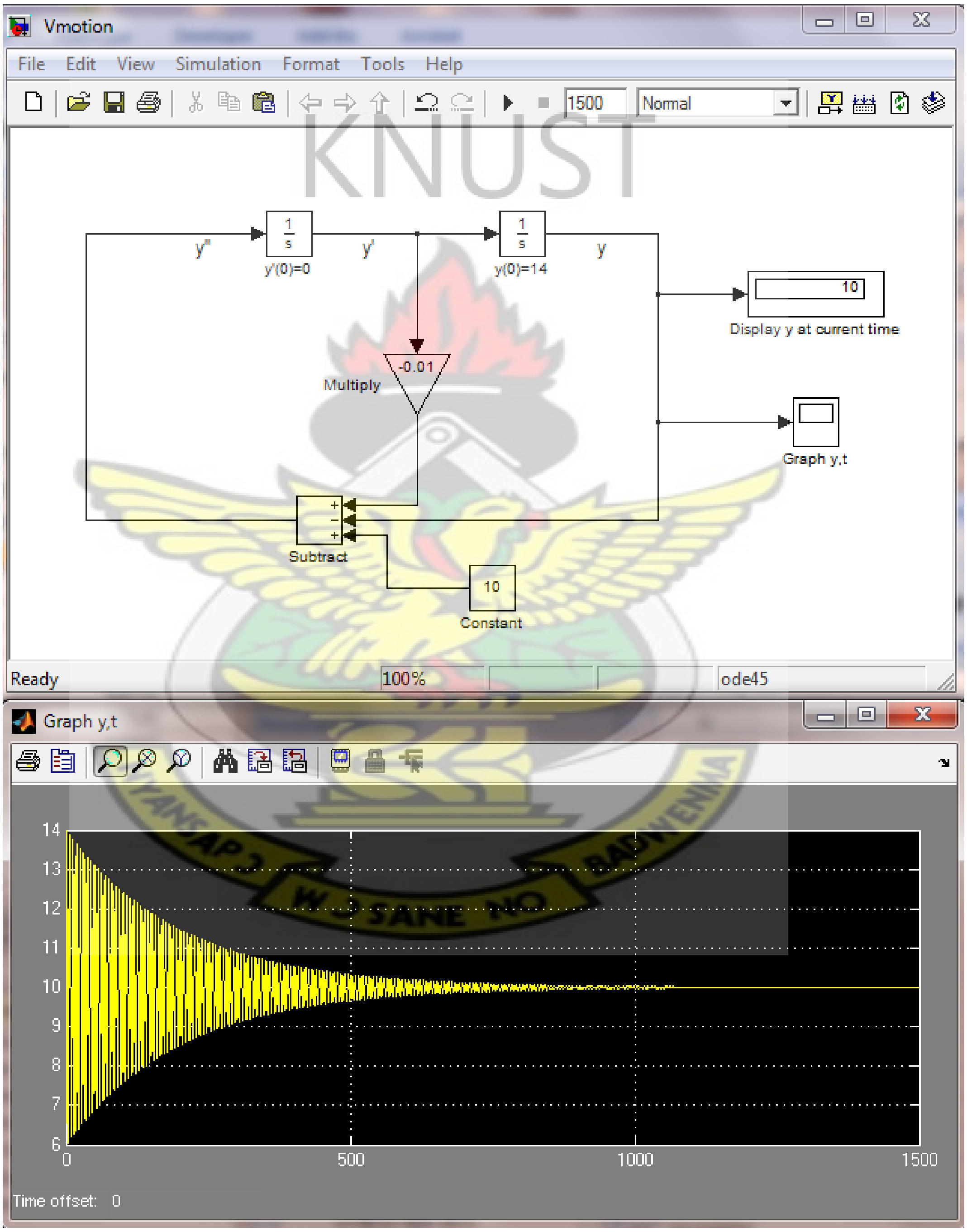

Figure 4.2: SIMULINK Scheme for vertical motion and the bridge response, y (0) =14

Table 4.1: Vertical motion results; numerical and analytical solution

Time	Numerical Solution SIMULINK	Analytical Solution (Closed form)	Absolute relative error
0	14.00000000	14.00000000	0.00000000
0.1	13.98002332	13.98002282	0.00000004
0.2	13.92031940	13.92031742	0.00000014
0.3	13.82152421	13.82151978	0.00000032
0.4	13.68466343	13.68465566	0.00000057
0.5	13.51114191	13.51112995	0.00000089
0.6	13.30272920	13.30271230	0.00000127
0.7	13.06154150	13.06151901	0.00000172
0.8	12.79002026	12.78999164	0.00000224
0.9	12.49090737	12.49087223	0.00000281
1	12.16721756	12.16717563	0.00000345
2	8.37007107	8.36998057	0.00001081
3	6.10178894	6.10176667	0.00000365
4	7.42221424	7.42236122	0.00001980
5	11.08769484	11.08792920	0.00002114
6	13.72166444	13.72174871	0.00000614
7	12.92479517	12.92457620	0.00001694
8	9.46021205	9.45983172	0.00004021
9	6.52390816	6.52372674	0.00002781
10	6.79679825	6.79705211	0.00003735
20	11.49433323	11.49351302	0.00007136
30	10.51277827	10.51405368	0.00012131
40	7.82926076	7.82803014	0.00015721
50	13.00144605	13.00197844	0.00004095
60	7.17257393	7.17322426	0.00009066
70	11.79796260	11.79607703	0.00015985
80	9.68804089	9.69069640	0.00027403
90	8.87115882	8.86858334	0.00029040
100	12.08438405	12.08594881	0.00012947
1000	10.01554304	10.01526855	0.00002741
1500	9.99970403	9.99974506	0.00000410
2000	9.99993837	9.99993411	0.00000043
3000	9.99999882	9.99999881	0.00000000
4000	9.99999999	9.99999999	0.00000000
5000	10.00000000	10.00000000	0.00000000
6000	10.00000000	10.00000000	0.00000000

For further verification Figure 4.3 show the SIMULINK scheme and the graph of y plotted against t for initial condition $y(0)=10$ and $y'(0)=0$. As expected the solution yields exactly $y(t)=10$

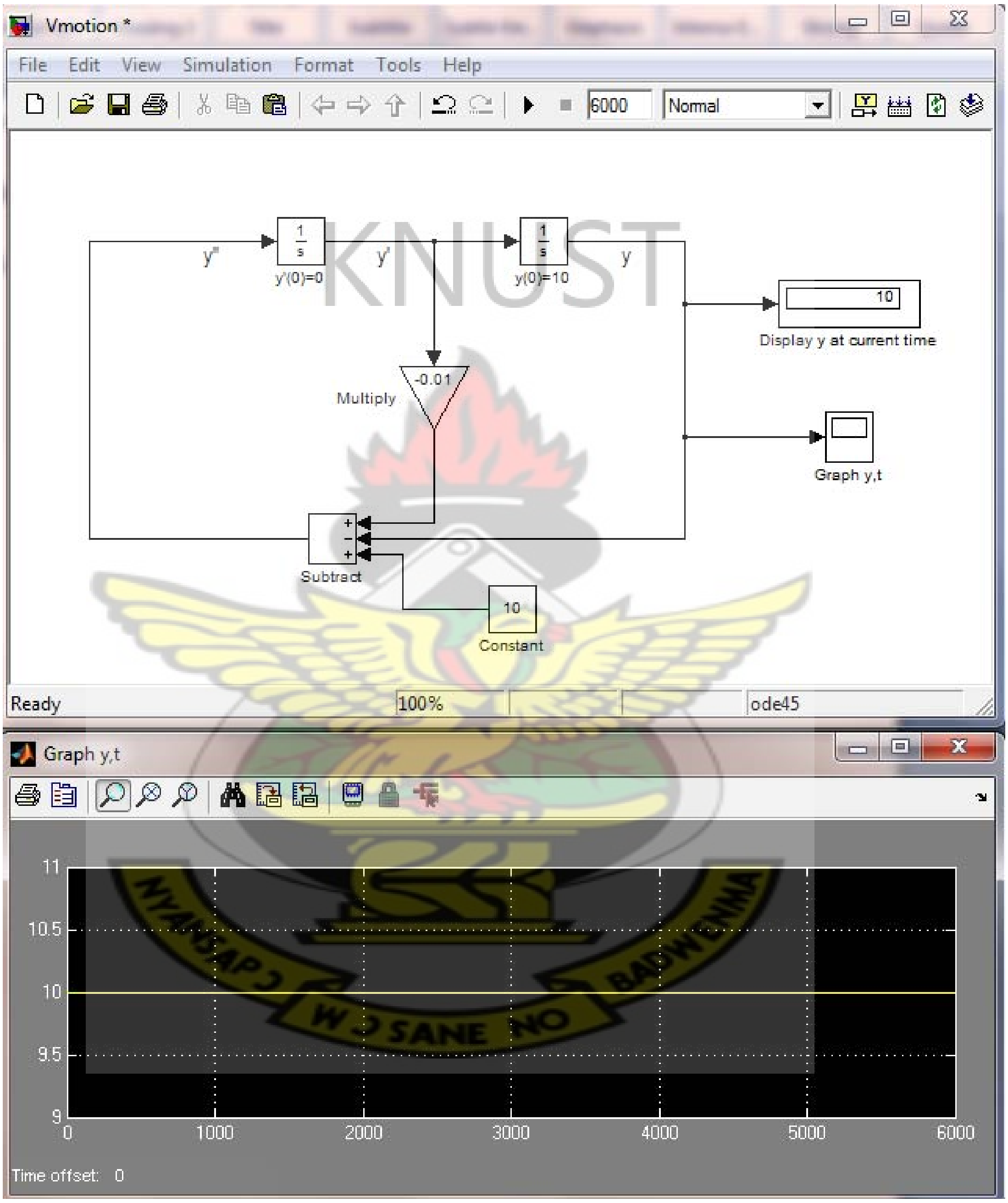

Figure 4.3: SIMULINK Scheme for vertical motion and the bridge response, y (0) =10

4.2.2 Torsional Motion

Now we consider the torsional motion of the bridge which is a non-linear second order differential equation of the form;

$$\ddot{\theta} = -\frac{6K}{m}\cos\theta\sin\theta - \delta_1\dot{\theta} + f(t) \Leftrightarrow \ddot{\theta} + \delta_1\dot{\theta} + \frac{6K}{m}\cos\theta\sin\theta = f(t) \qquad 4.8$$

Assuming we consider only small values of θ (an assumption engineers make for the motion of a bridge), then we can linearize equation 4.8 and rewrite it as

$$\ddot{\theta} + \delta_1\dot{\theta} + \frac{6K}{m}\theta = f(t) \qquad 4.9$$

Once again a forced harmonic oscillator with analytical solution of form

$$\theta(t) = e^{-\delta_1 t/2}\left(A\cos\left(\tfrac{1}{2}\sqrt{\tfrac{24k}{m} - \delta_1^2}\right)t + B\sin\left(\tfrac{1}{2}\sqrt{\tfrac{24k}{m} - \delta_1^2}\right)t\right) + \theta_p(t) \qquad 4.10$$

Where $\theta_p(t)$ is the particular solution dependent on forcing function $f(t)$.

A and *B* are constants determined by initial conditions $\theta(0)$ and $\dot{\theta}(0)$.

If the forcing function is assumed to be sinusoidal with small amplitude then no matter the initial conditions, the long-term behaviour of this linearized system will be sinusoidal with small amplitude signifying stability of the model and hence the stability of the suspension bridge regardless of the initial conditions.

We now proceed to investigate numerically the response of the non-linear system (equation 4.8), substituting $\delta_1 = 0.01, m = 6000$ in equation yields

$$\ddot{\theta} = -0.01\dot{\theta} - 0.001K\cos\theta\sin\theta + f(t) \qquad 4.11$$

Similarly as in the case of investigating the Tacoma Bridge (McKenna, 1999) we first of all simply consider the sinusoidal forcing function $f(t) = \lambda\sin\mu t = 0.05\sin 1.3t$

In our first trial we will verify our SIMULINK scheme and the accuracy of the ODE solver, as well as test the behaviour of the mathematical model. For this we choose K=0 which is equivalent to suspension bridge without cable stay! The result of this is obvious, the system just collapses.

Figure 4.4 shows the SIMULINK scheme for the differential equation (equation 4.11), figure 4.5 depicts the solution in form a graph of θ against t, for t up to 1800 seconds, as well as the phase portrait which is the graph of angular velocity $\dot{\theta}$ against torsional angle θ. The initial conditions considered are as in the case of the Tacoma Bridge; $\theta(0) = 1.2\,radians, \dot{\theta}(0) = 0$.The solver options in the program was set as follows; Type – Variable-step, solver – ode45 (Dormand-Prince), Max step size – auto, Min step size – auto, Initial step size – auto, Relative tolerance – 1e-9, Absolute tolerance – auto, Shape preservation – Disable all

The plots in figure 4.5 show results consistent with the expected outcome which is a total failure and unstable nature of a suspension bridge without cable stays! The torsional angle increases rapidly from the initial angle of 1.2 radians to over 3.142 radians (180 degrees) in 70 seconds. This means the bridge completely flips over! Signifying a complete destruction as expected. In our analysis of the results further on, any torsional angle exceeding 1.571 radians (90 degrees) will signify instability and ultimate failure.

The plots in figure 4.5 show the torsional angle settling around 5 radians in the long term. Of course the bridge would have long collapsed by then! The phase portrait in figure 4.5 shows that the angular velocity with an initial value of zero (0 rad/s) does not exceed 0.08rad/s over the entire period, meaning the system violently twists without accelerating which again indicate instant instability.

With K=0 and $f(t) = \lambda \sin \mu t = 0.05 \sin 1.3t$ equation 4.11 becomes

$$\ddot{\theta} + 0.01\dot{\theta} = 0.05 \sin 1.3t \qquad 4.12$$

Subject to initial conditions $\theta(0) = 1.2, \dot{\theta}(0) = 0$

The analytical solution of this equation is given as:

$$\theta(t) = 5.046154 - 3.845926e^{-0.01t} - 0.029584\sin(1.3t) - 0.000228\cos(1.3t)$$

In table 4.2, we present the values of the solution of the differential equation analytically (in closed form) and numerically by SIMULINK over time ranging from t=0 to t =1800 at varying intervals. A comparison of the values shows very little error which confirms the suitability of the SIMULINK scheme as well as the accuracy of the chosen algorithm in the SIMULINK program for the solution of the equation.

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Table 4.2: Trial torsional motion results; numerical and analytical solution

Time	Numerical Solution SIMULINK	Analytical Solution (Closed form)	Absolute relative error
0	1.20000000	1.20000000	0.00000000
0.1	1.20001083	1.20001082	0.00000001
0.2	1.20008636	1.20008633	0.00000002
0.3	1.20029012	1.20029007	0.00000004
0.4	1.20068341	1.20068333	0.00000007
0.5	1.20132429	1.20132418	0.00000010
1	1.20992871	1.20992835	0.00000030
2	1.26132722	1.26132643	0.00000062
3	1.33440468	1.33440403	0.00000048
4	1.37705822	1.37705808	0.00000010
5	1.38120925	1.38120927	0.00000001
10	1.55357913	1.55357914	0.00000001
20	1.87466908	1.87466910	0.00000001
30	2.16844823	2.16844832	0.00000004
40	2.43900148	2.43900142	0.00000003
50	2.68914931	2.68914928	0.00000001
100	3.65891689	3.65891735	0.00000013
200	4.50561804	4.50561635	0.00000037
400	5.00521705	5.00521440	0.00000053
1000	5.06296764	5.06296848	0.00000017
1200	5.01717679	5.01717563	0.00000023
1500	5.02259121	5.02258948	0.00000034
1800	5.03252323	5.03252001	0.00000064

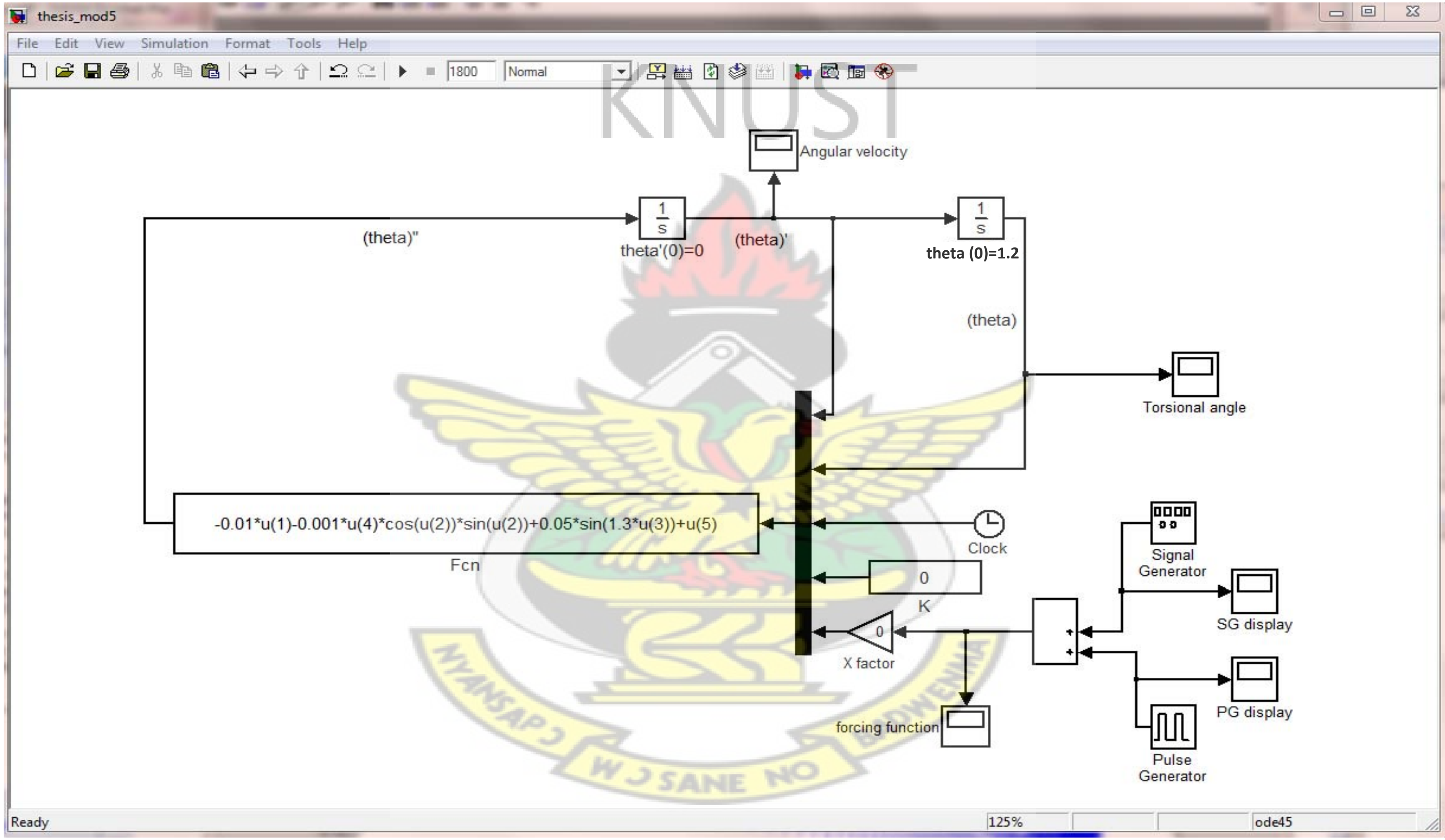

Figure 4.4: SIMULINK Scheme for torsional motion of bridge deck

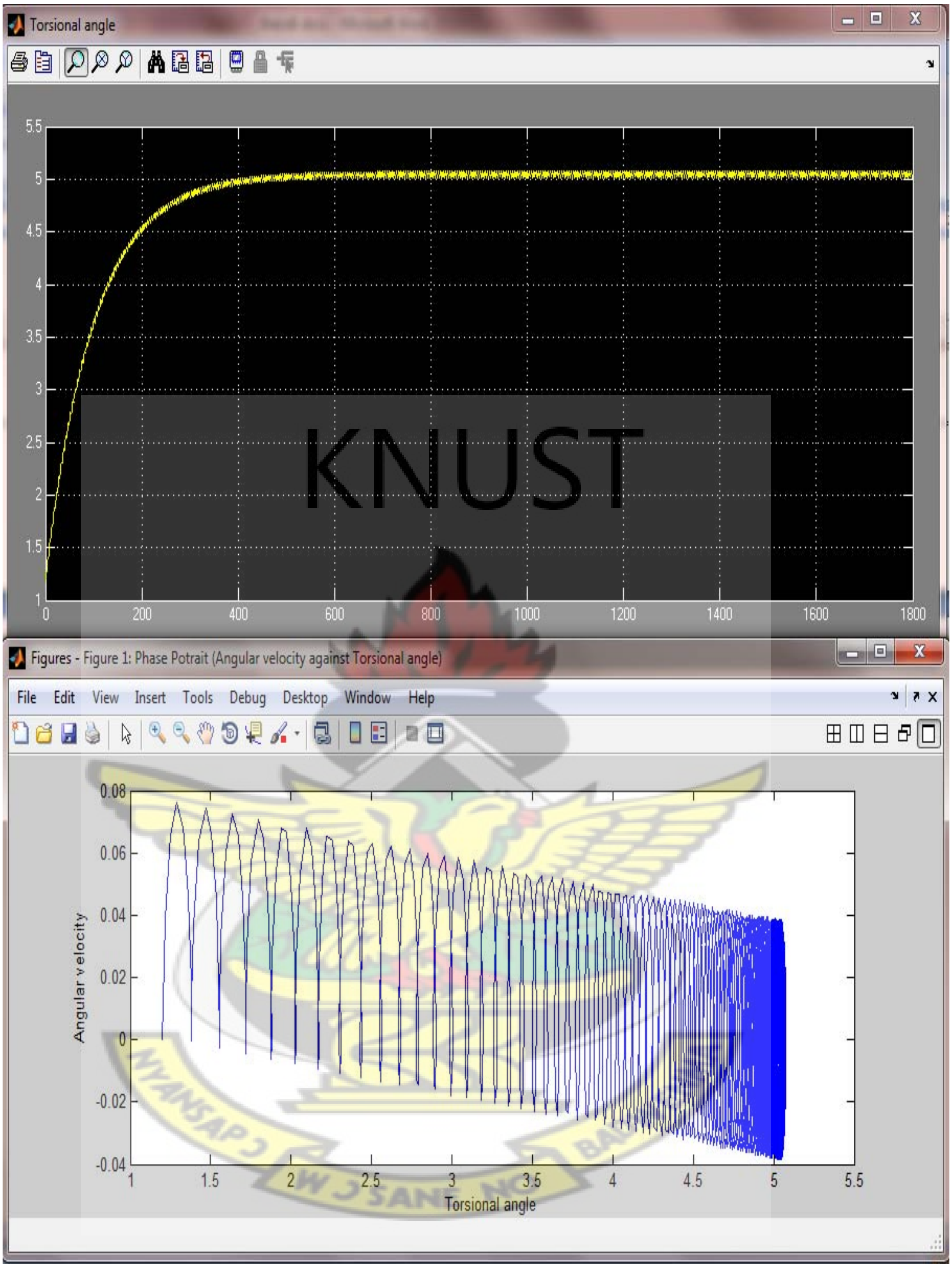

Figure 4.5: Torsional motion; Bridge response for *K*=0, *f(t)*=0.05sin1.3t and Phase potrait

4.2.3 Numerical result for torsional motion

In the mathematical model investigated by McKenna (1999) the forcing term was restricted to a sinusoidal form $f(t)=\lambda \sin \mu t$ which understandably does not accurately depict the nature of the forces acting the bridge. The forces acting on the bridge is of varying (random) nature and includes forces due to wind, earthquakes, hurricanes, dynamic impacts loads from vehicles etc. In this thesis, apart from the sinusoidal forcing term, additional forcing term from a signal generator (SG) and pulse generator (PG) available in the SIMULINK program are considered. These forces though periodic are more realistic and can simulate some of the actual forces acting on the bridge. Figure 4.6, figure 4.7 and figure 4.8 shows respectively the nature of the force SG, PG and the sum of the two referred to as forcing function (FF). FF is feed into the system after multiplication by the factor in X factor block (see SIMULINK scheme in figure 4.4)

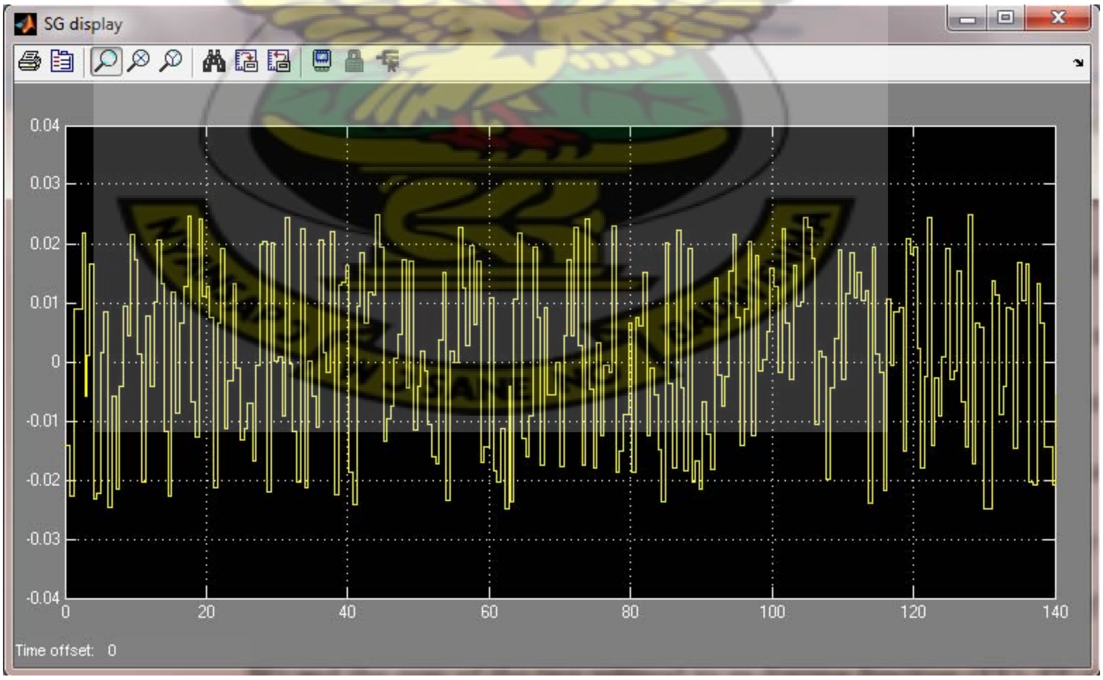

Figure 4.6: Signal generator; generates random periodic forces, amplitude = 0.025, frequency = 1.0 rad/sec

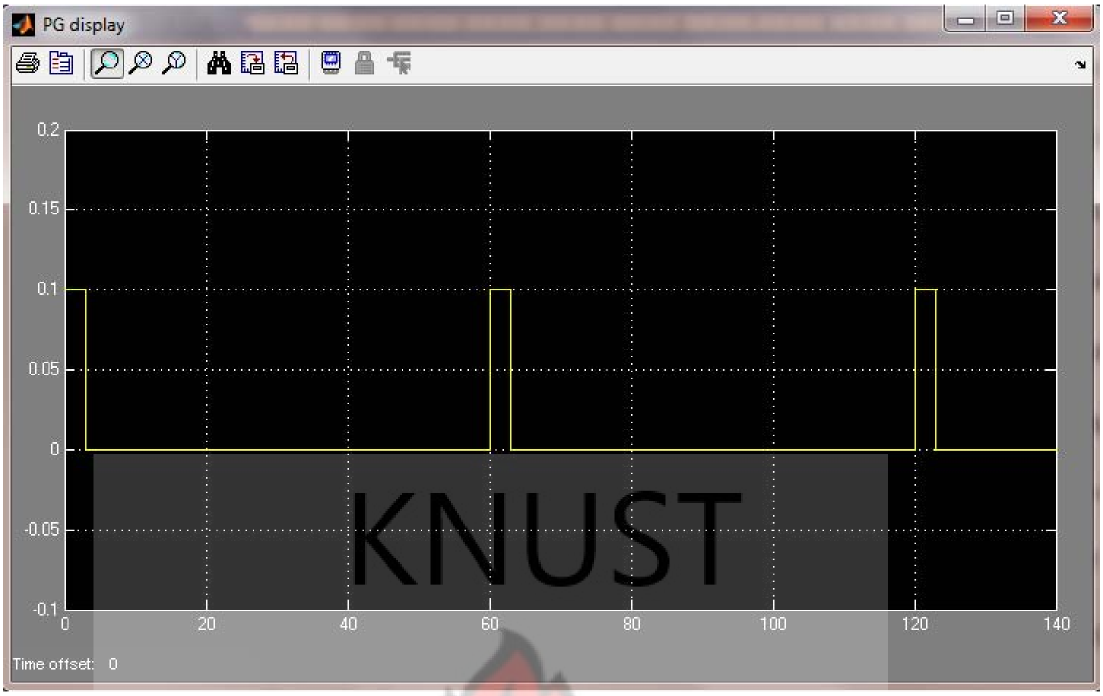

Figure 4.7: Pulse generator; generates random impulse forces, amplitude = 0.10, period = 60 seconds, pulse width = 3seconds

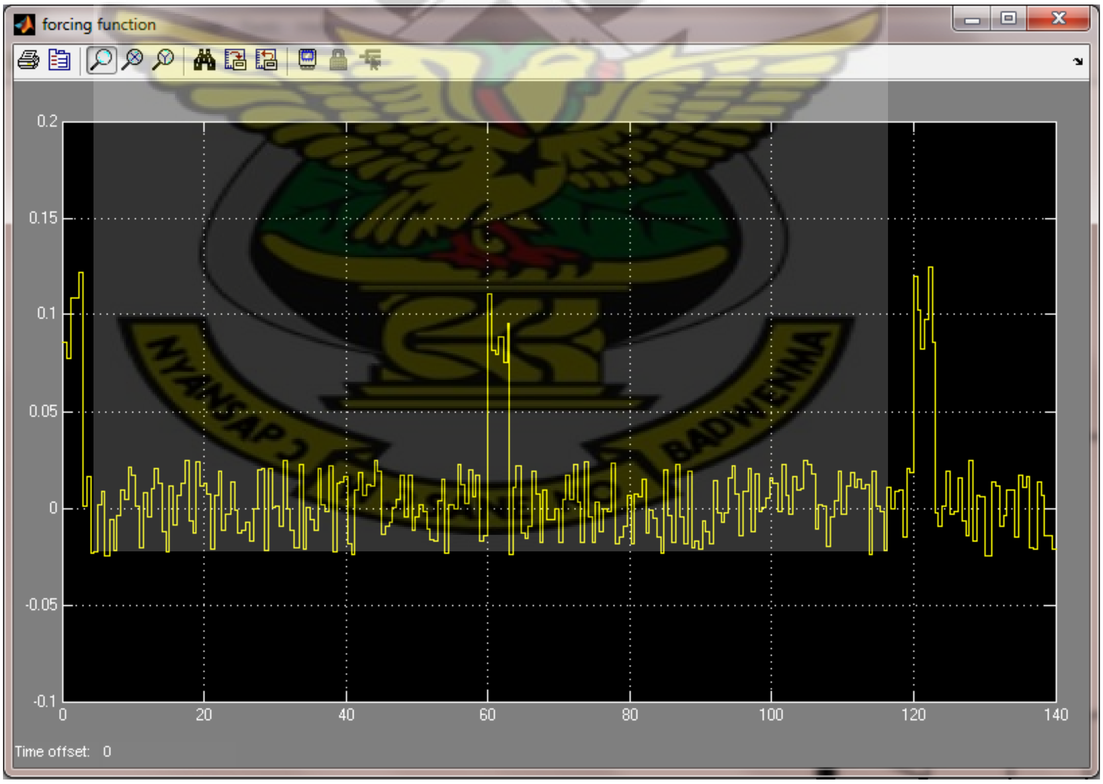

Figure 4.8: Combination of the Pulse generator and Signal generator

In this section we perform numerical experiment by using the SIMULINK scheme in figure 4.4, we vary the K values which corresponds to changing the stiffness of the cable stays. We also use different values for the X factor block which correspond to varying the forcing function acting on the bridge. The forcing function in McKenna (1999) which is $f(t) = 0.05\sin 1.3t$ is left unaltered throughout the whole set of experiment. All the simulations are performed for the period t=0 to t =*3600 secs*

- **Experiment 4.1:** K=1,000 , X factor = 0. This corresponds to stiffness of cable stays equals 1,000 $kgm^{-1}s^{-2}$ and $f(t) = 0.05\sin 1.3t$ as the only forcing function acting on the system. The results of this experiments is shown in figure 4.9, which is a graph of torsional angle (angle of rotation of the deck) in radians to time t in seconds. Figure 4.10 is a phase portrait which is a plot of angular velocity ($\dot{\theta}$) against torsional angle (θ)

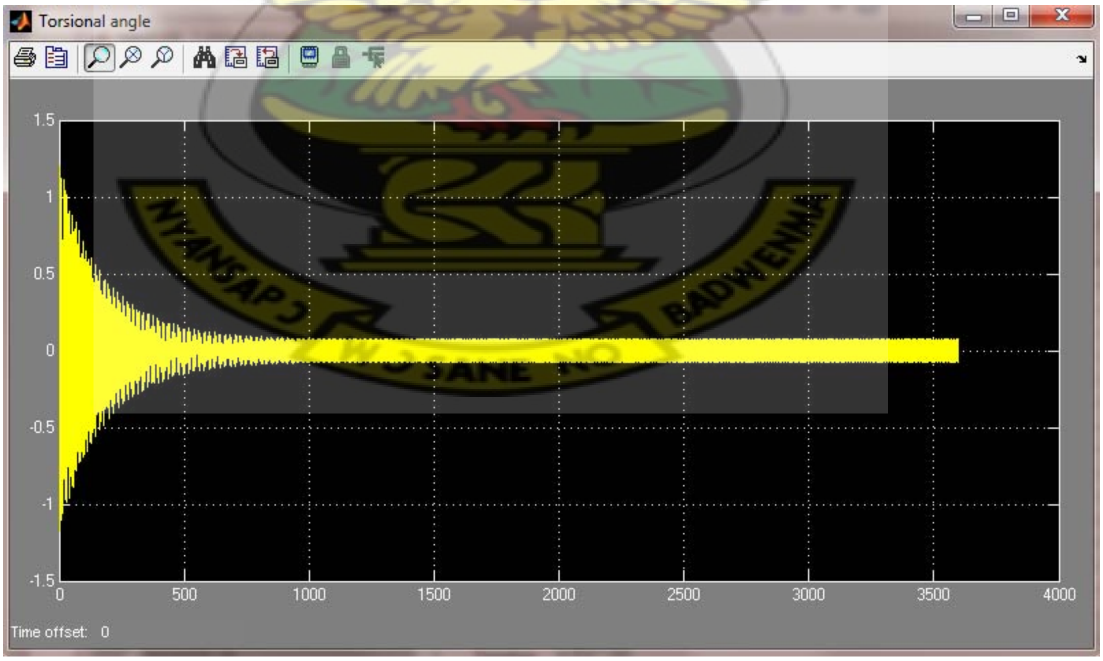

Figure 4.9: Experiment 4.1; Bridge response (Stable)

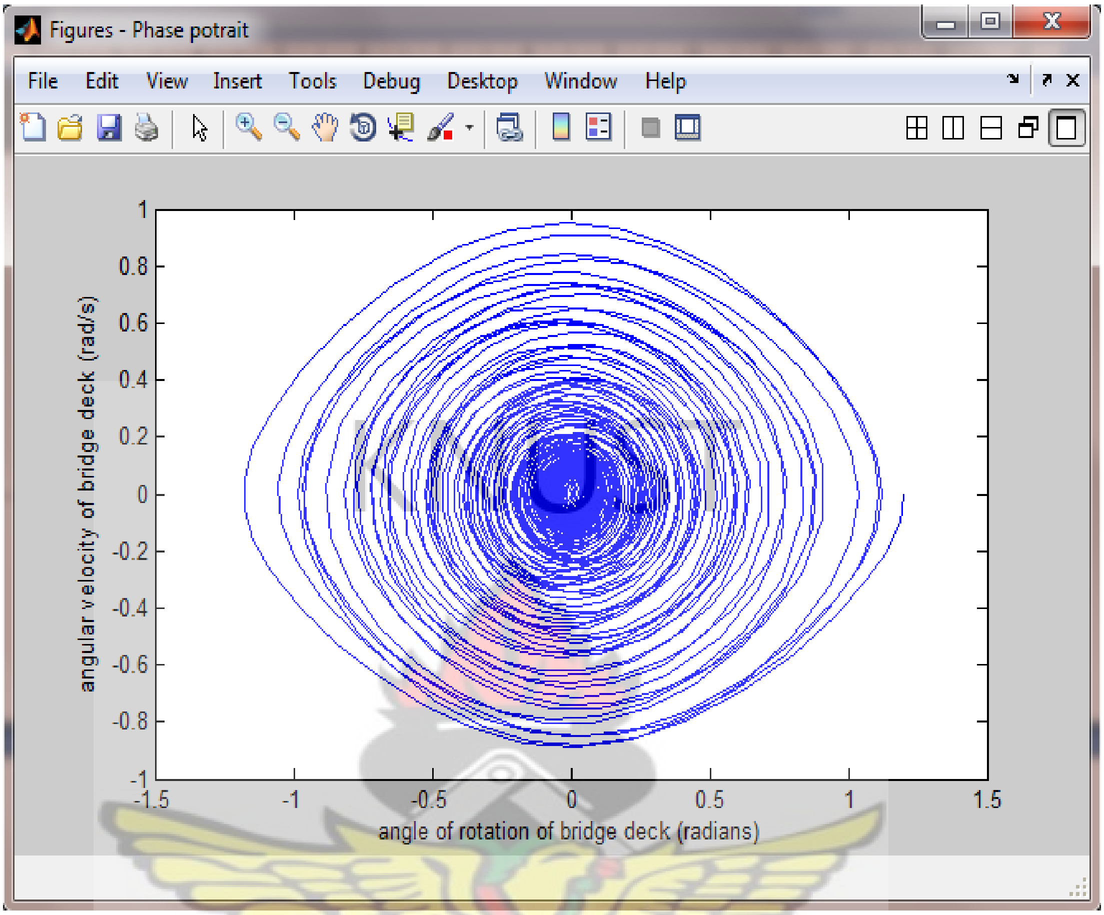

Figure 4.10: Experiment 4.1; Phase portrait (Spiral sink)

The plot in figure 4.9 indicates that, the amplitude of the oscillations of the bridge **subsides**, the peak value of torsional angle in the region close to of the end of the period (3600 seconds) is about 0.07 radians (4 degrees). The phase portrait of the system shown in figure 4.10 is that of a **spiral sink**. Here we observe that the long term behaviour of the bridge as **stable**.

- **Experiment 4.2:** K=2,400 , X factor = 0. This experiment corresponds to the system used to model the Tacoma Bridge collapse in McKenna (1999). The plot of torsional angle against t is shown in figure 4.11 and the phase portrait in figure 4.12

The plot in figure 4.11 indicates that, the amplitude of the oscillations of the bridge is **sustained**, the peak value of torsional angle in the region close to of the end of the period (3600 seconds) is about 0.8 radians (45 degrees). The phase portrait of the system shown in figure 4.12 is that of a **Limit cycle**. Here we observe the long term behaviour of the bridge as **Unstable,** which leads to ultimate failure (collapse).This was how Lazer and McKenna explained the reason for the collapse of the Tacoma Bridge.

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Figure 4.11: Experiment 4.2; Bridge response (Unstable)

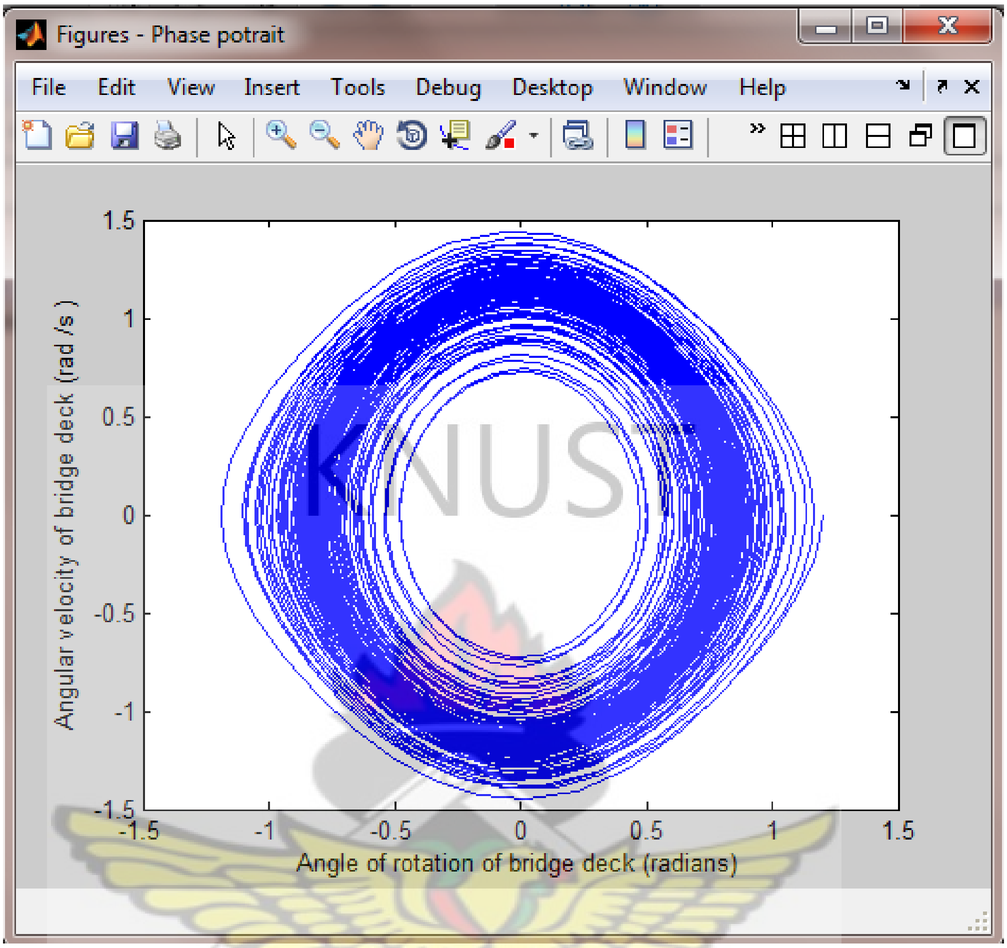

Figure 4.12: Experiment 4.2; Phase portrait (limit cycle)

- **Experiment 4.3:** *K*=100,000, X factor = 0. This experiment is equivalent to modeling the oscillations of a stiff bridge for example the Adomi Bridge. The plot of torsional angle against *t* is shown in figure 4.13 and the phase portrait in figure 4.14

The plot in figure 4.13 indicates that, the amplitude of the oscillations of the bridge **rapidly subsides**, the peak value of torsional angle in the region close to of the end of the period (3600 seconds) is about 0.0005 radians (0.03 degrees). The phase portrait of the system shown in figure 4.14 is that of a **spiral sink**. Here we observe that the long term behaviour of the bridge as **very stable**.

From the results of the three experiments, it can be seen that with only a sinusoidal forcing term acting on the bridge, a cable stay with K=1000 is stable and will withstand the initial large torsional angle, whilst that with K=2400 is unstable and will collapse. This is an unexpected result and is a kind of paradox. Such a result led Lazer, McKenna and other researchers to conclude that making the cable stay of suspension bridges stiffer does not always make it less prone to large oscillations.

Using the SIMULINK scheme in figure 4.4, additional numerical experiments are conducted (K between 1,000 and 300,000, X factor between 0 and 50) results of which are not included in this thesis. Conclusions drawn from these numerical experiments are discussed in chapter 5.

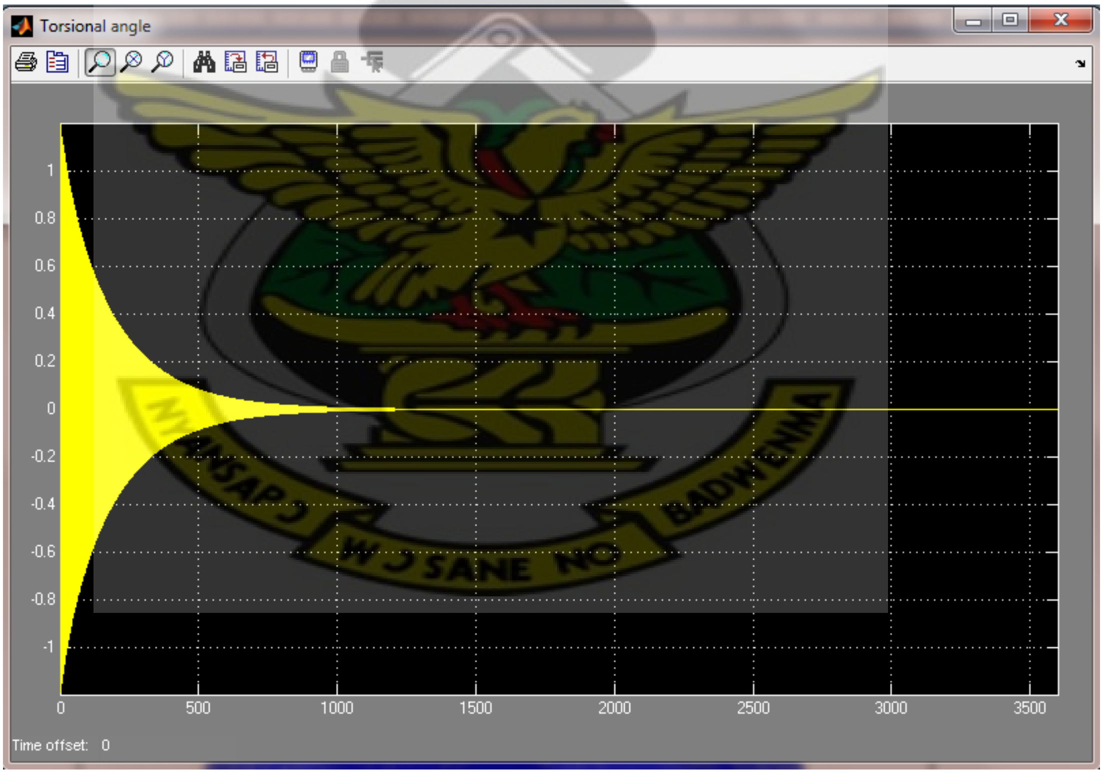

Figure 4.13: Experiment 4.3; Bridge response (Very stable)

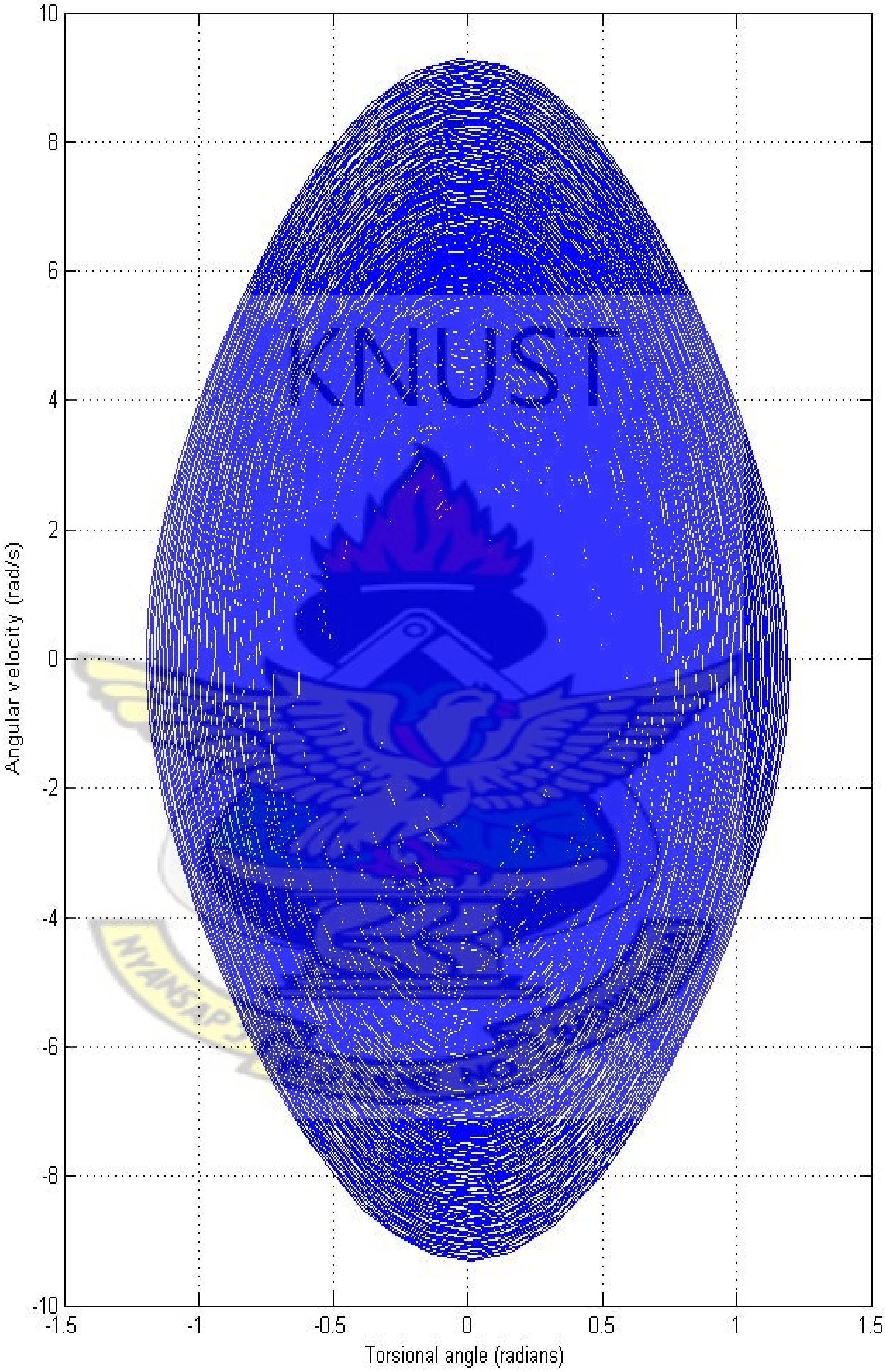

Figure 4.14: Experiment 4.2; Phase portrait (Spiral sink)

4.3 ADOMI BRIDGE RESULTS

As stated earlier on, Adomi Bridge is not truly a suspension bridge in a traditional sense. This is because the cable stays are connected to a rigid steel truss arches instead of being connected to another "vibrating flexible" main cable. This makes the Bridge very rigid and as a result, there are no noticeable oscillations under normal operating conditions. The cable stays of the Bridge are subjected to only small deformations, thus Hooke's law is applicable, and a good estimate of the stiffness of the cable stay is given by

$$K = \alpha \frac{AE}{ld}$$

α is coefficient that account for cable fatigue and imperfections (0.5)

A is effective cross sectional area of cable stay,

E is Young modulus of material used for the cable stay (steel -2 x10^{11} Nm^{-2}) and

l is length of the cable. (48.2 m)

d is the spacing between the cable stays (10.7 m)

For the Adomi Bridge, the value of K evaluated this way gives approximately

$K \approx 300{,}000\ kgm^{-1}s^{-2}$

For such values of K, and an X factor value set at 50, the response of the Bridge is similar to figure 4.13 of experiment 4.3. This indicates that, the Adomi Bridge is not affected by large torsional oscillations and any initial oscillation started under any condition will **quickly subside**.

CHAPTER FIVE

CONCLUSIONS AND RECOMMENDATIONS

From the various numerical experiments performed using the SIMULINK scheme in chapter 4, it was observed that, at a constant mass m of the deck of the bridge, if other small random or impulsive forcing terms are considered in addition to the sinusoidal force, then increasing the stiffness K of the cable stays of the suspension bridge always results in a more stable response to the initial torsional angle. This is a likely result, so we conclude that, it is certainly incorrect to consider only a sinusoidal forcing term as in the mathematical model of Lazer – McKenna which led to some paradoxical results discussed in chapter 4.

Keeping in mind that the magnitude of the non-linear term ($\sin\theta\cos\theta$) in the equation for the torsional motion (equation 4.1) is proportional to K/m (the ratio of the cable's spring constant (stiffness) to the mass of the roadbed). We expect then that by increasing m at a fixed value of K, we reduce the effect of the nonlinearity and therefore better control the oscillation of the roadbed.

We conclude that for steel arched-suspension bridge similar to the Adomi Bridge, their rigidity makes them withstand any form of large amplitude oscillations.

A major inadequacy of the dissertation is the inherent over simplification of the model adopted to represent the suspension bridge. Only a typical cross section at the centre of the Bridge's span is taken into account for the derivation of the system of non-linear differential equations.

We recommend a more accurate model which should take into consideration the full length of the Bridge. This will result in a system of non-linear partial differential equation as the model instead of the current system of non-linear ordinary differential equation.

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