# ON THE STUDY OF TOPOLOGICAL DYNAMICAL SYSTEMS

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THESIS

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# Declaration

I hereby declare that this submission is my own work towards the Doctor of Philosophy (PhD) in Pure Mathematics and that, to the best of my knowledge, it contains no material previously published by another person nor material which has been accepted for the award of any other degree of the University, except where due acknowledgement has been made in the text.

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# Dedication

This research work is dedicated to my Supervisors, Prof. I. K. Dontwi, Dean, Institute of Distance Learning, Kwame Nkrumah University of Science and Technology (K.N.U.S.T.) and Dr. S. A. Opoku, Senior Lecturer, Department of Mathematics, K.N.U.S.T

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# Abstract

The purpose of the study was to apply Topological Dynamics to Integral Equations. Topological Dynamical techniques were used to analyse it and confirmed the results. Sell developed methods which allowed one to apply the theory of topological dynamics to a very general class of nonautonomous ordinary differential equations. This was extended to nonlinear Volterra's Integral Equations. This research took off from there and applied the techniques of topological dynamics to an integral equation. The usage of limiting equations which were used by Sell on his application to integral equations were extended to recurrent motions and then studied the solution path. It thus confirmed the existence of contraction and the stationary point in the said paper. The study of Dynamical Systems of Shifts in the space of piece-wise continuous functions analogue to the known Bebutov system was embarked upon. The stability in the sense of Poisson discontinuous function was shown. It was proved that a fixed discontinuous function, $f$, is discontinuous for all its shifts, $\tau$, whereas the trajectory of discontinuous function is not a compact set. The study contributes to literature by providing notions of Topological Dynamic techniques which were used to analyse and confirm the existence and contractions and the stationary points of a special Integral Equation.

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# Chapter 1

# INTRODUCTION

## 1.1 Background

Topological dynamics was used to be called the qualitative theory of differential equations. In this scenario the problem of boundedness (Dontwi, 1988A, 1988B), periodicity, almost periodicity, stability in the sense of Poisson, and the problem of the existence of limit regimes of different types, convergence, dissipative are considered by Cheban (2009). The direction of the work is in the neighbourhood of asymptotic (Dontwi, 1990) stability in the sense of Poisson motions of dynamical systems and solutions of differential equations. Sell (1971) applied topological dynamics to differential and integral equations.

It is worthy to mention that a lot of authors have worked on the problem of asymptotical stability in the sense of Poisson. Initially, the concept of asymptotically almost periodicity of functions was launched in the works of Frechet (1941A, 1941B) . Soon after these results were generalized for asymptotically almost periodic sequences in the research work of Fan (1943) and Precupanu (1969) and for abstract asymptotically almost periodic functions in the works of Ararktsyan (1988), Precupanu (1969), Khaled (1983) and Cioranescu (1989).

Among other contributors include Dontwi (1988, 1989, 1990), Mambriani et al, (1971).

The most general and somewhat vague notion of a dynamical system includes the following ingredients:

A “phase space” **X**, whose elements or “points” represent possible states of the system.

“Time”, which may be discrete or continuous. It may extend either only into the future which may be termed irreversible or noninvertible Processes or into the past as well as the future and in other words, reversible or invertible processes. The sequence of time moments for a reversible discrete-time process is in a natural correspondence to the set of all integers; irreversibility corresponds to considering only nonnegative integers. Similarly, for a continuous-time process, time is represented by the set of all real numbers in the reversible case and by the set of nonnegative real numbers for the irreversible case (Katok and Hasselblatt,1995 ).

A *dynamical system* is a rule for time evolution on a *state space*. A dynamical system consists of an abstract *phase space* or state space, whose coordinates describe the state at any instant, and a dynamical rule that specifies the immediate future of all state variables, given only the present values of those same state variables. For example the state of a pendulum is its angle and angular velocity, and the evolution rule is Newton’s equation $F = ma$. Dynamical systems (2011).

Mathematically, a dynamical system is described by an *initial value problem*. The implication is that there is a notion of *time* and that a state at one time evolves to a state or possibly a collection of states at a later time. Thus states can be ordered by time, and time can be thought of as a single quantity.

Dynamical systems are deterministic if there is a unique consequent to every state, or stochastic or random if there is a probability distribution of possible consequents i.e., the idealized coin toss has two consequents with equal probability for each initial state.

### 1.1.1 The Time-Evolution Law

In the most general setting this is a rule that allows us to determine the state of the system at each moment of time *t* from its states at all previous times. Thus, the most general time-evolution law is time dependent and has infinite memory.

Different structures give rise to theories dealing with dynamical systems that preserve those structures. The most important of those theories are: Erogodic Theory, Topological Dynamics and Hamiltonian (Symplectic) Dynamics.

### 1.1.2 Ergodic Theory

Here the phase space **X** is a “good” measure space, that is, a Lebesgue space with a finite measure .

The origins of ergodic theory go back to the famous ergodic hypothesis of Boltzmann who postulated equality of time averages and space averages for systems in statistical mechanics. Within mathematics the notions of ergodic theory arose from the study of uniform distributions of sequences (Katok and Hasselblatt, 1995).

### 1.1.3 Topological Dynamics

The phase space in this theory is a good topological space, usually a metrizable compact or locally compact space. Topological dynamics concerns itself with groups of homeomorphisms and semi groups of continuous transformations of such spaces. Sometimes these objects are called topological dynamical systems. Application of topological strands are culled from Kelley, 1955.

Abstract topological dynamics is usually developed in the context of flows.

### 1.1.4 Hamiltonian or Symplectic Dynamics

This theory is a natural generalization of a study of differential equations of classical mechanics. The phase space here is an even-dimensional smooth manifold with a nondegenerate closed differential 2-form Ω.

The origin of Hamiltonian dynamics as an object of study from the point of view of dynamical systems is largely in the questions of celestial mechanics. Again Poincare introduced the fundamental approach of the qualitative study of the *n*-body problem.

## 1.2 Problem Definition

Weak asymptotically almost periodic solutions of linear differential equations and their perturbations was a paper that treated functions which are almost periodic in the sense of Frech`*e* (Dontwi, 2005).

Lots of works in diverse ways have considered the area of dynamical systems which culminated in the usage of topological methods. The interest in the study of Differential Equations with Impulse is increasing. Attempts to extend this study (Dontwi, 1994) to known topological methods of the Theory of Dynamical Systems (DS)(Sibiriskii 1970, Levitan and Zhikov 1982, Shcherbakov 1972 and 1975, Cheban 1977 and 1986) brought into fore the necessity of studying Dynamical Systems of shifts in the space of piece-wise-continuous functions which are solutions to some of these equations.

Introduction and application of notions of Recurrence motions of dynamical systems (Shcherbakov 1972, Bronshtein 1979, Levitan and Zhikov 1982, Pliss 1966, Sacker and Sell 1994) to various trajectories of Differental Equations with Impulse (Distributions) can be found in Hale (1977), Cheban (1999) and (2001), Dontwi (1994) and (2001).

The idea of minimal set is centrally located in topological dynamics. The first paper of Birkhoff was published in 1912 and developments of Topological Dynamicals appeared in journals. Matters related to general transformation groups can be found in Gottschalk and Hedlund (1955). The current topic employs the usage of recurrence (Gottschalk and Hedlund 1995, Nemyckii and Stepanov 1959, Floyd 1949) that stated that some minimal sets are homogeneous.

In his paper, Sell (1967 A) throws more light on the fact that topological dynamics could be applied to autonomous differential equations. In Sell (1967 B) asymptotic behaviour of solutions of nonautonomous differential equations were considered. Truly, humankind has been astonished by time-periodic, quasi-periodic, almost periodic and recurrent motions for centuries (Bongolan-Walsh, 2003). These motions have been observed in the solar system, that is, the earth rotates around the sun (Bongolan-Walsh, 2003). Limit points and limit sets play salient roles in topological dynamics (Katok and Hasselblatt, 1995). Topological dynamical system and recurrence in compact sets can be found in Furstenberg (1981). Topological entropy was done by Katok and Hasselblatt, (1995). Entropy as a dimension was treated in Pesin and Pitskel, (1984). Symbolic dynamical systems related to n-shifts were introduced in Lind and Marcus (1995). Alseda, *et al* (1993) covers one-dimensional topological dynamics to a very large extent. Flows and homeomorphisms on surfaces can be found in Nikolaev and Zhuzhoma (1999). Limiting equations and Lagrange stability were considered in Cheban (2009).

In Sell (1967 A and B), methods were developed which allowed one to apply the theory of topological dynamics to a very general class of nonautonomous ordinary differential equations. This was extended to non-linear Volterra's Integral Equations (Sell,1971).

This research takes off from here and we apply the techniques of topological dynamics to an integral equation in Dontwi (2005). Here we employed the usage of limiting equations which were used by Sell (1971) in the application to integral equations. In our bid to lend new innovations to our system we go further to apply recurrence motions to our systems and then to study the solution path.

Ergodic dynamical system on the finite measure space and its kronecker factor were considered in Assani (2004A). Pointwise convergence of ergodic averages along cubes was proved in Assani (2010). In Assani and Mauldin (2005A), negative solution to counting problem for measure preserving transformation was carried out. Full measures were treated in Assani (2004B). A question of H. Furstenburg on the pointwise convergence of the averages was answered in Assani (2005B). The pointwise convergence of some weighted averages linked to averages along cubes were studied in Assani (2007A). Two questions related to the strongly continuous semigroup were answered in Assani and Lin (2007B). Characteristics for certain nonconventional averages were studied in Assani and Presser (2011).Differentiable or smooth instead of topological gives a description of Differentiable Dynamics (de Vries, 2010; Shub and Smale, 1972; Smale, 1967; Smale, 1980).

The objects of study are integral equations taking our cue from periodic functions, almost periodic functions, asymptotic functions, asymptotic almost periodic functions, and weak asymptotically almost periodic functions.

## 1.3 Research Aim and Significance

The aim of this research is to identify various ways of establishing flows (and/or semiflows) and making it accessible to academia and users of dynamical systems portrayed in this write-up. This would represent an authentic contribution to knowledge.

## 1.4 Objectives

Basically, the objectives of the study are :

1. Use limiting equations on an Integral Equation in Dontwi (2005) based on topological dynamics and extended it to recurrence motions.

2. To identify types of family of maps

3. To identify ways of constructing flows and / or semiflows

4. To develop a method which can be applied to the Navier-Stokes equations in the sense of Volterra integral equations .

## 1.5 Study Design

The study design for addressing objective 1 and objective 3 will be multifaceted in dimension while objective 2 and objective 4 will be treated to an extent. It would use a wide range of resources covered by authors in the said field and furnish ourselves with the techniques used by them. The research would employ the usage of Matlab approach to come out with ways to draw curves for analysis.

## 1.6 Scope

The research will involve a study repertoire of works in the area under study recently. Papers and resources written by academicians would be consulted. Knowledge on Matlab would be of imperative advantage. Well established textbooks and resources in the said field would not be left out.

## 1.7 Structure of the Thesis

The dissertation has been divided into the following chapters:

- Chapter 1: Introduction
- Chapter 2: Literature Review
- Chapter 3: Theoretical Exposition and theorems of work
- Chapter 4: Applications of Topological Dynamics to Integral equations
- Chapter 5: Conclusion and Recommendation
- References

## 1.8 Notations Used

The following are notations from which the symbols in the research were culled from:

| | |
|---|---|
| $\forall$ | for every; |
| $\exists$ | exists; |
| $:=$ | equals (coincides) by definition; |
| 0 | zero, and also zero element of any additive group (semigroup); |
| N | the set of all natural numbers; |
| Z | the set of all integer numbers; |
| Q | the set of all rational numbers; |
| R | the set of all real numbers; |
| C | the set of all complex numbers; |
| $X \times Y$ | the Decart product of two sets; |
| $M_n$ | is the direct product of n copies of the set $M$; |
| $E_n$ | is the real or complex $n$-dimensional Euclidian space; |
| $\{x_n\}$ | is a sequence; |
| $x \in X$ | is an element of the set $X$; |
| $X \subseteq Y$ | the set X is a part of the set Y or coincides with it; |
| $X \cup Y$ | is the union of the sets X and Y; |
| $X\|Y$ | is the complement of the set Y in X; |
| $X \cap Y$ | is the intersection of the sets X and Y; |
| $\varphi$ | the empty set; |

| | |
|---|---|
| $(X,\rho)$ | is a full metric space with the metric $\rho$; |
| $M$ | is the closure of the set $M$; |
| $f^{-1}$ | is the mapping inverse to $f$; |
| $f(M)$ | is the image of the set $M \subseteq X$ in the mapping $f : X \to Y$ ; |
| $f \circ g$ | is the composition of the mappings $f$ and $g$; |
| $f(\cdot,x)$ | is the partial mapping defined by the function $f$; |
| $\|x\|$ or $\|\|x\|\|$ | is the norm of the element $x$; |
| $(x,y)$ | an ordered pair; |
| $C(X,Y)$ | is the set of all continuous mappings of the space $X$ in the space $Y$ ; |
| $f : X \to Y$ | is the mapping of $X$ into $Y$ ; |
| $\{x,y,...,z\}$ | is a set consisting of $x,y,...,z$; |
| $\{x \in X \| \mathrm{R}(x)\}$ | is the set of all elements from $X$ possessing the property R; |
| $f^{-1}(M)$ | is the preimage of the set $M \subseteq Y$ in the mapping $f : X \to Y$ ; |
| $\rho(\xi,\eta)$ | is a distance in the metric space $X$; |
| $\lim_{n\to+\infty} x_n$ | is the limit of a sequence; |
| $\lim_{x\to a} f(x)$ | is the limit of mapping $f$ as $x \to a$; |

| | |
|---|---|
| $\lambda \in \Lambda\}$ | is the union of the family of sets $\{M_\lambda\}_{\lambda\in\Lambda}$; |
| $\lambda \in \Lambda\}$ | is the intersection of the family of sets $\{M_\lambda\}_{\lambda\in\Lambda}$; |
| | is the set of all compacts from $X$; |
| | is a dynamical system; |
| $\Sigma_x^+$ | is a positive semitrajectory of the point $x$; |
| $\Sigma^+(M)$ | is a positive semitrajectory of the set $M$; |
| $H^+(x)$ | is a closure of the positive semitrajectory of the point $x$; |
| $\Sigma_x$ | is the trajectory of the point $x$; |
| $H(x)$ | is a closure of the trajectory of the point $x$; |
| $xt$ | is the position of the point $x$ in the moment of time $t$; |
| $X_0$ | is the space of all linear continuous functions on $X$; |
| $U(t,A)$ | is the operator of Cauchy; |
| $G_A(t,\tau)$ | is the function of Green; |
| $D(A)$ | is the domain of definition of the operator $A$; |
| $f\|_M$ | is the restriction of the mapping $f$ on the set $M$; |
| $D(f)$ | is the domain of definition of the function $f$; |
| $C^k(U,M)$ | set of all $k$ times continuously differentiable mappings of $U$ into $M$; |
| $\partial X$ | is the boundary of the set X; |
| $B(M,\epsilon)$ | is an open -neighbourhood of the set $M$ in the metric space $X$; |
| $B[M,\epsilon]$ | is a closed -neighbourhood of the set $M$ in the metric space $X$; |
| $\epsilon_k \downarrow 0$ | is a monotonically decreasing to 0 sequence; |
| $(\mathcal{H},\langle\cdot,\cdot\rangle)$ | is a Hilbert space with the scalar peoduct h·,·i; |

| | |
|---|---|
| $\omega_x(\alpha_x)$ | is the $\omega(\alpha)$-limit set of the point $x$; |
| $\Omega$ | is the closure of the union of all $\omega$-limit points of $(X,\mathrm{T},\pi)$; |
| $\mathrm{M}_x$ | is the set of all directing sequences of the point $x$; |
| $\mathrm{N}_x$ | is the set of all proper sequences of the point $x$; |
| $C_b(\mathrm{T},E^n)$ | Banach space of all continuous and bounded functions $f:\mathrm{T}\to E^n$ with sup-norm; |
| $(C_b^*(\mathrm{T},E^n))^n$ | is the space adjoint to $(C_b(\mathrm{T},E^n))^n$. |

## 1.9 Assumptions

Throughout this research work we shall assume that **X** is a uniform space with a Hausdorff topology generated by a directed set ($A$,≥) and a correspondence **V**. In *topology* and related branches of *mathematics*, a Hausdorff space, separated space or $T_2$ space is a *topological space* in which distinct points have *disjoint neighbourhoods*. Other assumptions would be indicated in the work.

## 1.10 Limitations

The researcher envisages some major problems in the process of carrying out the study. A crucial limitation is that of financial resources. Deciding upon the appropriate instrument to use would be major in order to come out with the desired result.

The scope would be limited to the existing works of academicians in the field under study.

## 1.11 Definition of Terms

The following are some notions and denotations used in the theory of dynamical systems which will be used in the work.

Let $X$ be a topological space, R(Z) a group of real (integer) numbers, $R_+(Z_+)$ a semigroup of nonnegative real (integer) numbers, S one of subsets of R or Z, and $T \subseteq S$ ($S_+ \subseteq T$, where $S_+ = \{s | s \in S, s \geq 0\}$ is a semi group of additive group S).



## Definition 1.11.1

The triplet $(X,T,\pi)$, where $\pi : X \times T \to X$ is a continuous mapping satisfying the following conditions:

$$\pi(0,x) = x \qquad (x \in X,\ 0 \in T), \tag{1.1}$$

$$\pi(\tau,\pi(t,x)) = \pi(t + \tau,x) \qquad (x \in X,\ t,\tau \in T), \tag{1.2}$$

are called a dynamical system. In that case if $T = R_+(R)$ or $Z_+(Z)$ then the system $(X,T,\pi)$ is called a semigroup (group) dynamical system. If $T = R_+(R)$ the dynamical system is called flow and if $T \in Z$ then $(X,T,\pi)$ is called cascade. **(Cheban, 2009.)** To be short we will write instead of $\pi(t,x)$ just $xt$ or $\pi^t x$.

Further, as a rule, $X$ will be a complete metric space with the metric $\rho$.

## Definition 1.11.2

The function $\pi(\cdot,x) : T \to X$ with fixed $x \in X$ is called motion of the point $x$ and the set $\Sigma_x := \pi(T,x)$ is called trajectory of this motion or of the point $x$.

Let $\mathsf{T} \subseteq \mathsf{T}^0$($\mathsf{T}^0$ is a subsemigroup from $\mathsf{S}$).

## Definition 1.11.3

If for any point $x \in X$ and $(\gamma_1, \mathsf{T}^0)$, $(\gamma_2, \mathsf{T}^{00}) \in \Phi_x$ from the equality $\gamma_1(t_0) = \gamma(t_0)$ it follows $\gamma_1(t) = \gamma_2(t)$ for all $t \in \mathsf{T}^0 \cap \mathsf{T}^{00}$, then $(X, \mathsf{T}, \pi)$ is said to be a semi group dynamical system with uniqueness.

## Definition 1.11.4

A nonempty set $\mathbf{M} \subseteq X$ is called positively invariant (resp., negatively invariant, invariant) if $\pi(t,\mathbf{M}) \subseteq \mathbf{M}$ (resp., $\pi(t,\mathbf{M}) \supseteq \mathbf{M}$, $\pi(t,\mathbf{M}) = \mathbf{M}$) for all $t \in \mathsf{T}$.

## Definition 1.11.5

A closed invariant set not containing proper subset which would be closed and invariant is called minimal.

## Definition 1.11.6

A point $p \in X$ is called $\omega$-limit point of the motion $\pi(\cdot,x)$ and of the point $x \in X$ if there exist a sequence $\{t_n\} \subset \mathsf{T}$ such that $t_n \to +\infty$ and $p = \lim_{n\to+\infty} \pi(t_n,x)$.

The set of all $\omega$-limit points of the motion $\pi(\cdot,x)$ is denoted by $\omega_x$ and is called $\omega$-limit set of this motion.

## Definition 1.11.7

A point $x$ and motion $\pi(\cdot,x)$ are called stable in the sense of Lagrange in positive direction and denoted st. $\overline{\phantom{x}}$ $L^+$ if $H^+(x) := \Sigma_x^+$ is a compact set., where $\Sigma_x^+ := \pi(\mathbb{T}_+, x)$ and $\mathrm{T}_+ :=$

$\{t | t \in \mathrm{T}, t \geq 0\}$.

## Definition 1.11.8

A point $x$ and motion $\pi(\cdot,x)$ are called stable in the sense of Lagrange and denoted st. $L$ if $H(x) := \overline{\Sigma_x}$ is a compact set, where $\Sigma_x := \pi(\mathrm{T},x)$.

## Definition 1.11.9

A point $x \in X$ is called fixed point or stationary point if $xt = x$ for all $t \in \mathrm{T}$ and $\tau$-periodic if $xt = x(t + \tau) = x$ ($\tau > 0, \tau \in \mathrm{T}$).

## Definition 1.11.10

Let $> 0$. A number $\tau \in \mathrm{T}$ is called -shift (-almost period) of $x$ if $\rho(x\tau, x) < \epsilon(\rho(x(t+\tau), xt) < \epsilon$ for all $t \in \mathrm{T}$). **(Cheban, 2009.)**

## Definition 1.11.11

A point $x \in X$ is called almost recurrent (almost periodic) if for every $> 0$ there exists $l = l(\epsilon) > 0$ such that on every segment from $\mathrm{T}$ of length $l$ there exists -shift (-almost period) of the point $x$.

## Definition 1.11.12

If a point $x \in X$ is almost recurrent and the set $H(x) = \overline{\Sigma_x}$ is compact, then the point $x$ is called recurrent.



## Definition 1.11.13

A point $x \in X$ is called positively Poisson stable if $x \in \omega_x$.

## Definition 1.11.14

A point $x \in X$ is called comparable by the character of recurrence with $y \in Y$ or, in short, comparable with $y$ if for every  $> 0$ there exists $\delta > 0$ such that $\delta$-shift of the point $y$ is -shift for $x \in X$.

## Definition 1.11.15

The motion $\pi(\cdot,x) : \mathsf{T} \to X$ of the semigroup dynamical system $(X,\mathsf{T},\pi)$ is called continuable onto $\mathsf{S}$, if there exists a continuous mapping $\phi : \mathsf{S} \to X$ such that $\pi^t\phi(s) = \phi(t + s)$ for all $t \in \mathsf{T}$ and $s \in \mathsf{S}$. In that case by $\alpha_\phi$ we will denote the set $\{\gamma | \exists t_n \to -\infty,\ t_n \in \mathsf{S}_-,\ \phi(t_n) \to y\}$ where $\phi$ is an extension onto $\mathsf{S}$ of the motion $\pi(\cdot,x)$. The set $\alpha_\phi$ is called $\alpha$-limit set of $\phi$ and its points are called $\alpha$-limit for $\phi$.

Along with the dynamical system $(X,\mathsf{T},\pi)$ let us consider $(Y,\mathsf{T},\sigma)$, where $Y$ is a complete metric space with metric $d$.

### Definition 1.11.16

The sequence $\{t_n\}$ is called proper sequence of the point $x$ if $x = \lim_{n\to+\infty} xt_n$.

### Definition 1.11.17

A point $x \in X$ is called uniformly comparable by the character of recurrence with $y \in Y$ or, in short, uniformly comparable with $y$ if for every  > 0 there exists $\delta > 0$ such that for any $t \in T$ every $\delta$-shift of the point $yt$ is $\epsilon$-shift for $xt$, that is, $\delta > 0$ is such that for every two numbers $t_1, t_2 \in T$ for which $d(yt_1, yt_2) < \delta$ is held the inequality $\rho(xt_1, xt_2) < \epsilon$.

**(Cheban, 2009.)**

### Definition 1.11.18

The triplet $((X,T_1,\pi),(Y,T_2,\sigma),h)$, where $h$ is a homomorphism of $(X,T_1,\pi)$ onto $(Y,T_2,\sigma)$, we will call nonautonomous dynamical system.

### Definition 1.11.19

Let $(X,T_1,\pi)$ and $(Y,T_2,\sigma)$ $(S_+ \subseteq T_1 \subseteq T_2 \subseteq S)$ be two dynamical systems. The mapping $h : X \to Y$ is called homomorphism (resp., isomorphism) of the dynamical system $(X,T_1,\pi)$ onto $(Y,T_2,\sigma)$ if the mapping $h$ is continuous (resp., homeomorphic) and $h(\pi(t,x)) = \sigma(t,h(x))$ for all $x \in X$ and $tT_1$. In that case $(X,T_1,\pi)$ is called an extension of the dynamical system $(Y,\mathrm{T}_2,\sigma)$ and $(Y,\mathrm{T}_2,\sigma)$ is the factor of $(X,\mathrm{T}_1,\pi)$. The dynamical system

$(Y,\mathrm{T}_1,\sigma)$ is also called base of the extension $(X,\mathrm{T}_1,\pi)$. **(Cheban, 2009.)**

## Definition 1.11.20

A dynamical system $(X\times Y,\mathrm{T},\lambda)$ is called direct product of the dynamical systems $(X,\mathrm{T},\pi)$ and $(Y,\mathrm{T},\sigma)$ if $\lambda(t,(x,y)) = (\pi(t,x),\sigma(t,y))$ for all $(x,y) \in X \times Y$ and $t \in \mathrm{T}$.

## Definition 1.11.21

A function $\phi \in C(\mathrm{R},E^n)$ is called bounded on $S \subseteq \mathrm{R}$, if the set $\phi(S) \subset E^n$ is bounded.

**(Cheban, 2009.)**

## Definition 1.11.22

A motion $\pi(\cdot,x)$ is called asymptotically stationary (resp., asymptotically $\tau$-periodic, asymptotically almost periodic, asymptotically recurrent, asymptotically Poisson stable) if there exists a stationary (resp., $\tau$-periodic, almost periodic, recurrent, Poisson stable) motion $\pi(\cdot,p)$ such that

$$\lim_{t\to+\infty} \rho(xt, pt) = 0 . \qquad (1.3)$$

## Definition 1.11.23

The dynamical system $(C(X,Y),\mathbb{T},\sigma)$ is called a dynamical system of shifts (dynamical system of translations or dynamical system of Bebutov) in the space of continuous functions

$C(X,Y)$.



## Definition 1.11.24

A function $f : S \to \mathbb{B}$ is said to be measurable if there exists a sequence $\{f_n\}$ of stepfunctions measurable and such that $f_n(s) \to f(s)$ with respect to the measure $\mu$ almost everywhere.

## Definition 1.11.25

A function $f : S \to \mathbb{B}$ is called integrable, if there exists a sequence $\{f_n\}$ of step-functions, integrable and such that for every $n$ the function $\phi_n(s) = |f_n(s) - f(s)|$ is integrable and

$$\lim_{n\to+\infty} \int |f_n(s) - f(s)| d\mu(s) = 0. \quad (1.4)$$

Then $\int f_n d\mu$ converges in the space $\mathbb{B}$ and its limit does not depend on the choice of the approximating sequence $\{f_n\}$ with the above mentioned properties. This limit is denoted by $\int f d\mu$ or $\int f(s) d\mu(s)$.

### Definition 1.11.26

A function $g \in C(\mathbb{R};E^n)$ is called $\omega$-limit for $f$, if there exists a sequence $t_n \to +\infty$ such that $f^{(t_n)} \to g$ in the topology of the space $C(\mathbb{R};E^n)$.

### Definition 1.11.27

A function $f \in (C^*(\mathbb{R}_+,E^n))^n$ is called weakly asymptotically almost periodic, if the set of shifts $\{\tau_h f : h \in \mathbb{R}_+\}$ forms a relatively compact set in the weak topology $(C^*(\mathbb{R}_+,E^n))^n$.

### Corollary 1.11.28

The point $x$ is asymptotically stationary if and only if the sequence $\{x(k\tau)\}_{k=0}^{\infty}$ converges for every $\tau \in \mathbb{T}_+$.

### Definition 1.11.29

A function $(C_b(\mathbb{R}_+ \times M;E))^n$ will be called weakly asymptotically almost periodic with respect to $t$ uniformly with respect to $p \in M$, if for every subsequence $\{t_k\} \subset \mathbb{R}_+$ there exist a subsequence $\{t_{k_m}\}$ and a function $g \in (C_b(\mathbb{R}_+ \times M;E^n))^n$ such that $\langle \varphi, f^{(t_{k_m})}(\cdot,p)\rangle \to \langle \phi, g(\cdot,p)\rangle$ as $m \to +\infty$ for every $\varphi \in (C_b^*(\mathbb{R}_+,E))^n$ uniformly with respect to $p$ on every compact subset $K \subset M$.

### Corollary 1.11.30

Every asymptotically almost periodic function is relatively compact and uniformly continuous on $\mathbb{R}_+$.

## Definition 1.11.31

Let $\phi \in C(\mathbb{R},\mathbb{B})$. The function $\phi$ is said to have an average value $M\{\phi\}$ on $\mathbb{R}_+$, if there exists a limit of the expression $(\frac{1}{L})\int_0^L \varphi(t)dt$ as $L \to +\infty$. So

$$M\{\varphi\} := \lim_{L\to+\infty} \frac{1}{L}\int_0^L \varphi(t)dt. \tag{1.5}$$

## Corollary 1.11.32

Every asymptotically almost periodic function is relatively compact and uniformly continuous on $\mathbb{R}_+$.

## Corollary 1.11.33

Let $\phi_k \in C(\mathbb{R},\mathbb{B})$ $(k = 1,2,...,m)$ be asymptotically almost periodic. Then $\phi := \phi_1 + \phi_2 + ... + \phi_m \in C(\mathbb{R},\mathbb{B})$ is asymptotically almost periodic too.

## Corollary 1.11.34

Let $\{\phi_k\} \subset C(\mathbb{R},\mathbb{B})$ be a sequence of asymptotically almost periodic functions and the series $\sum_{k=1}^{+\infty} \varphi_k$ converges uniformly with respect to $t \in \mathbb{R}_+$ and $S \in C(\mathbb{R},\mathbb{B})$ is the sum of this series. Then $S$ is an asymptotically almost periodic function.

## Definition 1.11.35

Let $(X,\mathbb{T},\pi)$ and $(Y,\mathbb{T},\sigma)$be dynamical systems, $x \in X, y \in Y$ . One will say that the point $x$ is comparable in limit in positive direction with respect to the character of recurrence with the point $y$, if $\mathbb{L}_y^{+\infty} \subseteq \mathbb{L}_x^{+\infty}$. If $x$ is comparable in limit with $y$ both in positive and negative direction, then we will say that $x$ is comparable in limit with respect to the character of recurrence with $y$. At last, if $L_y \subseteq L_x$, then we will say that $x$ is strongly comparable in limit with $y$. **(Cheban, 2009.)**

## Definition 1.11.36

Let $f \in C(R \times W, E^n)$ and $Q$ be a compact subset from $W$. One will say that the function $f$ is asymptotically stationary with respect to the variable $t \in R$ uniformly with respect to $x \in Q$, if the motion $\sigma(\cdot, f_Q)$ generated by the function $f_Q := f|_{R \times Q}$ in the dynamical system of shifts $\left(C(\mathbb{R} \times Q, E^n), \mathbb{R}, \sigma\right)$ is asymptotically stationary . **(Cheban, 2009.)**

## Corollary 1.11.37

For every fixed $t \in R$ the mapping $U_t : C\left(\mathbb{R}; [E^n]\right) \to [E^n]$ defined by the equality $U_t(A) := U(t,A)$ is continuous.

## Definition 1.11.38

Points $x_1$ and $x_2$ from $X$ are called positively proximal (distaL) if

$$\inf\{\rho(x_1t, x_2t) : t \in T_+\} = 0 \ (\inf\{\rho(x_1t, x_2t) : t \in T_+\} > 0). \tag{1.6}$$

## Lemma 1.11.39

Let $x \in X$ be asymptotically Poisson stable. If points from $\omega_x$ are mutually distal in positive direction, then $P_x$ consists from one point.



## Lemma 1.11.40

Let the point $x \in X$ be almost periodic, then on $\omega_x$ the dynamical system $(X,\mathrm{T},\pi)$ is distal, that is,

$$\inf\{\rho(pt,qt) : t \in \mathrm{T} > 0\} \tag{1.7}$$

for all $p,q \in H(x)$ $(p \neq q)$.

## Theorem 1.11.41

Let $f \in C(\mathrm{R}\times E^n,E^n)$ be continuously differentiable with respect to $x \in E^n$ and let exists $r_0 > 0$ such that

1. $|f(t,x)| \leq A(r) < +\infty$ for all $(t,x) \in \mathrm{R}_+ \times B[0,r]$ and $0 \leq r \leq r_0$;
2. $f$ is asymptotically Poisson stable with respect to $t \in \mathrm{R}$ uniformly with respect to $x \in B[0,r_0]$;
3. there exist positive numbers $m$ and $M(r)$ such that for all $(t,x) \in \mathrm{R}_+ \times B[0,r]$, $0 < r \leq r_0,\ mI \leq f'_x(t,x) \leq M(r)I$ ($I$ is a unit matrix from $[E^n]$) and the matrix $f'_x(t,x)$ is self-adjoint.

The proof of Theorem 1.11.41 bases upon the following lemma.

**Lemma 1.11.42**

Let $M > 0$ and $f \in C_b(\mathbb{R}_+, E^n)$. By the formula

$$\varphi(t) = -\frac{1}{2\sqrt{M}}\left\{e^{\sqrt{M}t}\int_t^{+\infty} e^{-\sqrt{M}\tau} f(\tau)d\tau + e^{-\sqrt{M}\tau}\int_0^t e^{\sqrt{M}\tau} f(\tau)d\tau\right\} \qquad (1.8)$$

there is defined a bounded on $\mathbb{R}_+$ solution of the equation

$$x^{00} = Mx + f(t) \qquad (1.9)$$

and this is a unique solution which may be estimated as follows:

$$||\varphi|| \leq \frac{1}{M}||f|| \qquad (1.10)$$

where $||f|| := \sup\{|f(t)| : t \in \mathbb{R}_+\}$.

If, besides, $f$ is asymptotically Poisson stable, then $\phi$ is compatible in limit.

# Chapter 2

# LITERATURE REVIEW

## 2.1 Historical Assertion

A dynamical system is a concept (Tabuada *et al*, 2007) in mathematics where a fixed rule describes the time dependence of a point in a *geometrical space* in Mallat, (2009). Examples include the *mathematical models* that describe the swinging of a clock pendulum in Holm *et al*, 2001, the flow of water in a pipe in Piela *et al*, (2008) and the number of fish each spring in a lake in Feger *et al*, (2010).

At any given time a dynamical system has a *state* given by a set of *real numbers* ,a *vector*, which can be represented by a *point* in an appropriate state space, a geometrical *manifold* (Nguyen *et al*, 1989) . Small changes in the state of the system correspond to small changes in the numbers (Purushothaman *et al*, 2005). The evolution (Ohtsuki *et al*, 2006) rule of the dynamical system is a *fixed rule* that describes what future states follow from the current state. The rule is *deterministic* (Ren and Zhang, 2009); in other words, for a given time interval only one future state follows from the current state.

It should be noted that the concept of a dynamical system has its origins in *Newtonian mechanics* in Sharipov, (2001). Detailed mechanisms of protein folding are not biased by nonnative contacts, typically argued to be a consequence of sequence design and/or topology (Levinthal, 1968). A record of the folding process is largely preserved in the final structure (Moult *et al*, 1991). The folding and refolding of mutant proteins can be used to map the formation of structure in transition states and folding intermediates (Fersht *et al*, 1992).

There is, as in other natural sciences and engineering disciplines, the evolution rule of dynamical systems is given implicitly by a relation that gives the state of the system only a short time into the future (Hufschmidt *et al*, 2005). The relation is either a *differential equation* (King *et al*, 2010), *difference equation* (Taixiang *et al*, 2005) or other *time scale*.

Topological Dynamics can be applied to biological systems (Hofbauar and Sigmund, 1988; May, 1973; Sigmund, 1993). To determine the state for all future times requires iterating the relation many times-each advancing time a small step. The iteration procedure is referred to as solving the system or integrating the system (Price *et al*, 2009). Once the system can be solved, given an initial point it is possible to determine all its future points, a collection known as a trajectory or *orbit* (Wells, 2011).

Before the advent of *fast computing machines* (Crouch, 2010), solving a dynamical system required sophisticated mathematical techniques (Powell, 2007) and could be accomplished only for a small class of dynamical systems. Numerical methods implemented on electronic computing machines (Khan *et al*, 2008) have simplified the task of determining the orbits of a dynamical system.

For simple dynamical systems, knowing the trajectory is often sufficient, but most dynamical systems are too complicated (Zimmermann *et al*, 2005 ) to be understood in terms of individual trajectories.

The difficulties arise because the systems studied may only be known approximatelythe parameters of the system may not be known precisely or terms may be missing from the equations. The approximations used bring into question the validity or relevance of numerical solutions. To address these questions several notions of stability have been introduced in the study of dynamical systems, such as *Lyapunov stability* or *structural stability* (Cortes, 2008). The stability of the dynamical system implies that there is a class of models or initial conditions for which the trajectories would be equivalent. The operation for comparing orbits (Fre *et al*, 2011) to establish their *equivalence* changes with the different notions of stability.

The type of trajectory may be more important than one particular trajectory (Sanz and Miret-Artes, 2008). Some trajectories may be periodic, whereas others may wander through many different states of the system. Applications often require enumerating these classes or maintaining the system within one class. Classifying all possible trajectories has led to the qualitative study of dynamical systems (Elezovic *et al*, 2009), that is, properties that do not change under coordinate changes. *Linear dynamical systems* and *systems that have two numbers describing a state* are examples of dynamical systems where the possible classes of orbits are understood.

The behavior of trajectories as a function of a parameter may be what is needed for an application. As a parameter is varied, the dynamical systems may have *bifurcation* (Di *et al*, 2006) *points* where the qualitative behavior of the dynamical system changes. For example, it may go from having only periodic motions to apparently erratic behavior, as in the *transition to turbulence of a fluid*.

The trajectories of the system may appear erratic, as if random. In these cases it may be necessary to compute averages using one very long trajectory or many different trajectories. The averages are well defined for *ergodic* (Lansberg, 1961) *systems* and a more detailed understanding has been worked out for *hyperbolic systems*. Understanding the probabilistic aspects of dynamical systems has helped establish the foundations of *statistical mechanics* and of chaos (Habib, 2005; Lorenz, 1993; Hunt *et al*, 2004; de Vries, 1993; Stewart, 1989).

Poincare did a lot of work through which these dynamical systems themes developed (Araujo *et al*, 2008).

In his paper, Dontwi (2005) introduced Bounded on $R^+$ generalized functions (distributions) together with their properties. Results of equations satisfying the conditions of exponential dichotomy on $R^+$ bounded functions were extended to bounded on $R^+$ generalized functions. In a related paper, Dontwi (2005) devoted the paper to the introduction of weak asymptotically almost periodic functions. Properties of those functions were then applied to some class of linear, weakly linear and non linear differential equations with the right hand sides exhibiting the dynamics.

Further demonstrations of the existence of weak asymptotically almost periodic functions which are not asymptotically periodic in the sense of Freche were presented by Dontwi (2005). Cheban *et al* (2004) proved the existence of recurrent or Poisson stable motions in the Navier-Stokes fluid (Galeazzo *et al*, 2011) system under recurrent or Poisson stable forcing, respectively and used an approach based on nonautonomous dynamical systems. In a paper by Tao *et al* (2006) noncommutative topological dynamical sinumodal and bimodal maps of the interval were studied and they gave a statistical interpretation to the topological numerical invariants associated to bimodal maps.

Yaacov (2008), defined a dynamical system as a continuous self-map of a compact metric space. It was added that topological dynamics studies the iterations of such a periodic map or equivalently the trajectories of points of the state space. The basic concepts of topological dynamics are minimality, transitivity, recurrence, shadowing property, stability, equicontinuity ,sensitivity, attractors, and topological entropy. A time series of the catalytical reaction of carbon mono oxide and oxygen on a surface was used to model the topological features (Gilmore, 1998) of the underlying attractor by reconstructing unstable periodic orbits in a three dimensional imbedding space.

In *physics*, the Navier-Stokes equations, named after *Claude-Louis Navier* and *George Gabriel Stokes*, describe the motion of *fluid* substances. These equations arise from applying *Newton's second law* to *fluid motion*, together with the assumption that the fluid *stress* is the sum of a *diffusing viscous* term which is proportional to the gradient of velocity, plus a *pressure* term.

The equations are useful because they describe the physics of many things of academic and economic interest. They may be used to *model* the *weather*, *ocean currents*, water *flow in a pipe* and air flow around a wing. The Navier-Stokes equations in their full and simplified forms help with the design of aircraft and cars, the study of blood flow, the design of power stations, the analysis of pollution, and many other things. Coupled with *Maxwell's equations* they can be used to model and study *Magnetohydrodynamics*.

The Navier-Stokes equations are also of great interest in a purely mathematical sense. Somewhat surprisingly, given their wide range of practical uses, mathematicians have not yet proven that in three dimensions solutions always exist (*existence*), or that if they do exist, then they do not contain any *singularity* (smoothness). These are called the *Navier-Stokes existence and smoothness problems*. The *Clay Mathematics Institute* has called this one of the *seven most important open problems in mathematics* and has offered a US$1,000,000 prize for a solution or a counter-example. [Navier-Stokes equations, Wikipedia, the free encyclopedia, 01/10/2011].

The Navier-Stokes equations dictate not *position* but rather *velocity*. A solution of the Navier-Stokes equations is called a velocity field or flow field, which is a description of the velocity of the fluid at a given point in space and time. Once the velocity fields are solved for, other quantities of interest, such as flow rate or drag force may be found. This is different from what one normally sees in *classical mechanics*, where solutions are typically trajectories of position of a *particle* or deflection of a *continuum*. Studying velocity instead of position makes more sense for a fluid; however for visualization purposes one can compute various *trajectories*.

## A Survey of Topological Dynamics

The many branches of dynamical systems theory are outgrowths of the study of differential equations and their applications to physics, especially celestial mechanics. The transition from the differential equations to the dynamical systems viewpoint is important. The difference can be illustrated by considering the *initial value problem* in ordinary differential equations:

$$\frac{dx}{dt} = \xi(x) \tag{2.1}$$

$$x(0) = p$$

Here $x$ is a vector variable in a Euclidean space $X = \mathrm{R}^n$ or in a manifold $X$, and the initial point $p$ lies in $X$. The infinitesimal change $\xi(x)$ is thought of as a vector attached to the point $x$ so that $\xi$ is a vector field on $X$.

The associated *solution path* is the function $\varphi$ such that as time $t$ varies, $x = \varphi(t,p)$ moves in $X$ according to the above equation and with $p = \varphi(0,p)$ so that $p$ is associated with the initial time $t = 0$. The solution is a curve in the space $X$ along which $x$ moves beginning at the point $p$. A theorem of differential equations asserts that the function $\varphi$ exists and is unique, given mild smoothness conditions, for example Lipschitz conditions on the function $\xi$.

Because the equation is autonomous, i. e. $\xi$ may vary with $x$, but is assumed independent of $t$, the solutions satisfy the following *semigroup identities*, sometimes also called the *Kolmogorov equations*:

$$\varphi(t,\varphi(s,p)) = \varphi(t + s,p). \tag{2.2}$$

Suppose we solve equation (2.1), beginning at $p$, and after $s$ units of time, we arrive at $q = \varphi(s,p)$. If we again solve the equation, beginning now at $q$, then the identity (2.2) says that we continue to move along the old curve at the same speed. Thus, after $t$ units of time we are where we would have been on the old solution at the same time, $t + s$ units after time 0.

The initial point $p$ is a parameter here. For each solution path it remains constant, the fixed base point of the path. The solution path based at $p$ is also called the *orbit* of $p$ when we want to emphasize the role of the initial point.

For each fixed $t$ value we define the time-$t$ map $\varphi^t : X \to X$ by $\varphi^t(x) = \varphi(t,x)$. For each point $x \in X$ we ask whither it has moved in $t$ units of time. The function $\varphi : TxX \to X$ is called the *flow* of the system and the semigroup identities can be rewritten:

$$\varphi^t \circ \varphi^s = \varphi^{t+s}, \qquad \forall t,s \in T. \tag{2.3}$$

These simply say that the association $t$ 7→ $\varphi^t$ is a group homomorphism from the additive group $T$ of real numbers to the automorphism group of $X$. In particular, observe that the time-0 map $\varphi^0$ is the identity map $1_X$ (Akin, 2007).

# Chapter 3

# THEORETICAL EXPOSITION

# AND THEOREMS

## 3.1 Topological Spaces

In this section, we explore the definitions and examples of topological spaces, metric spaces, properties of dynamical systems, limit sets, compact motions, attractors, flows and semi-flows.

### 3.1.1 Definition: A topological space

$(X,T)$ is a set X endowed with collection $T \subset P(X)$ of subsets of $X$, called the topology of $X$, such that

1. $\emptyset, X \in T$
2. If $\{O_a\}_{a \in A} \subset T$ then $\cup_{a \in A} O_a \in T$ for any set A
3. If $\{O_i\}_{i=1}^{k} \subset T$ then $\cap_{i=1}^{k} O_i \in T$

that is, $T$ contains $X$ and $\emptyset$ and is closed under arbitrary unions and finite intersections. The sets $O \in T$ are called open sets, and their complements are called closed sets. If $x \in X$ then an open set containing $x$ is called a neighbourhood of $T$ (Katok and Hasselblatt, 1995).

This calls for the definition and explanations of interior, exterior and boundary of a set.

The sequence $\{x_i\}_i \in \mathbb{N} \subset X$ is said to converge to $x \in X$ if for every open set $O$ containing $x$ there exists $N \in \mathbb{N}$ such that $\{x_i\}_{i>N} \subset O$.



$(X,T)$ is called a *Hausdorff space* if for any two $x_1,x_2 \in X$ there exist $O_1,O_2 \in T$ such that $x_i \in O_i$ and $O_1 \cap O_2 = \emptyset$. It is called normal if it is Hausdorff and for any two closed $X_1,X_2 \subset X$ there exist $O_1,O_2 \in T$ such that $X_i \subset O_i$ and $O_1 \cap O_2 = \emptyset$.

$\{O_a\}_{a\in A} \subset T$ is called an open cover of $X$ if $X = \cup_{a\in A} O_a$, and a finite open cover if $A$ is finite. $(X,T)$ is called compact if every open cover has a finite subcover, locally compact if every point has a neighborhood with compact closure, and sequentially compact if every sequence has a convergent subsequence, $X$ is called compact if it is a countable union of compact sets.

If $(X_a,T_a), a \in A$ are topological spaces and $A$ is any set, then the product topology on $\prod_{a\in A} X$ is the topology generated by the base $\{\prod_a O_a | O_a \in T_a, O_a \neq X_a\}$ for only finitely many $a\}$.

Let $(X,T)$ be a topological space. A set $D \subset X$ is said to be dense in $X$ if $\overline{D} = X$. $X$ is said to be separable if it has a countable dense subset.

$R^n$ with the usual open and closed sets is a familiar example. The open balls (open balls with rational radius, open balls with rational center and radius) form a base. Points of a Hausdorff space are closed sets.

## Proposition 3.1.1



A closed subset of a compact set is compact.

## Proof

If $K$ is compact, $C \subset K$ is closed, and $T$ is an open cover for $C$ then $T \cup \{K \cap C\}$ is an open cover for $K$, hence has a finite sub cover $T \cup \{K \cap C\}$, so $T$ is a finite sub cover of $T$ for $C$.

## Proposition 3.1.2

A compact subset of a Hausdorff space is closed.

## Proof

If X is Hausdorff and $C \subset X$ compact fix $x \in \times C4$ and for each $y \in C$ take neighborhoods $U_y$ of $y$ and $V_y$ of $x$ such that $U_y \cap V_y = \emptyset$. The cover $U_{y \in C} U_y \supset C$ has a finite sub cover $\{U_x | O \leq i \leq n\}$ and hence $N_x = U^n_{i=0} V_{yi}$ is a neighborhood of $x$ disjoint from $C$.

Thus $X\ C = U_{x \in \times\ C} N_x$ is open and $C$ is closed.

## Proposition 3.1.3 [Katok and Hasselblatt, 1995]

A compact Hausdorff space is normal.

## Proof

First we show that a closed set $K$ and a point $p \notin K$ can be separated by open sets. For $x \in K$ there are open sets $O_x$,$U_x$ such that $x \in O_x$,$p \in U_x$ and $O_x \cap U_x = \emptyset$. Since $K$ is compact there is a finite sub cover $O : U_{i=1}^{n} O_x \supset K$ ,and $U : U_{i=1}^{n} U_x$ is an open set containing

$p$ disjoint from $O$. Now suppose $K$, $L$ are closed sets and for $p \in L$ consider open disjoint

sets $O_p \supset K$,$p \in U_p$. By compactness of $L$ there is a finite sub cover $U := U_{j=1}^{m} U_p, \supset L$, and $O : \cap_{j=1}^{m} O_p, \supset K$, is an open set disjoint from $U$.

A useful consequence of normality is the following extension result:

## Theorem 3.1.1 [Katok and Hasselblatt, 1995]

If $X$ is a normal topological space, $Y \subset X$ closed, and $f : Y \to <$ is continuous, then there is a continuous extension of $f$ to $X$.

A collection of sets is said to have the finite intersection property if every finite subcollection has nonempty intersection.

## Proposition 3.1.4 [Katok and Hasselblatt, 1995]

A collection of compact sets with the finite intersection property has nonempty intersection.

**Proof**

It suffices to show that in a compact space every collection of closed sets with the finite intersection property has nonempty intersection. To that end consider a collection of closed sets with empty intersection. Their complements form an open cover. Since it has a finite sub cover the finite intersection property does not hold.

## 3.2 Metric Spaces

For several quite natural notions a topological structure is not adequate, but one rather needs a uniform structure, that is, a topology in which one can compare neighbuorhoods of different points. This can be defined abstractly and is realized for topological vector spaces but it is a little more convenient to introduce these concepts for metric spaces [Katok and Hasselblatt, 1995].

### 3.2.1 Definition: Metric Space

If $X$ is a set then $d : X \times X \to \mathrm{R}$ is called a metric if

1. $d(x,y) = d(y,x)$
2. $d(x,y) \geq 0 if x,y \in X$
3. $d(x,y) = 0 \Leftrightarrow x = y$,
4. $d(x,y) + d(y,z) \geq d(x,z)$(triangle inequality)

If $d$ is a metric then $(X,d)$ is called a metric space. The set $B(x,r) := \{y \in X / d(x,y) < r\}$ is called the (open) r - ball round $x$.

$O \subset X$ is called open if for every $x \in O$ there exists $r > 0$ such that $B(x,r) \subset O$.

For $A \subset X$ the set $\bar{A} := \{x \in X / \forall r > 0, B(x,r) \cap A \neq \emptyset\}$ is called the closure of $A$. $A$ is called closed if $\bar{A} = A$.

Let $(X,d)$, $(Y,dist)$ be metric spaces. A map $f : X \to Y$ is said to be uniformly continuous if for all > 0 there is a $\delta > 0$ such that all $x,y \in X$ with $d(x,y) < \delta$ we have $dist(f(x), f(y)) < \epsilon$. A uniformly continuous bijection with uniformly continuous inverse

is called a uniform homeomorphism.

A family $F$ of maps $X \to Y$ is said to be equicontinuous if for every $x \in X$ and $\epsilon > 0$ there is a $\delta > 0$ such that $d(x,y) < \delta$ implies $dist(f(x), f(y)) < \epsilon$ for all $y \in X$ and $f \in F$.

A map $f : X \to Y$ is said to be Holder continuous with exponent $\alpha$, or $\alpha - Holder$, if there exist $C$, > 0 such that $d(x,y) < \epsilon$ implies $d(f(x),f(y)) \leq C(d(x,y))^{\alpha}$, Lipschitz continuous if it is 1-Holder, and biLipschitz if it is Lipschitz and has a Lipschitz inverse.

It is called an isometry if $d(f(x),f(y)) = d(x,y)$ for all $x,y \in X$ (Katok and Hasselblatt, 1995).

## 3.3 Exploring Basic Properties of Dynamical Systems

In this section we shall give the traditional definition of a dynamical system, or flow, and discuss some of the elementary properties.



### 3.3.1 Flows and Semi-Flows

Throughout this chapter we shall assume that X is a uniform space with a Hausdorff topology generated by a directed set $(A, \geq)$ and a correspondence $V$ . In topology and related branches of mathematics, a Hausdorff space, separated space or $T_2$ space is a topological space in which distinct points have disjoint neighbourhoods *Hausdorff, 2011*.

Points $x$ and $y$ in a topological space $X$ can be separated by neighbourhoods if there exists a neighbourhood $U$ of $x$ and a neighbourhood $V$ of $y$ such that $U$ and $V$ are disjoint ($U \cap V = \emptyset$). $X$ is a Hausdorff space if any two distinct points of $X$ can be separated by neighborhoods. This condition is the third separation axiom (after $T_0$ and $T_1$), which is why Hausdorff spaces are also called $T_2$ spaces. The name separated space is also used.

Many of our statements will be valid in even greater generality, but this viewpoint will be adequate.

A **dynamical system** on $X$ is defined to be a mapping

$$\pi : X \times T \to X$$

Where $T$ is a topological group subject to the conditions:

1. (Identity property); $\pi(x,0) = x$
2. (Group property); $\pi(\pi(x,t),s) = \pi(x,t+s)$
3. $\pi$ is continuous. [*Sell, 1971*]

In this write up we will only be interested in the two groups R, the reals, and I, the integers, with the usual topologies. The dynamical system $T$ is sometimes referred to as a (continuous) flow when $T$ = R, and a (discrete) flow when $T$ = I. In the sequel we shall use the symbol $T$ to refer to either R or I. Many of our statements will be valid in either situation. Let $\pi$ be a flow on $X$ and define $\pi_t(x) = \pi(x,t)$.

## Proposition 3.3.1

For each $t$ in $T$, the mapping $\pi_t$ is a homomorphism of $X$ onto $X$, that is a bijection.

## Proof

The group property (Property 2) implies that $(\pi_t)^{-1} = \pi_{-t}$, and therefore both $\pi_t$ and $(\pi_t)^{-1}$ are continuous by the continuity property (Property 3). Furthermore, $\pi_t$ is an injection.

Indeed, if $\pi_t(x_1) = \pi_t(x_2)$ then by the group property one has

$$x_1 = \pi(\pi(x_1,t),-t) = \pi(\pi(x_2,t),-t) = x_2$$

Finally $\pi_t$ is surjective for if $y \in X$, then $y = \pi_t(x)$ where $x$ is given by $x = \pi(y,-t)$. Discrete flow can be characterized in terms of homeomorphism as the following statement asserts:

## Proposition 3.3.2

A necessary and sufficient condition that a mapping $\pi : X \times I \to X$ be a discrete flow is that there exists a homeomorphism $F : X \to X$ of $X$ *onto* $X$ such that

$$\pi(x,n) = F^n(x), \qquad n \in I$$

where $F^0$ is defined to be the identity.

Therefore the study of discrete flows is equivalent to the study of homeomorphisms and their iterates.

If $\pi : X \times R \to X$ is a continuous flow on $X$ and we define $F : X \to X$ by $F(x) = \pi(x,1)$ then $F$ is a homeomorphism of $X$ onto $X$ and $\pi(x,n) = F^n(x)$ defines a discrete flow on $X$. In this case we say that the homeomorphism $F$ is generated by the continuous flow $\pi$. The problem of characterizing those homeomorphisms that are generated by continuous flows is basically unresolved [*Sell, 1971*].

If $T$ is $R$ of $I$, we define $T^+ = \{t \in T : t \geq 0\}$, so $T^+$ is then a semi-group. We define a semi-flow as a mapping $\pi : X \times T^+ \to X$ satisfying.

1. (Identity Property) $\pi(x,0) = x$;

2. (Group Property) $\pi(\pi(x,t),s) = \pi(x,t+s)$;
3. $\pi$ is continuous

We see then that by restricting $t$ to $T^+$ any flow gives rise to a semi-flow. The converse is not true simply because one may be unable to "back-up" a semi-flow. In other words, if $\pi$ is a semi-flow and $t \in T$, the inverse mapping $(\pi_t)^{-1}$ may fail to be defined, or if defined, it may fail to be continuous. As a result of this, proposition 3.3.1 is no longer valid. Proposition 3.3.2 takes on the following form:

## Proposition 3.3.3

A necessary and sufficient condition that a mapping $\pi : X \times I^+ \to X$ be a discrete semi-flow is that there exist a continuous mapping $F : X \to X$ (not necessarily surjective) such that

$$\pi(x,n) = F^n(x), \qquad n \in I^+$$

We see then that the study of discrete semi-flows is equivalent to the study of continuous mappings and their iterates [*Sell, 1971*].

### 3.3.2 An Important Example

Here we want to examine a basic example to show the inadequacy of the above definition of a flow, and to motivate a modified definition.

Consider the differential equation.

$$x^0 = f(x) \tag{3.1}$$

on Euclidcan space $R^n$ where $f$ is a $C^1$-function. Then for every point $x \in R^n$, there is one and only one solution $\varphi(x,t)$ of (3.1) that satisfies the initial value problem $\varphi(x,0) = x$. It would appear that one could define a continuous flow $\pi$ on $R^n$ by setting $\pi(x,t) = \varphi(x,t)$, and indeed this is the *case provided* every solution [*Sell, 1971*].

$\varphi(x,t)$ can be continued for all time $t$. As is well known, though, this global existence property is not shared by all differential equations. The scalar equation

$$x_0 = x_2$$

has precisely one solution that is defined for all time $t$.

One could remedy this defect by multiplying the right-side of (3.1) by an appropriate scalar-valued function. For example, the differential equation.

$$x' = \frac{1}{1+|f(x)|} f(x) \tag{3.2}$$

does have the global existence property and the solution curves of (3.1) agree with those of (3.2). This is of course, equivalent to introducing a new time parameter.

This change of time parameter has two defects. First, it can destroy certain dynamical properties that depend on a specific time parameter. Secondly, it has no obvious counterpart for nonautonomous differential equations.

It seems then that it is more appropriate to modify the definition of a flow so that a motion $\pi(x,t)$ does not have to be defined for all time [*Sell, 1971*].

## 3.4 Modified Definition of a Flow

Let $X$ be a uniform space.

For each point $x$ in $X$ let $I_x = (\alpha_x,\beta_x)$ be an open interval in $T$ containing 0. We shall assume that the intervals $1_x$ have the following continuity property: If $x_n \to x$ in $X$, then

$1_x \subset$ *lim* in $fl_{x_n}$. Let $D \subset X \times T$ be defined by

$$D = \{(x,t) \in X \times T : (t \in I_x)\}$$

A function $\pi : D \to X$ is said to be (*local*) flow on $X$ if the following properties hold:

1. (Identity property) $\pi = (x,0) = x$ for all $x$ in $X$:

2. (Group property) if $t \in I_x$, and $s \in I_{\pi(x,t)}$ then $(t + s) \in 1_x$ and $\pi(\pi(x,t),s) = \pi(x,t + s)$;

3. $\pi$ is continuous
4. (Maximal property) Each interval $1_x$ is maximal in the sense that either $1_x = T$, or the set $\{\pi(x,t) : 0 \leq t < \beta_x\}$ is not conditionally compact if $\beta_x < +\infty$ or the set $\{\pi(x,t) : \alpha_x < t \leq 0\}$ Is not conditionally compact if $-\infty < \alpha_x$.

5. The intervals $1_x$ are lower semi-continuous in $x$, that is if $x_n \to x$, then $1_x \subset lim$ in $fI_{x_n}$. It should be clear from the maximal property that if $E \subset D$ and $E \to X$ is a local flow, then $E = D$. Because of this there is reason to distinguish between a local flow and a flow, and therefore we shall drop the modifier "local". Nevertheless, it will be convenient to have a way of referring to those flows that satisfy the first definition, viz, those flows for which $1_x = T$ for all $x$, we shall call such a flow a global flow. The following formulation of the continuity of $\pi$ will be needed in the sequel [*Sell, 1971*].

## Proposition 3.4.1

Let $\pi$ be a flow on $X$. If $\{x_n\}$ is a (generalized) sequence in $X$ with $x_n \to x$, then the sequence of functions $\{\pi(x_n,t)\}$ converge to $\pi(x,t)$, and the convergence is uniform on compact sets in $1_x$.

## Proof

Let $k$ be a compact set in $1_x$. By the semicontinuity of $I_x$ we see that there is an index $m$ such that $K \subset I_{x_n}$ for $n \geq m$. The uniform convergence on $K$ means that for every neighborhood $V_a(.)$, there is an index $p$ such that $\pi(x_n,t) \in V_a(\pi(x,t))$ for all $n \geq p$ and all $t$ in $K$. Assume, on the contrary, that this is false. Then there is a neighborhood $V_a(.)$ such that for every index $n \geq m$ there is a $t_n$ in $K$ such that

$$\pi(x_n,t_n) \notin V_a(\pi(x,t_n)) \tag{3.3}$$

We use the fact that since $K$ is compact, there is a convergent subsequence of $\{t_n\}$ which we shall denote by $\{t_n\}$. Say that $t_n \to t_0$, one then has $\pi(x,t_n) \to \pi(x,t_0)$.

With $a \in A$ given by (3) choose $b \in A$ by Proposition $I.4$ so that one has

$$\pi(x,t_0) \in V_b(\pi(x,t_n)) \text{ and } z \in V_b(\pi(x,t_0)) \Rightarrow z \in V_a(\pi(x,t_n)) \tag{3.4}$$

It follows from the continuity of $\pi$ that $\pi(x_n,t_n) \to (x,t_0)$. This means that we can find an index $N$ so that

$$z = \pi(x_n,t_n) \in V_b(\pi(x,t_0))$$

For $n \geq N$. By applying (3.4) we get $\pi(x_n,t_n) \in V_a(\pi(x,t_n))$ and this contradicts (3.3).QED

Let $\pi$ be a flow on $X$ and let $x \in X$ be fixed. The function of $t, \pi(x,t)$ is said to be the motion through $x$. The trajectory through $x$ is the set

$$\gamma(x) = \{\pi(x,t) : t \in I_x\}$$

(This is also referred to as the orbit through $x$.) The positive semitrajectory and the negative semitrajectory are defined by

$$\gamma^+(x) = \{\pi(x,t) : 0 \leq t < \beta_x\}$$

and

$$\gamma^-(x) = \{\pi(x,t) : \alpha_x < t \leq 0\}$$

respectively. The hull of a point $x$ is defined as

$$H(x) = Cl\gamma(x)$$

and the positive and negative hulls are defined by

$$H^+(x) = Cl\gamma(x) \text{ and } H^-(x) = Cl\gamma^-(x)$$

respectively [*Sell, 1971*].

A set $E \subset X$ is said to be invariant if $\gamma(x) \subset E$ whenever $x \in E$. The set $E$ is said to be positively invariant, or negatively invariant, if $\gamma^+(x) \subset E$,or $\gamma^-(x) \subset E$, whenever $x \in E$. It is easy to see that a set $E$ is invariant if and only if $E$ is the union of a collection

of full trajectories [*Sell, 1971*].



## Proposition 3.4.2

The closure of an invariant set is an invariant set.

## Proof

Let $E$ be an invariant set and let $y \in ClE$. We want to show that $\gamma(y) \subset ClE$, that is $\pi(y,t) \in ClE$ for each $t$ in $I_y$. Since $y \in ClE$ there is a (generalized) sequence $\{y_n\} \subset E$ such that $y_n \to y$. If $t \in I'_y$ then by the continuity property of the intervals $I_x$, there is an index $m$ such that $t \in I_{y_n}$ for $n \geq m$. By the continuity of $\pi$ one has $\pi(y_n,t) \to \pi(y,t)$. Since $E$ is invariant, the subsequence $\{\pi(y_n,t) : n \geq m\}$ is in $E$, hence $\pi(y,t) \in ClE$.

By the same argument we also see that the closure of a positively invariant set is positively invariant, and the closure of a negatively invariant set is negatively invariant.

Perhaps a "principle" should be noted here. It is apparent that there is a duality between the positive and negative properties of a flow. A statement valid for positive trajectories has an obvious counterpart for negative trajectories. In the sequel we shall examine the behavior for positive $t$ and leave unstated, but noted, the corresponding facts for negative time.

## 3.5 Limit sets and Compact Motions

Let $\pi$ be a flow on $X$. Define the sets

$$LB^+ = \{x \in X : \beta_x = +\infty\}$$

$$LB^- = \{x \in X : \alpha_x = -\infty$$

$$LB = LB^- \cap LB^+$$

We note that if $LB$ is nonempty, then $LB \times T \subset D$, the domain of definition of $\pi$, and $\pi$ restricted to $LB \times T$, is a global flow on $LB$

If $x \in LB^+$ we define the $\omega$-limit set by

$\Omega_x = \cap_{t\geq 0} H^+ (\pi(x,t))$ Similarly, if $x \in LB^-$ we define the $\alpha$-limt set by

$$A_x = \cap_{t\leq 0} H^- (\pi(x,t))$$

### Proposition 3.5.1

The limit sets are described equivalently by

$\Omega_x = \{y \in X : y = \lim\pi(x,t_n)$ for some sequence $\{t_n\}$ with $t_n \to +\infty\}$ and

$A_x = \{y \in X : y = \lim\pi(x,t_n)$ for some sequence $\{t_n\}$ with$t_n \to -\infty\}$

### Proof

By the duality principle it will suffice to prove the equality for the $\omega$-limit set. The sequences noted above are, of course, generalized sequences.

If $y = \lim\pi(x,t_n)$ for some sequences $\{t_n\}$ with $t_n \to +\infty$ then $y \in Cl\gamma^+(x) = H^+(x)$.

However,

$$y = \lim\pi(x,t_n) = \lim\pi(\pi(x,t),-\tau + t_n)$$

where $(-\tau + t_n) \to +\infty$. Hence $y \in H^+(\pi(x,t))$ for every $\tau \geq 0$. Therefore $y \in \Omega_x$. Conversely, if $y \in \Omega_x$ then $y \in Cl\gamma^+(\pi(x,t))$ for each $\tau \geq 0$. It follows that there is a sequence $x_a$, defined on $A$ with range $\gamma^-(\pi(x,t))$, such that $x_a \to y$. We can write this as $x_a = \pi(x,\tau + t_a)$ for some $t_a \geq 0$. With the usual ordering on $T^+$ we define the product ordering on $T^+ \times A$, that is $(t,a) \geq (\sigma,b)$ whenever $\tau \geq \sigma$ and $a \geq b$.

We define a sequence $t$ on $T^+ \times A$, with range in $T^+$ by

$$t : n = (\tau,a) \to \tau + t_a = t_n$$

First we note that $t_n \to +\infty$. To prove this we pick an integer $N$ and fix an index $(\sigma,b)$ in $T^+ \times A$, with $\alpha \geq N$. It follows that if $n = (\tau,a) \geq (\sigma,b)$ then

$$t_n = \tau + t_a \geq \sigma \geq N$$

Hence $t_n \to +\infty$. Secondly we note that $y = \lim\pi(x,t_n)$.

## Theorem 3.5.2

The $\alpha$-and $\omega$-limit sets are closed and invariant.

## Proof

Consider the $\omega$-limit set $\Omega_x$. Since $\Omega_x$ is defined as the intersection of a family of closed sets, $\Omega_x$ is closed. If $y \in \Omega_x$, then there is a sequence $\{t_n\}$ with $t_n \to +\infty$ and $x_n = \pi(x,t_n) \to y$. By the continuity of $\pi$ for any $t$ in $I_y$ one has $\pi(x_n,t) \to \pi(y,t_n)$. However $\pi(x_n,t) = \pi(x,t + t_n)$. Since $(t + t_n) \to +\infty$ it follows that $\pi(y,t) \in \Omega_x$ from the last

proposition. Hence $\Omega_x$ is invariant.

It should be noted here that the limit sets may be empty. This will be shown in the examples below. We would like to find a sufficient condition that the limit sets be nonempty. For this we introduce the following concept.

A motion $\pi(x,t)$ is said to be compact if the trajectory $\gamma(x)$ lies in a compact set, that is $\pi(x,t)$ is compact if the hull $H(x)$ is a compact set. Similarly a motion $\pi(x,t)$ is positively compact if $\gamma^+(x)$ lies in a compact set, that is if the positive hull $H^+(x)$, is a compact set. Negative compactness defined similarly.

Because of the maximality of $I_x$ we see that if $\pi(x,t)$ is positively compact, then $x \in LB^+$. Similarly if $\pi(x,t)$ is negatively compact {compact}, then $x \in LB^-\{x \in LB\}$.

The following result shows the relationship between positively compact motions and their $\omega$-limit sets [*Sell, 1971*].

## Theorem 3.5.3

Let $\pi(x,t)$ be a positively compact motion. Then $\Omega_x$ is nonempty, compact and invariant.

Moreover, for every $y$ in $\Omega_x$ one has $I_y = T$. If, in addition, the group $T$ is R, then the set $\Omega_x$ is connected.

## Proof

We have already observed that $\Omega_x$ is invariant. If $\pi(x,t)$ is positively compact, then $H^+(x)$ is a compact set in $X$. For $\tau \geq 0$ one has $H^+(\pi(x,\tau)) \subset H^+(x)$, is compact set in $X$. Since the family of sets $\{H^+(x,\tau)\}$ is decreasing and since the intersection of a decreasing family of nonempty compact sets is nonempty and compact, it follows that $\Omega_x$ is nonempty and compact.

If $y \in \Omega_x$ then $\pi(y,t)$ lies in the compact set $\Omega_x$ for all $t$ in $I_y$. Therefore from the maximality of $I_y$ we have $I_y = T$.

Now assume that $T = R$ and that $\Omega_x$ is not connected. Then there exist disjoint, nonempty, open sets $A$ and $B$ such that $\Omega_x \subset A \cup B$. We then can find (ordinary) sequences $\{t_n\}$ and $\{s_n\}$ so that

$$0 < s_1 < t_1 < \cdots < s_n < t_n < s_{n+1} < \cdots, S_n \to \infty, t_n \to \infty$$

$$\pi(x,s_n) \in A \text{ and } \pi(x,t_n) \in B$$

Since the path $\pi$ $(x,[s_n,t_n])$ is connected it follows that there is a point $t_n^*$ in $[s_n,t_n]$ such that $\pi(x,t_n^*) \in A \cup B$. From the compactness of $\pi(x,t)$ it follows that the sequence $\{\pi(x,t_n^*)\}$ contains a convergent subsequence, say that $\pi(x,t_n^*) - y$. The limit point $y$ lies in the close set $X - (A \cup B)$ and since $t_n^* \to -\infty$ one has $y\Omega_x$. This is a contradiction and

so $\Omega_x$ is connected.

### Corollary 3.5.4

Let $\pi$ be a flow on $X$. If there exists a positively compact motion, then the set $LB$ is nonempty.

### Proof

It follows from the last theorem that $\Omega_x$ is nonempty and lies in $LB$.

## 3.6 Minimal Sets

A set $E \subset X$ is said to be a minimal set if it is nonempty, closed and invariant and if it contains no proper subset with these three properties.

A point $x$ in $X$ is said to be a rest point (equilibrium point) for the flow $\pi$ if $\pi(x,t) = x$ for all $t$. If $x$ is a rest point, then $E = x$ is a minimal set.

A point $x$ is said to be a periodic point, and the motion $\pi(x,t)$ is said to be periodic, if there is a $\tau > 0$ such that $\pi(x,\tau + t) = \pi(x,t)$ for all $t$. In this case we say that the motion $\pi(x,t)$ is $\tau$-periodic. The number $\tau$

is said to be a period of the motion $\pi(x,t)$. If $x$ is a periodic point, then $E = \gamma(x)$ is a minimal set. We shall see later that if $\pi(x,t)$ is compact, then $\gamma(x)$ is a minimal set if and only if $\pi(x,t)$ is periodic.

It should be noted that a periodic point may be a rest point. If it is not then one can show that every period is a multiple of a minimal period [*Sell, 1971*].

## Proposition 3.6.1

Let $x$ be a periodic point and define $\sigma$ by

$$\omega = inf\,\{\tau > 0 : \pi(x,\tau) = x\}$$

Then $x$ is a rest point if and only if $\sigma = 0$. Furthermore if $\sigma > 0$, then any period $\tau$ of the motion $\pi(x,t)$, is an integral multiple of $\sigma$.

The existence of minimal sets, or better, compact minimal sets is assured by the following result.

## Theorem 3.6.2

If a flow $\pi$ admits a nonempty, compact, invariant-set, then the flow has a compact minimal set. In particular, if there exists a positively compact motion $\pi(x,t)$, then the limit set $\Omega_x$ contains a compact minimal set.

**Proof**

This is a simple application of Zorn's lemma. Let $\epsilon = E$ denote the collection of all nonempty, compact, invariant sets. Order by set inclusion. If $\{E_\alpha\}$ is a chain in , that is $E_\alpha \subset E_\beta$ whenever $\beta \leq \alpha$ (the indices denote ordinal numbers), then the intersection $E = \cap_\alpha E_\alpha$ is an element of . Thus every chain in has a lower bound, and we conclude that has a minimal element $E_0$. It follows then that $E_0$ is a minimal set for the flow $\pi$.

If we apply the same argument to the collection $f = \{E \cap \Omega_x : E \in \epsilon\}$ when $\pi(x,t)$, is a positively compact motion, we conclude that the minimal set $E_0$ can be chosen as a subset of $\Omega_x$.

## 3.7 Semi-Flows, Revisited

We define a (local) by the same properties that define a (local) flow but now restricting $t$ to be in the semi-group $T^+$. In other words, the intervals $I_x$, now are of the form $I_x = [0,\beta_x)$. Where $\Omega_x > 0$. Because of the maximality of the intervals $I_x$ we can, and do, drop the modifier "local" from this term.

The observations made for flows are also valid for semi-flows, with the appropriate modifications since the sets $\gamma^-(x)$and $LB^-$ are not defined in a semi-flow. For examples if $\pi(x,t)$, is a positively compact motion in a semi-flow on $X$, then $\Omega_x$ is a nonempty, compact, positively invariant set in $X$.

## 3.8 Attractors

Let $\pi$ be a flow on a uniform space $X$ and let $M$ be a closed invariant subset of $X$. Assume that the restriction of $\pi$ to $M$ is a global flow. $M$ is said to be an attractor if there is an open set $U_0$ such that $M \subset U_0$ and for each $x \in U_0$

1. $\pi(x,t) \in U_0$ for all $t \geq 0$.

2. $\pi(x,t) \to M$ as $t \to \infty$ that is, for every open neighborhood $V$ of $M$ there is a $\tau$ such that $\pi(x,t) \in V$ for all $t \geq \tau$.

An attractor $M$ is said to be stable if for every neighborhood $U$ of $M$ there is a neighborhood $V$ of $M$ such that $\pi(V,t) \subset U$ for all $t \geq 0$. If $M$ is an attractor we shall let $A(M)$ denote the region of attraction of $M$, that is $A(M)$ is the largest open set satisfying (1) and (2) above [*Sell, 1971*].

### Lemma 3.8.1

Let $M$ be a compact attractor for a flow $\pi$ on a uniform space $X$. If $x \in X$ has the property that $\gamma^{+}(x) \cap A(M)$ 6= $\emptyset$, then $\omega_x \subset M$ if in addition the space $X$ is locally compact, then $\omega_x$ is nonempty.

## Proof

Assume that $y = \pi(x,\tau) \in A(M)$ for some $\tau \geq 0$. Then $\pi(y,t) = \pi(x,\tau + t) \rightarrow M$. This means that if $z \in \omega_x = \omega_y$, then $z = \lim\pi(x,\tau + t_n)$ for some sequence $\{t_n\}$ with $t_n \rightarrow \infty$.



Since the sequence $\{\pi(x,\tau+t_n)\}$ is eventually in every neighborhood of $z$ and eventually in every neighborhood of $M$, it follows from the Hausdorff property that $z \in M$.

If $X$ is locally compact, then there is a conditionally compact neighborhood $V$ of $M$ such that $M \subset V \subset \bar{V} \subset A(M)$. It follows from (2) that $\pi(x,t) \in \bar{V}$ for $t \geq \tau$, hence $\pi(x,t)$ is positively compact. Hence $\omega_x$ is nonempty .

# Chapter 4

# APPLICATION OF TOPOLOGICAL DYNAMICS TO INTEGRAL EQUATIONS

## 4.1 Introduction

In this section examples of topological dynamical systems are given in (1)-(4).

Let R = $(-\infty,+\infty)$, R$^+$ = $[0,+\infty)$ $\pi$ = R or R$^+$ and $E^n$ − $n$−dimensional Euclidean space endowed with the norm |.|. Denote by $(C(\pi;E))^n$ the Banach space of all continuous and bounded functions. The mapping $f : \pi \to E^n$ has the norm k$f$k = sup$\{|f(t)| : t \in \pi\}$.

For $h \in \pi$, $\tau_h f \in (C(\pi;E))^n$ and $\tau_h f(t) = f(t + h)$. The adjoint of $(C(\pi;E))^n$ is represented by $(C^*(\pi;\pi))^n$. If $\phi \in (C^*(\pi;E))^n$ and $f \in (C(\pi;E))^n$, then h$\phi$,$f$i $\in E^n$. Let $\{h_n\}$,$\{k_n\}$ denote strictly increasing sequence of natural numbers with $\{\tau_h F\}$ being a *weakly* sequence of translates of $f$. From Dontwi (2005), then $\tau_{h_n}f \longrightarrow g$ implies $\{\tau_{h_n}f\}$ weakly converges to $g$ in $(C(\pi;E))^n$.

1. A function $f \in (C(\mathbb{R}^+;E))^n$ is weakly asymptotically almost periodic (waap), if the set of its translates $\{\tau_{h_n}f : h \in \mathbb{R}^+\}$ forms a relatively compact set in the weak topology $(C(\mathbb{R}^+;E))^n$. The set of all waap functions are represented by $W_a$ in Dontwi (2005). The function $f$ defines a topological dynamical system.

2. The systems $x = A(t)x + f(t)$ in Dontwi (2005) and $x = A(t)x$ in Cheban (1980) satisfy the hyperbolic or satisfies the condition of exponential dichotomy. There is at least one bounded solution on $\mathbb{R}^+$ of equation $x = A(t)x + f(t)$.

3. Periodic functions e.g. the sine function.

4. Almost periodic functions are topological dynamical systems.

## 4.2 Almost Periodic Functions

In 1924, Henald Bohr published in Acta Mathematica the first of a series of three fundamental papers entitled 'Zur Theorie der fusperiodis chen Functionen'. In this, Bohr defines the property of being almost periodic. A function $f(x)$, real or complex, defined for all arguments $x$, is said to possess a translation number, $\tau$, pertaining to the positive number

, if for all values of $x$ from $-\infty$ to $\infty$

$$|f(x + \tau_\epsilon) - f(x)| \leq \epsilon$$

The continuous function $f(x)$ is then said to be almost periodic if, whenever is given, there exists a finite number $l$ such that if $y$ is any real number, the interval ( $y, y + l_\epsilon$)contains at least one translation number $\tau$ pertaining to .

Bohr shows that if $f(x)$ is almost periodic, then

$$A(\lambda) = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau f(x) e^{i\tau x} d\tau$$

exists for every real $\tau$.

He then shows that there is at most a denumerable set $\{\lambda_n\}$ of values of $\lambda$ for which $A(\lambda)$ 6= 0. Let us write $A(\lambda_n) = A_n$; then Bohr's fundamental theorem asserts that

$$\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau |f(x)|^2 dx = \sum_1^\infty |A_n|^2,$$

or in another form, that

$$\lim_{N \to \infty} \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau |f(x) - \sum_1^N A_n e^- i\lambda_n x|^2 dx.$$

This is an analogue of Parseval's theorem for Fourier series. Its simplicity gives a motive for the definition of almost periodic functions.

Bohr's second paper makes this motive even clearer by showing that the class of almost periodic functions is identical with the class of functions which may be uniformly approximated by polynomials of the form

$$\sum_{1}^{N} B_n e^{-} i\lambda_n x,$$

where $N$ is a positive number, $B_1,\ldots,B_n$ are arbitrary complex numbers, and $\lambda_1,\ldots,\lambda_n$ are arbitrary real numbers. Bohr's third paper is devoted to the extension of almost periodic functions to complex arguments. [The Mathematical Gazette, 1932].

## 4.3 Bounded Functions

In mathematics, a function $f$ defined on some set $X$ with real or complex values is called bounded if the set of its values is bounded. In other words, there exists a number $M > 0$ such that $|f(x)| \leq M$ for all $x$ in $X$.

Intuitively, the graph of a bounded function stays within a horizontal band whiles the graph of an unbounded function does not.

### Examples

1. The function $f : R \to R$ defined by $f(x) = \sin x$ is bounded . The sine graph is no longer bounded if it is defined over the set of all complex numbers.

2. The function $f(x) = \frac{1}{x^2-1}$ defined for all $x$ which do not equal -1 or 1 is not bounded. As $x$ gets closer to -1 or to 1, the values of this function gets larger in magnitude. This function can be made bounded if one considers its domain to be, for example, $[2,\infty)$.

3. The function $f(x) = \frac{1}{x^2+1}$ defined for all real is bounded.

Thus, almost periodic functions are those functions defined on the real line which can be approximated by trignometric polynomials. From the definition of almost periodic functions, it follows that any trignometric polynomial is an almost periodic function. From the theorem of approximation of periodic functions by trignometric polynomials, it follows that any periodic function is also almost periodic.

## Theorem 4.3.1

There exists almost periodic functions which are not periodic.

## Proof:

It is sufficient to show that there exists at least one trignometric polynomial which is not a periodic function. Set $f(x) = e^{ix} + e^{inx}$ and let us assume the existence of a real number $\omega \neq 0$ such that $f(x+\omega) = f(x)$ for any $x$. This means that

$$(e^{i\omega} - 1)e^{ix} + (e^{in\omega} - 1)e^{inx} = 0 \tag{4.1}$$

But the functions $e^{ix}$ and $e^{inx}$ are linearly independent. Therefore $\omega$ must satisfy the conditions $e^{i\omega} = e^{in\omega} = 1$. Then we have $\omega = 2k\pi, \pi\omega = 2h\pi$ where $k$ and $h$ are integers. However these two equations are incompatible, so that the theorem is proved.

Almost periodic functions have many properties of periodic functions. Certain almost periodic functions have properties that the periodic functions do not have.

Here are some of them:

## Theorem 4.3.2

An almost periodic function is continuous and bounded on the real line.

## Proof

Let $f(x)$ be an almost periodic function and $\tau_n(x)$ a trignometric polynomial such that

$$|f(x) - \tau_n(x)| < \frac{1}{n}; \qquad -\infty < x < +\infty$$

The sequence $\tau_n x$ converges to the function $f(x)$ on the whole number line. But the limit of a uniformly convergent sequence of continuous functions is a continuous function.

Since the trignometric polynomials are continuous functions, $f(x)$ is continuous at every point of the real line.

Assume that $|\tau_n(x)| \leq M$, where $M > 0$. Then we have

$$|f(x)| \leq |f(x) - \tau_1(x)| + |\tau_1(x)| \leq M + 1, \qquad -\infty < x < +\infty$$

which proves the theorem.

### Theorem 4.3.3

An almost periodc function is uniformly continuous on the real line.

### Theorem 4.3.4

If $f(x)$ is an almost periodc function, $C$, a complex number, and $a$ a real number, then the function $f(x)$,$cf(x)$,$f(x + a)$ and $f(ax)$ are almost periodic.

### Theorem 4.3.5

If $f(x)$ and $g(x)$ are almost periodic functions then $f(x) + g(x)$ and $f(x) \cdot g(x)$ are almost periodic functions.

### Theorem 4.3.6

The limit of a uniformly convergent sequence of almost periodic functions is an almost periodic function.

## 4.4 The Periodic, Almost Periodic and Recurrent Limit Regimes of Some Class of Nonautonomous Differential Equations.

### 4.4.1 Introduction

In this section, we acquaint ourselves with notions from Cheban (1999).

Let $(E,|.|)$ be a Banach space, $C(\mathbb{R} \times E,E)$ is a space of all continuous mappings from $\mathbb{R} \times E$ into $E$ equipped with the topology of convergence on every compact (opencompact topology). For $f \in C(\mathbb{R} \times E,E)$ and $\tau \in \mathbb{R}$ we denote by $f_\tau$ the $\tau$ trans-

lation of $f$ with respect to $t$, i.e. $f_\tau(t,x) = f(t+\tau,x)$, $H^+(f) = \overline{\{f_\tau | \tau \in \mathbb{R}_+\}}$ and $\omega_f = \{g | \exists \tau_n \to +\infty, g = \lim_{n\to+\infty} f_{\tau_n}\}$.

Consider the differential equation

$$x^0 = f(t,x), \qquad (4.2)$$

where $f \in C(\mathbb{R} \times E,E)$, and a family of equations

$$y^0 = g(t,y), \qquad (4.3)$$

with $g \in H^+(f)$ or $\omega_f$. Throughout this section we suppose that $f \in C(\mathbb{R} \times E,E)$ is regular, i.e. for all $g \in H^+(f)$ and $x \in E$ the equation $y^0 = g(t,y)$ admits a unique solution $\phi(t,x,g)$ defined on $\mathbb{R}_+$ with condition that $\phi(0,x,g) = x$ and a mapping $\phi$ :

$\mathbb{R}_+ \times E \times H^+(f) \to E$ is continuous.

The solution $\phi(t,x_0,f)$ is called uniformly stable, if for any $> 0$ there exists $\delta(\epsilon) > 0$ so that $|\phi(t_0,x,f) - \phi(t_0,x_0,f)| < \delta$ implies $|\varphi(t,x,f) - \varphi(t,x_0,f)| < \epsilon$ for all $t \geq t_0 \geq 0$.

The solution $\phi(t,x_0,f)$ is called globally asymptotically stable, if $\phi(t,x_0,f)$ is uniformly stable and for all $> 0$ and $K \in C(X)$ there is $T(\epsilon, K) >0$ so that $|\phi(t,x,f) - \phi(t,x_0,f)| <$ for all $t \geq t_0 + T(\epsilon, K)(\varphi(t_0,x,f) \in K)$.

We will call the equation $x^0 = f(t,x)$ convergent if it admits at least one compact solution on $\mathbb{R}_+$ which is globally asymptotically stable.

Bounded (on $\mathbb{R}_+$ or $\mathbb{R}$) solutions of the equation

$$x^{\cdot} = A(t)x \tag{4.4}$$

with recurrent (in particular, almost Bohr periodic) cefficients.

The well-known Cameron-Johnson theorem for equation (4.4) in a finite dimensional space states that this equation can be reduced by a Lyapunov transformation to an equation

$$y^{\cdot} = B(t)y, \tag{4.5}$$

where B$(t)$ is a skew-symmetric matrix, if all the solutions of equation (4.4) and the solutions of all its limit equations are bounded by R.

In 1962, W. Hahn posed the problem of whether asymptotic stability implies uniform stability for linear equation

$$x^0 = A(t)x \ (x \in \mathrm{R}^n) \tag{4.6}$$

with almost periodic coefficients.

The matrix $A(t)$ in (4.6) is recurrent (in particular, almost periodic), and the asymptotic stability holds for the null solution of (4.6) and for the null solutions of all systems

$$x^0 = B(t)x, \tag{4.7}$$

where $B \in H(A) = \overline{\{A_\tau : \tau \in \mathrm{R}\}}$, with $A_\tau$ denoting the translation of the matrix $A$ by $\tau$ and the 'bar' denoting the closure in the topology of the uniform convergence on compact subsets of R.

We study the limit regimes of almost periodic equations.

$$x^0 = f(t,x), \tag{4.8}$$

where $x \in E\Big((E, |.|$)is a Banach space$\Big)$, $f : \mathbb{R} \times E \to E$ is a closed mapping and for any

$t_0 \in R$ and $x_0 \in E$ defined for all $t \geq t_0$ and satisfying the initial condition $x(t;t_0,x_0) = x_0$.

# 4.5 Main Results of Thesis

## 4.5.1 Introduction

This section constitutes the main facet of the research work. Topological Dynamical techniques were applied to an Integral Equation. It was then shown that by using Limiting

Equations, Poisson Stability, and Asymptotic Stability the conclusion arrived in Dontwi (2005) was confirmed in terms of contraction and stability of the stationary point.

## 4.5.2 The Main Work

We now consider the Volterra integral equation from Dontwi (2005),

$$(Sy)(t) = \int_0^{\tau} G_A(t,\tau)F(\tau, y(t))d\tau. \qquad (4.9)$$

We introduce arbitrary value that is $\tau$ 7→ $+\infty$ where $y$ is an $n$-vector, $F : R^{+} \times R^{n} \to R^{n}$, $G_A : R^{n} \times R^{+} \to R^{n}$ and $A$ is an $n \times n$ matrix. $G_A$ and $F$ satisfy the concept of a group (Fraleign, 2003) and a topological group (Munkres, 1975).

If $\phi = \phi(f,g;t), 0 \le t < a$, is a solution of (4.10) we shall define the function $T_\tau f = T_\tau(f,g)$ by

$$(T_\tau f)(\theta) = \int_0^\tau g(\tau+\theta,\tau) f(\phi(\tau),t) d\tau. \tag{4.10}$$

The functions $g$ and $f$ are dynamical systems (Robinson (1995), Devaney (1989), Alligood *et al* (1996), Hirsch *et al*(2004)) and hence $Sy(t)$ is a dynamical system from (4.9). See Definition 1.11.1.(Page 17)

Assume that $C$ has a topology of uniform convergence on compact sets. That is, a sequence $\{f_k\}$ in $C$ is said to converge to a limit $f$ if for every compact set $K \subset R$ the sequence of restrictions $\{f_k|K\}$ converges to $f|K$ uniformly (Sell, 1971).

Let us show that this topology is metrizable. We let $I_n = [-n,n]$ and let $|.|$ denote any norm on $R^n$. Now define

$$\begin{aligned}
\sigma_n(f,g) &= \sup\{|f(\theta) - g(\theta)| : \theta \in I_n\} \\
\rho_n(f,g) &= \min\{1, \sigma_n(f,g)\} \\
\rho(f,g) &= \sum_{n=1}^{\infty} 2^{-n} \rho_n(f,g)
\end{aligned}$$

when $f,g \in C$. We claim that $\rho$ is a metric on $C$ and that $\rho(f_k,f) \to 0$ if and only if $\{f_k\}$ converges to $f$ uniformly on every compact set in $R$.

To prove that $\rho$ is a metric we first note that

1. $\sigma_n(f,g) = \sigma_n(g,f)$

2. $\sigma_n(f,f) = 0$

3. $\sigma_n(f,g) \leq \sigma_n(f,h) + \sigma_n(h,g)$



where $f,g,h \in C$. In other words, $\sigma_n$ is a pseudo-metric (Kelley, 1955; Sell, 1971). Similarly $\rho_n$ satisfies the same three conditions so $\rho_n$ and $\sigma_n$ are equivalent, which means that for any sequence $\{f_k\}$ in $C$ and any $f$ in $C$ one has $\rho_n(f_k,f) \to 0$ if and only if $\sigma_n(f_k,f) \to 0$. Since $|\rho_n(f,g)| \leq 1$, it follows that the series defining $\rho$ converges absolutely for every pair $f,g$ in $C$. It follows then that $\rho$ satisfies the three conditions (1,2,3) given above. We need only prove that $\rho(f,g) = 0$ for every $n$. Hence $f(\theta) = g(\theta)$ on every set $I_n$. Since $\cup_n I_n = R$, it follows that $f = g$.

Hence $\rho$ is a metric on $C$.

The next thing to show is that the metric topology generated by $\rho$ agrees with the topology of uniform convergence on compact sets. Let $K$ be a compact set in $R$. Then for some integer $n$ one has $K \subset I_n$. Now if $\rho(f_k,f) \to 0$ for some sequence $\{f_k\}$, then $\rho_n(f_k,f) \to 0$, which implies that $\sigma_n(f_k,f) \to 0$. Hence the sequence $\{f_k\}$ converges uniformly to $f$ on the compact sets $I_n$ and $K$. (Katok and Hasselblatt, 1995; Sell, 1971).

Going the other way, assume that $\{f_k\}$ converges to $f$ in the topology of uniform convergence on compact sets. We then want to show that for every  > 0 there is an index

$K$ such that $\rho(f_k, f) \leq 2\epsilon$ whenever $k \geq K$. Let  > 0 be given. Then choose $N$ so that

one has

$$\sum_{n=N+1}^{\infty} 2^{-n} \leq \epsilon$$

Since $\{f_k\}$ converges to $f$ uniformly on compact sets it converges uniformly on $I_N$. Therefore an index $K$ can be found so that the pseudo-metrics

$$\sigma_1(f_k,f),\ldots,\sigma_N(f_k,f)$$

are small whenever $k \geq K$. We choose $K$ so that

$$\sum_{n=1}^{N} 2^{-n}\rho_n(f_k,f) \leq \epsilon$$

Then one has

$$\rho(f_k,f) \leq \sum_{n=1}^{N} 2^{-n}\rho_n(f_k,f) + \sum_{n=N+1}^{\infty} 2^{-n} \leq 2\epsilon$$

whenever $k \geq K$.

We define a mapping $\pi : C \times R \to C$ by

$$\pi(f,t) = f_t$$

where $f_t(\theta) = f(t + \theta)$. For $0 \le \tau < a$ and $0 \le \theta$, define $g_\tau$ by

$$g_\tau(\tau,t) = g(t,\tau + t)$$

Consider $\lim_{t\to+\infty} \pi(f,t)$ (Definition 1.11.6). Then all points under this motion denoted by $\omega_s$ is called the $\omega$-limit of the motion $\pi(f,t)$.

A point $P$ in $C$ is almost recurrent if we can choose $> 0$ where $l = l(\epsilon) > 0$ (Definition 1.11.11). According to Definition 1.11.12, if a point $p \in C$ then $p$ is recurrent (Dontwi and Denteh, 2012A) then it follows that the motion is asymptotically stationary. (Cheban, 2005; Dontwi and Denteh, 2012B, 2012C). Now define a mapping $\pi$ (formally) by

$$\pi(f,g,\tau) = (T_\tau f,g_\tau) \tag{4.11}$$

where $0 \le \tau < a$. We will show shortly that the mapping $\pi$ defines a semiflow on a space $X$ that consists of ordered triples $(f,g)$.

Let us be more precise about (4.9). First we shall assume that the function $f$ lies in $C = C(R^+,R^n)$.

We want to show that

$$\pi(T_\tau f, g_\tau; \sigma) = (T_{\tau+\sigma} f, g_{\tau+\sigma}) \qquad (4.12)$$

where $\tau \geq 0$ and $\sigma \geq 0$. The left side of (4.12) suggests that we want to solve the integral equation.

$$x(t) = \int_0^\tau g_\tau(t,\tau) f_\tau(\tau, y(t)) d\tau \qquad (4.13)$$

and then compute $T_\sigma(T_\tau f)$ by (4.10). Let $\psi(t), 0 \leq t \leq \sigma$, be the solution of (4.13), and $\varphi(s), 0 \leq \tau < \sigma$, the solution of (4.9).

Since

$$\begin{aligned}
\varphi(t+\tau) &= \int_0^{t+\tau} f(t+\tau, t) g(\varphi(\tau), \tau) d\tau \\
&= \int_0^{\tau} d\tau + \int_\tau^{t+\tau} [f(t+\tau, s) g(\varphi(\tau), \tau)] d\tau \\
&= \int_0^{t} [f(\tau+t, \tau+s) g(\varphi(\tau+t), \tau+t] d\tau = \int_0^{t} [f_\tau(t,\tau) g_\tau(\varphi(\tau+t), t) d\tau
\end{aligned}$$

the uniqueness of solutions implies that $\varphi(t+\tau) = \psi(\tau)$ for $0 \leq \tau \leq \sigma$.

Next we have

$$
\begin{aligned}
T_\sigma(T_\tau f)(\theta) &= \int_0^\sigma f_\tau(\sigma\theta,t)g_\theta(\psi(\tau),\tau)d\tau \\
&= \int_0^\tau f(\tau+\sigma+\theta,\tau)g(\varphi(\tau),\tau)d\tau \\
&+ \int_\tau^{\tau+\sigma} f_\tau(\sigma+\theta,t-\tau)g_\tau(\psi(t-\tau),t-\tau)ds \\
&= \int_0^{\tau+\sigma} f(\tau+\sigma+\theta,\tau)g(\varphi(\tau),\tau)dt \\
&= (T_{\tau+\sigma}f)(\theta)
\end{aligned}
$$

Since the mapping $(g,\tau) \to g_\tau$ and $(f,\tau) \to f_\tau$ satisfy the semigoup property, we see that $\pi$ does as well *[Sell, 1971]*.

## Theorem 4.5.1

Assume that the spaces C,$L_p$ and $\mathbb{C}_p^*$ have the prescribed topologies. Then the solution $\varphi(f,g;t)$ depends continuously on the four variables. More precisely, if $f_n \to f$ in $C(R^+,R^n), g_n \to g \text{ in } (\mathcal{L}_p,\mathcal{T}_b), \tau_n \to \tau \text{ in } (\mathbb{C}_p^*,\mathcal{T}_w)$ and $t_n \to t$ in $R^+$, where $t_n$ is a point in the maximal interval of definition of the solution $\varphi(f_n,g_n,\tau_n,t)$, then $t$ lies in the interval of definition of $\phi(f,g,\tau;t)$ and

$$\varphi(f_n,g_n,\tau_n,t_n) \to \varphi(f,g,\tau;t)$$

One consequence of this theorem is that if $[0,a_n)$ denotes the maximal interval of definition of $\varphi(f_n,g_n,\tau_n,t_n)$, then $a \leq \liminf \tau_n$.

Another consequence of this theorem is that the mapping $T_\tau$ , which actually depends on the three terms *f,g* and $\tau$, is continuous in these variables. That is if $f_n \to f, g_n \to g, \tau_n \to \tau$ then $T_{\tau_n} f_n(\theta)$ converges to $T_\tau f(\theta)$ uniformly for $\theta$ in compact sets in $R^+$ [Sell, 1971].



A third consequence is the following:

## Theorem 4.5.2

Assume that the spaces C,$L_p$ and $\mathcal{C}_p^*$ have the prescribed topologies. Then the mapping $\pi$ given by (4.11) defines a semiflow on $X = C \times \mathcal{L}_p \times \mathcal{C}_p^*$.

## Theorem 4.5.3

Let (G,C) be a compatible pair of spaces. Let $\varphi(f,g;t)$ be the unique solution of

$$(Sy)(t) = \int_0^\tau g(t,\tau) f(y(\tau),\tau) d\tau,$$ Dontwi (2005)

where $f \in C, g \in$ G and $a \in$ C, on the maximal interval $[0,a)$. Then the mapping $\pi$ given by (4.11) is a semi flow on $C \times$ G $\times$ C, where the interval of definition of the motion $\pi(f,g,a;t)$ is $[0,a)$.

Equation (4.9) then agrees with both Theorems 5.6.1 and 5.6.2 in Cheban (2005), where $f$ is asymptotically almost periodic. From (4.9), it is inferred from Cheban (2009, pp. 160), that a resolvent of integral equation 4.9 is called a Matrix Function R $\in C(R^+,R^n)$ satisfying the equation

$$\mathcal{R}(t) = \int_0^\tau A(\tau - x)\mathcal{R}(\tau)d\tau.$$

This implies that the solution is given by

$$(Sy)(t) = -\int_0^\tau \mathcal{R}(\tau - s)f(\tau)d\tau,$$

where R is the resolvent.

It is then declared that R of (4.9) is hyperbolic (Cheban, 2005; Dontwi, 2005) satisfies the condition of exponential dichotomy on R.

*S* is a contraction from Katok and Hasselblatt (1995), *f* and *g* are called contraction if there exist $\lambda < 1$ such that for any $x,y \in X$

$$d\Big(f(x), f(y)\Big) \leq \lambda d(x,y) \cdots\cdots\cdots\cdots (*)$$

The inequality (*) implies that the map *f* is continuous and therefore its positive iterates form a discrete-time topological dynamical system (Dontwi and Denteh, 2012C).

By iteration equation (*), one sees that for any positive integer *n*,

$$d\Big(f^n(x), f^n(y)\Big) \leq \lambda^n d(x,y) \cdots\cdots\cdots\cdots (**)$$

so

$$d\Big(f^n(x), f^n(y)\Big) \to 0 \text{ as } n \to \infty.$$

This means that the asymptotic behaviour of all points is the same. On the other hand, for any $x \in X$, the sequence $\{f^n(x)\}_{n \in \mathbb{N}}$ is a Cauchy sequence because for $m \geq n$

$$\begin{aligned} d\Big(f^m(x), f^n(y)\Big) &\leq \sum_{k=0}^{m-n-1} d\Big(f^{n+k+1}(x), f^{n+k}(x)\Big) \\ &\leq \sum_{k=0}^{m-n-1} \tau^{n+k} d\Big(f(x), x\Big) \qquad \cdots\cdots(***) \\ &\leq \frac{\tau^n}{1-\tau} d\Big(f(x), x\Big) \longrightarrow 0 \text{ as } n \to \infty \end{aligned}$$

Thus, for any $x \in X$ the limit of $f^n(x)$ as $n \to \infty$ exists if the space is complete, and by equation (**) this limit is the same for all $x$. Let us denote this limit by $p$ and show that $p$ is fixed for $f$. For any $x \in X$ and any integer $n$ one has

$$\begin{aligned} d\Big(p, f(p)\Big) &\leq d\Big(p, f^n(x)\Big) + d\Big(f^n(x), f^{n+1}(x)\Big) + d\Big(f^{n+1}(x), f(p)\Big) \\ &\leq (1+\tau) d\Big(p, f^n(x)\Big) + \tau^n d\Big(x, f(x)\Big) \end{aligned}$$

Since $d\Big(p, f^n(x)\Big) \to 0 \text{ as } n \to \infty$, we have $f(p) = p$. Taking the limit in inequality (***) as $m \to 0$ we obtain that

$$d\Big(f^n(x), p\Big) \leq \frac{\tau^n}{1-\tau} d\Big(f(x), x\Big).$$

We will say that the two sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ of points in a metric space converge exponentially (or with exponential speed) to each other if $d(x_n,y_n) < c\tau^n$ for some $c > 0$, $\tau > 1$. In particular, if one of the sequences is constant, that is, $y_n = y$, we will say that $x_n$ converges exponentially to $y$.

### 4.5.3 The Mapping $T_\tau f$.

If one sets $\theta = 0$, then Equation (4.10) becomes

$$T_\tau f(0) = \varphi(f,g;\tau),$$

in other words, $T_\tau f(0)$ agrees with the solution of (4.9). Secondly, if the kernel $g(t,\tau)$ depends only on $s$ and if $f$ is constant, then $T_\tau f(\theta)$ is a constant function, that is

$$T_\tau f(0) = \varphi(\tau)$$

for all $\theta \geq 0$. This situation occurs precisely when one changes the initial value problem of the differential equation

$$x^0 = g(x,t), \qquad x(0) = x_0$$

into the integral equation

$$x(t) = x_0 + \int_0^\tau g(x(s),s)ds.$$

Consequently the semiflow $\pi(f,g;\tau)$ in (4.11) is an extension of the (semi) flow defined.

The following reformulations of $T_\tau f$ can be established by means of a routine change of variables.

$$T_\tau f(\theta) = \int_{-\tau}^{0} f_\tau(\theta,\tau) g_\tau(\phi_\tau(\tau),\tau) d\tau \tag{4.14}$$

$$T_\tau f(\theta) = -\int_{0}^{\theta} f_\tau(\theta,\tau) g_\tau(\phi_\tau(\tau),\tau) d\tau \tag{4.15}$$

Equation (4.14) is always valid, whereas Equation (4.15) is valid only when the solution $\varphi(\tau)$ can be continued to $t = \tau + \theta$.

### 4.5.4 Limiting Equations

Let (G,C) be a compatible pair of spaces and let $\pi$ be given by (4.11). Let $\Omega$ denote the $\omega$-limit set of the motion $\pi(f,g;t)$ in $X = C \times G \times C$. The collection of all Volterra integral equations of the form

$$X(t) = \int_{0}^{t} F(t,s) G(Y(\tau),\tau) d\tau \tag{4.16}$$

where $(F,G) \in \Omega$ is said to be the set of limiting equations of

$$x(t) = \int_{0}^{t} f(t,s) g(y(\tau),\tau) d\tau \tag{4.17}$$

One can give a comparison between the solutions (4.17) and those of (4.16) much in the same spirit as we can see in Limiting Equations. It is obvious that the mapping $(f,t) \to f_t$ defines a continuous flow on $C(W \times R, R^n)$. Let $\Omega_f$ denote the $\omega$-limit set of the motion $f_t$. We define the set of limiting equations for

$$x^0 = f(x,t)$$

to be the set of all differential equations of the form

$$x^0 = f^*(x,t) \qquad f^* \in \Omega_f$$

For example, if the motion $\pi(f,g;t)$ in $X$ is positively compact, then the set of limiting equations is a nonempty compact connected subset of $X$. Another comparison is the following:

### Theorem 4.5.4

Let $(F,G) \in \Omega$ where $\Omega$-limit set of the motion $\pi(f,g;t)$. Then there is a sequence $\{\tau_n\}$ with $\tau_n \to \infty$ and

$$\varphi(f,g,\tau_n + t) \to \varphi(F,G;t)$$

where the convergence is uniform on compact subsets of $[0,a)$, the maximal interval of definition of $\varphi(F,G;t)$.

## Proof

It follows that there is a sequence $\{\tau_n\}$ with $\tau_n \to \infty$ and

$$\pi(f,g;\tau_n + t) \to \pi(F,G;t)$$

uniformly on compact subsets of $[0,a)$. In particular, this implies that

$$T_{\tau_n+t}f \to T_tF \tag{4.18}$$

uniformly for $t$ in compact subsets of $[0,a)$. However, the convergence in (4.18) means that

$$T_{\tau_n+t}f \to T_tF(\theta)$$

uniformly for $\theta$ in compact sets in $R^+$. If we set $\theta = 0$, this becomes

$$\varphi(f,g;\tau_n + t) \to \varphi(F,G;t)$$

One of the problems that arises with this flow is to describe the limiting behavior of $T_\tau f$ that is, to describe the function $F(t)$ in (4.16).

## Theorem 4.5.5

Let (G,C) be a compatible pair and let $g_t$ be a positively compact motion in G and let $f(t,s) = f(t - s)$ be in C where $a$ is in $L_1[0,\infty)$. Assume that for some $f \in C$ the motion $f_t$ is positively compact and the solution $\varphi(f,g;t) = \varphi(t)$ of

$$x(t) = \int_0^t f(t-\tau)g(y(\tau),\tau)d\tau \tag{4.19}$$

## Proof

Choose a sequence $\{t_n\}$ so that $t_n \to \infty$ and $g_{t_n} \to g^*$ (in G), $f_{t_n} \to f^*$ (in $C$) and

$\varphi_{t_n}(t) \to X(t)$, where the last limit is uniform for $t$ in compact sets. Then $T_{t_n}$ becomes

$$T_{t_n} f(\theta) = \int_{-t_n}^{0} f(\theta - s) g_{t_n}(\phi_{t_n}(\tau),\tau)d\tau$$

or

$$T_{t_n} f(\theta) = f_{t_n}(\theta) + \int_{-\infty}^{0} f(\theta - \tau) g_{t_n}(\phi_{t_n}(\tau),\tau)d\tau$$
$$- \int_{-\infty}^{-t_n} f(\theta - \tau) g_{t_n}(\phi_{t_n}(\tau),\tau)d\tau.$$

If we let $t_n \to \infty$, the last equation becomes

$$F(\theta) = f^* \int_{-\infty}^{0} f^*(\theta - s) g^*(Y(\tau),\tau)d\tau$$
$$- \lim_{t_n \to \infty} \int_{-\infty}^{-t_n} f(\theta - s) g_{t_n}(\phi_{t_n}(\tau),\tau)d\tau.$$

### 4.5.5 Conclusion of the Proof

However, $g_{t_n}(\varphi_{t_n}(\tau),\tau)$ is uniformly bounded. Since $a$ is in $L_1[0,\infty)$ the last limit above is zero. Hence $F$ is given by (4.19) and this agrees with the conclusion drawn by Dontwi (2005) that

$$\lim_{t\to+\infty} |\omega(t)| = 0.$$

It has thus been confirmed that Topological Dynamical techniques have been applied to the given integral equation in Dontwi (2005) and is a contraction and its stationary point gives the required solution.

## 4.6 Applications of Topological Dynamics to the Navier-Stokes Equations



### Introduction

The Navier-Stokes equations would be delved into in a dimension.

### 4.6.1 Central Object of Topological Dynamics Revisited

The central object of study in topological dynamics (Akin (1993, 1997), Ellis (1969), Furstenberg (1981), de Vries (1993)) is a topological dynamical system, i.e. a topological space, together with a continuous transformation, a continuous flow, or more generally, a semigroup of continuous transformations of that space. The origins of topological dynamics lie in the study of asymptotical properties of trajectories of systems of autonomous ordinary differential equations, in particular, the behavior of limit sets and various manifestations of "repetitiveness" of the motion, such as periodic trajectories, recurrence and minimality, stability, non-wandering points. George Birkhoff is considered to be the founder of the field. A structure theorem for minimal distal flows proved by Hillel Furstenberg in the early 1960s inspired much work on classification of minimal flows. A lot of research in the 1970s and 1980s was devoted to topological dynamics of one-dimensional maps, in particular, piecewise linear self-maps of the interval and the circle.

## 4.7 Examples of Flows and Semi-flows

### 4.7.1 Autonomous differential equations

Let $v(x)$ be a $C^1$-vector field on an open set $W$ in $R^n$ and consider the ordinary differential equation

$$x' = \frac{dx}{dt} = v(x) \tag{4.20}$$

Let $\varphi(x,t)$ be the solution of (4.20) that satisfies $\varphi(x,0) = x$. Then $\varphi$ is a flow on $W$. If one only assumes $v(x)$ to be continuous and that (4.20) has unique solutions, then $\varphi$ is still a flow on $W$.

It is possible to replace $W$ with a smooth manifold $M^n$, where $v(x)$ is a $C^1$-vector field on $M^n$. If $\varphi$ is defined as above, then $\varphi$ is a flow on $M^n$. It is worth noting that if the manifold $M^n$ is compact, then $\varphi$ defines a global flow (Sell, 1971).

### 4.7.2 Nonautonomous differential equations

Let $v : W \times R \to R^n$ be a $C^1$-vector field, where $W$ is an open set in $R^n$ and consider the ordinary differential equation

$$x^0 = v(x,t) \tag{4.21}$$

Let $\varphi(x_0,t_0,t)$ be the solution of (4.21) that satisfies $\varphi(x_0,t_0,t_0) = x_0$ and let $I_{(x_0,t_0)}$ be the interval of definition of this solution.

Define

$$J_{(x_0,t_0)} = \left\{t : t_0 + t \in I_{(x_0,t_0)}\right\}$$

and let $X = W \cup R$. Define $\pi$ formally by

$$\pi(p,t) = (\varphi(x_p,t_p,t_p + t),t_p + t)$$

where $p = (x_p,t_p) \in X$. The mapping $\pi$ defines a flow on $X$ where the interval of definition of the motion $\pi(p,t)$ is $J_p$.

This construction is equivalent to reducing Example B to Example A by writing (4.21) as

$$x^0 = v(x,t), \qquad t^0 = 1 \tag{4.22}$$

It should be noted that the flow $\pi$ for this case has no rest points, no periodic points, no compact motions and all the limit sets are empty. One of the objectives of this research is to overcome these deficiencies by giving a more appropriate definition of a flow for (4.21).

### 4.7.3 Parametric equations

Modify Example A so that the vector field $v$ depends continuously on a parameter $\mu$, where $\mu$ ranges over a uniform space $M$. Let $\varphi(x,\mu,t)$ be the solution of

$$x^0 = v(x,\mu) \qquad (4.23)$$

that satisfies $\emptyset(x,\mu,0) = x$. Let $X = W \times M$ and define $\pi$ by

$$\pi(x,\mu,t) = (\varphi(x,\mu,t),\mu)$$

Then $\pi$ is a flow on $X$.

### 4.7.4 Differential Equations

Let $W$ be an open set in $\mathrm{R}^n$ and $f : W \to \mathrm{R}^n$ a continuous function. Then the difference

equation

$$u_{n+1} = f(u_n) \qquad (4.24)$$

defines a discrete semi-flow on $W$ as follows: Let $\varphi(u,n)$ be the solution of (4.24) that satisfies $\varphi(u,0) = u$. Let $I_u$ be the maximal interval of definition of $\varphi$, that is either $I_u = I^+$ or $I_u = \{0,1,...,N\}$ where $\varphi(u,n) \in W$ for $0 \le n \le N$ and $f(\varphi(u,N)) \notin W$. (The set $I_u$

could consist of the single point $\{0\}$.) Then $\varphi$ is a discrete semi-flow on $W$. (Sell, 1971).

## 4.8 Volterra Integral Equations

**Definition of an integral equation**: An integral equation is an equation in which the unknown function appears under one or more integral signs. Naturally, in such an equation there can occur other terms as well. For example, for $a \le s \le b$, $a \le t \le b$, or $a \le s ,t \le b$

the equations

$$f(s) = \int_a^b K(s,t)g(t)dt$$

$$g(s) = f(s) + \int_a^b K(s,t)g(t)dt$$

$$g(s) = \int_a^b K(s,t)[g(t)]^2dt$$

where the function $g(t)$ is the unknown function while all the other functions are known, are integral equations. These functions may be complex-valued functions of the real variables $s$ and $t$.

An important interjection is that integral equations arise as a representation formulas for the solutions of differential equations. Virtually, a differential equation can be replaced by an integral equation which incorporates its boundary conditions. In lieu of that each solution of the integral equation automatically satisfies these boundary conditions.

We have Fredholm and Volterra integral equations. In all Fredholm integral equations the upper limit is fixed while the upper limit of the Volterra integral equations is a variable. (Kanwal, 1971).

## 4.9 Functions

One of the basic concepts in mathematics are functions. Let two sets $X$ and $Y$ be given and suppose that to each element $x \in X$ corresponds an element $y \in Y$, which is denoted by $f(x)$. In this case one says that a function $f$ is given on $X$ (and also that the variable $Y$ is a function of the variable $x$, or that $y$ depends on $x$) and one writes $f : X \to Y$ .

In ancient mathematics the idea of functional dependence was not expressed explicitly and was not an independent object of research, although a wide range of specific functional relations were known and were studied systematically. The concept of a function appears in a rudimentary form in the works of scholars in the Middle Ages, but only in the work of mathematicians in the 17th century.

### 4.9.1 Set Theoretic Definition of a Function

One says that the number of elements of a set $A$ is equal to 1 or that the set $B$ consists of one element if it contains an element and no others (in other words, if after deleting the set $\{a\}$ from $A$ one obtains the empty set). A non-empty set $A$ is called a set with two elements, or a pair, $A = \{a,b\}$, if after deleting a set consisting of only one element $a \in A$ there remains a set also consisting of one element $b \in A$ (this definition does not depend on the choice of the chosen element $a \in A$ ).

If a pair $A = \{a,b\}$, is given, then the pair $\{a,\{a,b\},b\}$, is called the ordered pair of elements $a \in A$ and $b \in A$ and is denoted by $\{a,b\}$ . The element $a \in A$ is called its first element and $b \in A$ is called the second element.

Given sets $X$ and $Y$ , the set of all ordered pairs $(x,y)$, $x \in X$, $y \in Y$ , is called the product of the sets $X$ and $Y$ and is denoted by $X \times Y$ . It is not assumed that $X$ is different from $Y$ , that is, it is possible that $X = Y$ .

Each set is $f = \{(x,y)\}$ of ordered pairs $(x,y)$ , $x \in X$, $y \in Y$ , such that, if $(x^0,y^0) \in f$ and if $(x^{00},y^{00}) \in f$ , then $y^0$ 6= $y^{00}$ implies that $x^0$ 6= $x^{00}$ , is called a function or, what is the same as a mapping.

The set of all first elements of ordered pairs $(x,y)$ of a given function $f$ is called the <u>domain of definition</u> (or the set of definition) of this function and is denoted by $X_f$, and the set of all second elements is called the range of values (the set of values) and is denoted by $Y_f$. The set of ordered pairs itself, $f = \{(x,y)\}$, considered as a subset of the product

$X \times Y$ , is called the graph of $f$. The element $x \in X$, is called the argument of the function, or the independent variable, and the element, $y \in Y$ is called the dependent variable.

### 4.9.2 Iterating Functions

The focus here is on dynamical systems defined by repeated application of a function that maps a space to itself. Specifically, let $X$ be a topological space and $f : X \to X$ be a function mapping $X$ to itself. For every $n \in \mathbf{Z}_+$, define $f^n(x) = fofo...f(x)$, the composition of $n$ copies of the function $f$. The idea is that we start with $x$, then apply $f$ to $x$, then apply $f$ to $f(x)$, and continue this iterative process until we obtain $f^n(x)$. (Adams *et al*, 2008).

## Definition 4.9.1

The dynamical system defined by $f : X \to X$ is the family of functions $\{f^n\}_{n\in Z^+}$, with each $f^n$ mapping $X$ to $X$.

## Example 4.9.2

Let $f : R \to R$ be given by $f(x) = \frac{x}{5}$. Then the dynamical system defined by $f$ is the family of functions given by $f^n(x) = \frac{x}{5^n}$.

In an application of a dynamical system defined by iterating a function $f : X \to X$ , we think of $f(x)$ as describing the new state of the system one unit of time after it was at state $x$. For example, in the modeling of a bacteria population growing by the hour, one might have a function $f(x)$ representing the population size that results one hour after the population was $x$. Also in modeling the position and velocity of a rocket, one might have a function $f(x,v)$ representing the position and velocity of the rocket one second after it had position and velocity $(x,v)$.

Below are few more examples of functions defining a dynamical system.

## Example 4.9.3

Consider the following four functions defined on $R$:

1. $f$ : defined by $f(x) = -2x$,

2. $g$ : defined by $g(x) = \frac{1}{2}x$,

3. $h$ : defined by $h(x) = -x$

4. $k$ : defined by $k(x) = 0$.



When each of these functions is evaluated at $x = 0$, the result is 0. We say that 0 is a fixed point for the associated dynamical system.

First, consider the function $f$, if we take a particular point $x_0$, then $f^n(x_0) = (-2)^n x_0$. Therefore if $x_0$ 6= 0, then the repeated iteration of $f$ on $x_0$ results in values that move further and further and further and further from 0, bouncing back and forth between positive and negative values. The dynamics of $f$ on R are qualitatively depicted in Figure 4.1 in what is called a phase diagram for the dynamical system.

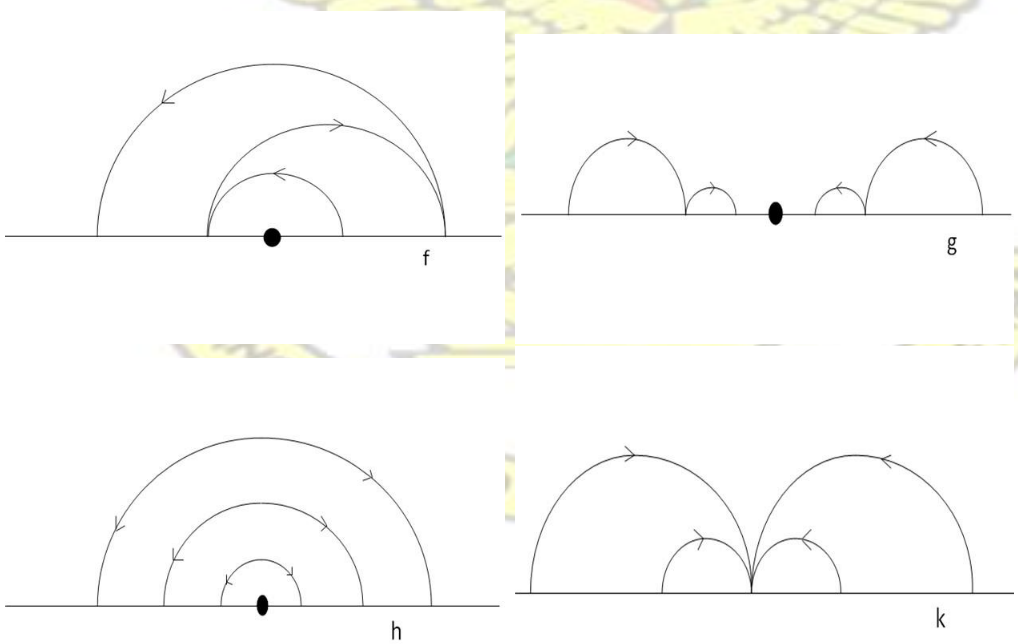

Figure 4.1: Phase diagrams for $f$, $g$, $h$ and $k$

Next consider $g$. Hence iteration of $g$ on a nonzero value results in values that move progressively closer to 0, approaching 0 in the limit. In this case, 0 is referred to as an asymptotically stable fixed point. With $h$ we see a different dynamic picture. We have the fixed point at 0, but with any other $x$ the result of iterating $h$ is an oscillation between the values $-x_0$ and $x_0$. Each nonzero value $x_0$ is called a period-2 point of the dynamical system.

Finally, consider $k$. Here the dynamics are simple. Every point, upon application of $k$, is sent immediately to the fixed point at 0. So 0 is a fixed point and every other point is referred to as an eventual fixed point.

## Definition 4.9.4

Let $f : X \to X$ , and assume $x \in X$.

1. The orbit of $x$ under $f$ is the sequence $(x, f(x), f^2(x), ..., f^n(x), ...)$ And is denoted $O(x)$.
2. We say that $x$ is a fixed point of $f$ if $f(x) = x$. So the orbit of a fixed point $x$ is a constant sequence at the point $x$.
3. We say that $x$ is an eventual fixed point of $f$ if $x$ is not a fixed point of f but $f^n(x)$ is a fixed point for some $n \in Z_+$.

4. Assume $m \in \mathrm{Z}_+$. We say that $x$ is a periodic point or a -$m$ point if $f^m(x) = x$ and $fj(x)$ 6= $x$ for $j = 1,...,m -$ 1. Under these circumstances the orbit of $x$ is called a periodic orbit or a period - $m$ orbit. Also, we say that $m$ is the period of the periodic point or the periodic orbit.



5. We say that $x$ is an eventual periodic point if $x$ is not a periodic point but $f^n(x)$ is a periodic point for some $n \in \mathrm{Z}_+$.

## Example 4.9.5

Here we consider two simple examples of savings accounts. First, suppose that we deposit money in a savings account that earns 10% interest, compounded annually. After the initial deposit, we do not make any further deposits to the account nor do we make any withdrawals from it. We simply let the amount in the account accrue the earned interest. The function $f : [0,\infty) \to [0,\infty)$, given by $f(x) = 1.10x$, defines a dynamical system that models the amount in the account as it changes year by year.

The dynamics of $f$ are straightforward: there is a fixed point at 0, and every other point has an orbit that increases away from 0 upon successive iteration of $f$. (See figure 4.2)

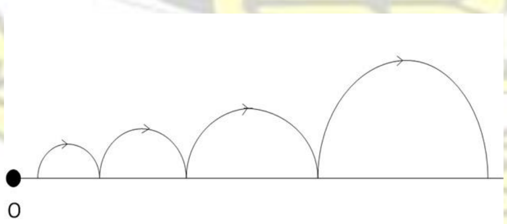


Figure 4.2: The dynamics of $f$

Now assume that after the interest is applied each year, we withdraw either GHC5,000 from the account (if there is at least that much in the account) or the balance of the account (if it is less than GHC5,000). In this case the function $g : [0,\infty) \rightarrow [0,\infty)$,given by



$$g(x) = \begin{cases} 1.10x - 1000 & \text{if } 1.10x \geq 5000, \\ 0 & \text{if } 1.10x \leq 5000, \end{cases}$$

defines a dynamical system modelling how the amount in the account changes. Here too, 0 is a fixed point. There is another fixed point at $x$ = 50,000 that we find by solving $g(x) = x$. We can also find the fixed point at 50,000 by reasoning that the amount in the account will be fixed when it is such that the interest of 10% provides exactly the GHC 5,000 needed for the annual withdrawal. Since 10% of 50,000 is 5,000, it follows that the fixed point occurs at $x$ = 50,000. For values of $x$ greater than 50,000, the interest on $x$ provides more than the amount needed for the GHC 50,000 withdrawal; so the amount in the account will grow without bound under successive iteration of $g$. If $x$ < 50,000, then eventually the amount in the account will equal 0. So nonzero values of $x$ that are less than 50,000 are eventual fixed points of $g$. This is illustrated by the dynamics of $g$ in figure 4.3

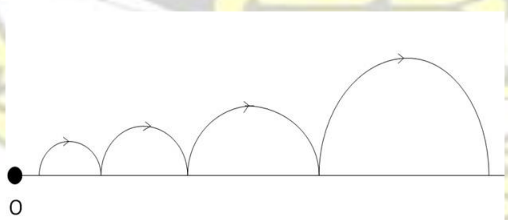


Figure 4.3: The dynamics of $g$

## Example 4.9.6

Imagine a stretch of bread dough lying across the interval [0,1]. Suppose that we uniformly stretch the dough to thrice its length and then fold the dough over, pressing it together so that it again lies across the interval [0,1]. (See figure 4.4)



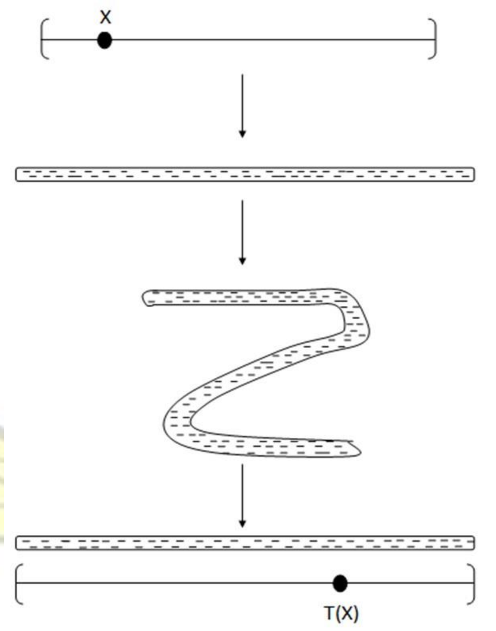


Figure 4.4: Stretching, folding and pressing the bread dough

Let :[0,1] → [0,1] be defined by setting $T(x)$ equal to the new position of a loaf of bread that was originally at $x$, after the stretching, folding, and pressing of the dough (Dontwi and Denteh, 2012D). More precisely, $T$ is defined by

$$T(x) = \begin{cases} 3x & \text{if } x \in \left[0, \frac{1}{3}\right] \\ 2 - 3x & \text{if } x \in \left[\frac{1}{3}, \frac{2}{3}\right] \end{cases}$$

$$3x - 2 \quad \text{if} \quad x \in \left[\tfrac{2}{3}, 1\right]$$

The following definition establishes a notion of equivalence for dynamical systems defined by function iteration:

## Definition 4.9.7

The functions $f : X \to X$ and $g : Y \to Y$ (and the dynamical systems defined by them) are said to be topologically conjugate if there exists a homeomorphism (Banks *et al* (1992) showed the importance of homeomorphisms in topological transitions, Akin (1993)and Fathi and Herman (1977)) $h : X \to Y$ such that $g \circ h = h \circ f$. The function $h$ is called a topological conjugacy between $f$ and $g$.

We illustrate the topological conjugacy condition $g \circ h = h \circ f$ in Figure 4.5. The idea is that both routes from the upper-left $X$ to the lower-right $Y$ - across the top, then down the right side, and down the left side, then across the bottom - give the same result. We say that the diagram commutes. Essentially, $h$ is mapping the function $f$ to the function $g$.

## Example 4.9.8

The dynamics of the functions $f(x) = 2x$ and $g(x) = 3x$ appear qualitatively the same. In both cases there is a fixed point at 0, and all other orbits stay either on the positive or negative side of 0 and move outward from 0. In fact, these two functions are topologically conjugate. The function $h : \mathbb{R} \to \mathbb{R}$,defined by $h(x) = x^{\log_2(3)}$, is a homeomorphism that satisfies $g \circ h = h \circ f$. This is illustrated as follows:

A topological conjugacy between two functions $f$ and $g$ naturally maps orbits of $f$ to

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & X \arrow[d, "h"] \\ Y \arrow[r, "g"] & Y \end{tikzcd}$$

Figure 4.5: Topological conjugacy requires that $g \circ h = h \circ f$

orbits of $g$, as the following theorem indicates:

## Theorem 4.9.9

Let $h$ be a topological conjugacy between $f : X \to X$ and $g : Y \to Y$. For each $x \in X$ and $n \in \mathbb{Z}_+$,we have $h(f^{n-1}(x))$ $= g^n(h(x))$, and consequently $h$ maps the orbit of $x$ under $g$.

## Proof

We prove this by induction on $n$. Then $n = 1$ case holds by the definition of topological conjugacy. Assume that the result holds for $n - 1$. Then,

$$
\begin{aligned}
h\left(f^{n}(x)\right) &= h\left(f^{(n-1)}(f(x))\right) \\
&= g^{n-1}\left(h(f(x))\right) \\
&= g^{n-1}\left(g(h(x))\right) \\
&= g^{n}(h(x))
\end{aligned}
$$

### Corollary 4.9.10

Let $h$ be a topological conjugacy between $f : X \to X$ and $g : Y \to Y$, and assume that $x \in X$. Then the following implications hold:

1. If $x$ is a fixed point of $f$, then $h(x)$ is a fixed point of $g$.
2. If $x$ is a period - $m$ point of $f$, then $h(x)$ is a period - $m$ point of $g$.
3. If $x$ is an eventual fixed point of $f$, then $h(x)$ is an eventual fixed point of $g$.
4. If $x$ is an eventual periodic point of $f$, then $h(x)$ is an eventual periodic point of $g$.

The corollary implies that important dynamic features of $f$ are mirrored in functions that are topologically conjugate of $f$.

## 4.10 The Application to the Navier-Stokes Equations

In this section topological dynamics is applied to the Navier-Stokes Equation to determine that it defines an integral equation. The Navier-Stokes Equations cover the study of fluids.

The equations hinge on the assumption that the fluid is a continuum, which means it is not made up of discrete particles but rather a continuous substance. (Batchelor, 2000; White, 2006; Derivation of the Navier-Stokes Equations, 2012.).

Conservation of mass, momentum and energy are used in its derivation. The convective or material derivative concept is adopted:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v.\nabla \qquad (4.25)$$

Here *v* is the velocity of the fluid. The first term on the right-hand side of the equation is the ordinary Eulerian derivative which is the derivative on a fixed reference frame, representing changes at a point with respect to time, whereas the second term denotes changes of a quantity with respect to position. The operator *D/Dt* has meaning only when applied to a field variable, that is a function of *x* and *t* and is said to give a time derivative following the motion of the fluid, or a material derivative. If a material surface in the fluid is specified geometrically by the equation $F(x,t)$ =constant. *F* is a quantity which is invariant for a fluid particle on the surface, or that $\frac{DF}{Dt} = 0$ at all points on the surface. In particular, the equation to any surface bounding the fluid must satisfy the equation $\frac{DF}{Dt} = 0$. (Batchelor, 2000).

For instance, the measurement of changes in wind velocity in the atmosphere can be obtained with the help of an anemometer in a weather station or by mounting it on a weather balloon. Using the Reynolds transport theorem the conservation laws would give rise to the following integral equation:

$$\frac{d}{dt}\int_{\Omega} LdV = -\int_{\partial\Omega} LvndA - \int_{\Omega} QdV \tag{4.26}$$

In this scenario $v$ is the velocity of the fluid and $Q$ represents the sources and sinks in the fluid. $\Omega$ denotes the control volume and $\partial\Omega$ its bounding surface.

The divergence theorem when applied changes the surface integral into a volume integral:

$$\frac{d}{dt}\int_{\Omega} LdV = -\int_{\Omega} \nabla.(Lv)dV - \int_{\Omega} QdV \tag{4.27}$$

Using Leibniz's rule on the left and combining yields:

$$\int_{\Omega} \frac{dL}{dt}dV = -\int_{\Omega} \nabla.(Lv)dV - \int_{\Omega} QdV \qquad \frac{\partial L}{\partial t} + \nabla.(Lv) + Q = 0 \tag{4.28}$$

In another sense the Navier-Stokes equations in the form of elemental nature can be obtained from using conservation of momentum:

$$\frac{\partial}{\partial t}(\rho v) + \nabla.(\rho vv) + Q = 0 \tag{4.29}$$

$$\frac{\partial \rho}{\partial t}v + \frac{\partial v}{\partial t}\rho + \nabla.(\rho v).v + \rho v\nabla.v = b \tag{4.30}$$

$$v\frac{\partial \rho}{\partial t} + \rho\frac{\partial v}{\partial t} + \nabla(\rho)v.v + \rho\nabla(v).v + v\rho\nabla.v = b \tag{4.31}$$

$$v\frac{\partial \rho}{\partial t} + \rho\frac{\partial v}{\partial t} + v.v\nabla(\rho) + \rho v\nabla(v) + \rho v\nabla.v = b \tag{4.32}$$



The covariant derivative as special case brings out the gradient of a vector:

$$v.\nabla\rho + \nabla\rho.v = \nabla.(\rho v) \tag{4.33}$$

$$v\left(\frac{\partial \rho}{\partial t} + v.\nabla\rho + \rho\nabla.v\right) + \rho\left(\frac{\partial v}{\partial t} + v.\nabla v\right) = b \tag{4.34}$$

$$v\left(\frac{\partial \rho}{\partial t} + v.\nabla(\rho v)\right) + \rho\left(\frac{\partial v}{\partial t} + v.\nabla v\right) = b \tag{4.35}$$

The convective derivative now becomes:

$$\rho\left(\frac{\partial v}{\partial t} + v.\nabla v\right) = b \qquad \Rightarrow \rho\frac{Dv}{Dt} = b \tag{4.36}$$

(4.36) is an expression for the Newton's Second Law in the Navier-Stokes equations and can be written as:

$$\rho\frac{d}{dt}\left(v(x,y,z,t)\right) = b \quad \Rightarrow \quad \rho\left(\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial t}\right) = b \tag{4.37}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = b \quad \Rightarrow \rho\left(\frac{\partial v}{\partial t} + v.\nabla v\right) = b \tag{4.38}$$

where $v = (u,v,w)$

$$v = \left(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}\right)$$

The derivative is in a way "following" a fluid "particle", and in order for Newton's second law to work, forces must be summed following a particle. The convective derivative is also called the particle derivative.

## 4.11 Application

The Navier-Stokes equation can be generalized in the form depicted in the theorem

**Theorem 4.11.1**

Consider the equation

$$x(t) = f(t) + \int_{-t}^{0} \big(\alpha(t-s)\big) g\big(x(s)\big) ds \tag{4.39}$$

which implies that $f(t)$ is continuous for $t \to 0$ and $f(t) \to f_0$ as $t \to -\infty$, $g(x)$ is locally Lipschitian and strictly decreasing. Suppose the solution $\varphi(t) = \varphi(f,g;t)$ of equation (4.44) is bounded and uniformly continuous for all $t \leq 0$ and let $x_0$ be the solution $x_0 = f_0 - Ag(x_0)$. Then $\varphi(t) \to x_0$ as $t \to -\infty$.

## 4.12 Dynamic Systems of Shifts in the Space of Piecewise Continuous Functions

### 4.12.1 Introduction

In this section we embark on the study of Dynamic Systems of Shifts in the space of piece-wise continuous functions analogue to the known Bebutov system. We give a formal definition of a topological dynamic system in the space of piece-wise continuous functions and show, by way of an example, stability in the sense of Poisson discontinuous function. We prove that a fixed discontinuous function, $f$, is discontinuous for all its shifts, $\tau$, whereas the trajectory of discontinuous function is not a compact set.

The interest in the study of Differential Equations with Impulse is increasing. Attempt to extend this study (Dontwi 1994) to known topological methods of the Theory of Dynamic Systems (DS) (Sibiriskii 1970, Levitan and Zhikov 1982, Shcherbakov 1972 and 1975, Cheban 1977 and 1986) brings into fore the necessity of studying DS of shifts in the space of piece-wise-continuous functions which are solutions of these equations.

We extend the study DS of shifts in the space of piece- wise-continuous functions analogue to Bebutov Systems. We give a formal definition of a topological dynamic system in the space of piece-wise continuous functions and show, by way of an example, stability in the sense of poisson discontinuous function. We prove that a fixed discontinuous function, $f$, is discontinuous for all its shifts, $\tau$, whereas the trajectory of discontinuous function is not a compact set. These should prepare the way for the introduction and application of notions of Recurrence (Gottschalk and Hedlund (1955) introduced the idea of distinguishing different sets of recurrence) motions of dynamic systems (Shcherbakov 1972, Bronshtein 1979, Levitan and Zhikov 1982, Pliss 1966, Sacker and Snell 1994) to various trajectories of Differential Equations with Impulse(Distributions) (Hale 1977, Chehan 1999 and 2001, Dontwi 1988A, 1988B, 1988C, 1988D, 1994 and 2001). Fu and Duan, (1999) have shown that the Bebutov's shift dynamical system is a chaotic system.

### 4.12.2 Notions and Preliminaries

Let **R** and **N** be the set of real numbers and the set of natural numbers respectively, $f(t_0+0)$, $f(t_0+0)$ be the left and right sided limits of the function $f(t)$ at the point $t = t_0$.

We consider **PC[R]**- the space of piece-wise-continuous real-valued functions defined on the number line **R** with the following properties:

i. The set of points of discontinuity of every function $f \in$ **PC[R]** represented as **D**$(f))$is either empty or has points of discontinuity of the first kind;

ii. The point of discontinuity of every function, if it is more than one, is distinct from each other at a distance not less than some fixed positive number for a given function.

The jump or discontinuity of the function $f(t)$ at the point $t = t_0$ is the number

$$h = \max\left\{|f(t_0-0) - f(t_0)|, |f(t_0+0) - f(t_0)|, |f(t_0+0) - f(t_0-0)|\right\}.$$

In **PC[R]**, (or simply **PC**), we consider countable partitions of family of semi-norms

$$P_n(f) = \sup_{|t|\leq n}\left|f(t)\right| \quad (n \in N),$$

defined for every function $f \in$ **PC** and induces metrisable topology in this space. Further we shall represent this metrisable space by **PC**.

## Remark 4.12.1

The sequence of functions $\{f_n(t)\}$ from **PC** is convergent if in **PC** there exist a function $f(t)$ such that $f_n(t)$ converges uniformly to $f(t)$ in every interval $[-k,k]$, where $k \in N$. We write this in the form

$$\lim_{n\to\infty} f_n = f$$

The following hold:

## Lemma 4.12.2

If the function $f(t)$ at the point $t = t_0$ is a jump of magnitude $h > 0$ while the function $g(t)$ is continuous at this point, then for every $n \in N$ and $n > |t_0|$ the following is true:

$$P_n(f-g) \geq \frac{h}{2}.$$

## Lemma 4.12.3:

Let $\lim_{n\to\infty} f_n(t) = f$. Then

a. If the function $f(t)$ is discontinuous at the point $t = t_0$, then all functions $f_n(t)$ (except, maybe, for a finite number of points) are also discontinuous at the same point. As a consequence we have the following:

b. If beginning from some number, all functions $f_n(t)$ are continuous at the point $t = t_0$, then the function $f(t)$ is also continuous at this point.

The reverse of the above statements hold.

## Example 4.12.4

Let $f(t) = 0$ and $f_n(t) = \begin{cases} \frac{1}{n} & \text{for } t > 0; \\ 0 & \text{for } t \leq 0. \end{cases}$ then $\lim_{n\to\infty} f_n = f$.

## Remark 4.12.5

The space **PC** is not complete.

For any $f \in$ **PC** and $\tau \in$ **R** we represent by the symbol $f^\tau$ the shifts of the function $f(t)$ by $\tau$, that is $f^\tau(t) = f(t + \tau)$.

Following Bebutov dynamic systems in the space **PC** we consider the family of shifts (or translates) $\phi$ : **PC**×**R** $\longrightarrow$ **PC**, defined by the formula $\phi(f,\tau) := f^\tau$ for all $f \in$ **PC**, $\tau \in$ **R**

### 4.12.3 Main Results

## Theorem 4.12.6

The mapping $\phi$ defined above satisfies the following conditions:

a. $\phi(f,0) = f$, for any $f \in$ **PC**;

b. $\varphi(\varphi(f,s)) = \varphi(f,t+s)$, for any $f \in$ **PC** and $t,s \in$ **R**;

c. $\phi(f,\tau)$ is continuous in $f$ for any fixed $\tau$ and for a fixed $f$-continuous function, the mapping $\phi(f,\tau)$ is continuous in $\tau$, however, if for a fixed $f$ it is a discontinuous function then $\phi(f,\tau)$ is discontinuous at all points of $\tau$.



## Proof:

(a.) and (b.) are obvious.

Continuity of $\phi(f,\tau)$ in $f$ for a fixed $\tau$ by Remark (4.12.1) implies uniform convergence of the function $f_n(t) \to f(t)$ as $n \to \infty$ in every interval $|t| \le m$, $m \in N$, which in turn implies uniform convergence of the function $f_n(t+\tau) \to f(t+\tau)$ as $n \to \infty$ in every interval $|t| \le k$, $k \in N$.

If $f_0$ is a continuous function, then $\phi(f_0,\tau)$ is continuous in $\tau$ by the known property of Bebutov Dynamic System.

The motion corresponding to the continuous function $f$ is continuous (Abott, 2001), and if it is discontinuous it will be discontinuous at every point.



## Theorem 4.12.7

For any arbitrary discontinuous function $f$ from **PC**, its trajectory is not a compact set.

## Proof:

Let $f$ be any discontinuous function in **PC**. Consider $\{f^{\tau m}\}$, where $\tau_n = \frac{1}{n}$. The given sequence converges point-wise to the function $f$ of points of continuity of $f$. However, no sub-sequence of the given sequence converges to $f$ in **PC**, and this means, in general it does not converge in **PC**.



### 4.12.4 Concluding Remarks

The topological dynamical system in the space of piece-wise continuous functions has been shown by way of an example, as well as stability in the sense of the Poisson discontinuous function. It has also been proved that a fixed discontinuous function, $f$, is discontinuous for all its shifts, $\tau$, whereas the trajectory of discontinuous function is not a compact set.

## 4.13 Using Differential Equations in Modeling Bats' Numbers in a Habitat

### 4.13.1 Introduction

The use of differential equations in real life problems cannot be overemphasized. Differential equations have been used to model the dynamics of systems such as aeroplanes, the hierarchy of a political or religious system. Ordinary differential equations are again used in modeling biological, social, physical, engineering, as well as systems. In fact most things behave and evolve in ways determined by some rules. For instance, the Newtonian revolution lies in the fact that the principles of nature can be expressed in terms of mathematics and physical events can be predicted and designed with mathematical certainty. In the social sciences quantitative deterministic descriptions also have taken a hold. Some important expressions lend themselves to the quintessence of truth.



In order to develop a systematic approach to calculate or predict the number of bats in an environment this study was embarked upon to come up with a differential equation model that will be able to do that easily.

The subjects included in this study were assumed to be bats and the following factors were outlined: the current population of the habitat, the rate of animals entering the forest, the rate of departure of the number of animals. These were used to develop the model.

### 4.13.2 Assumptions

All natural and climatic conditions are assumed such that the population of the species would continue to exist continuously. Assume the current population of the said environment which would be referred to as the forest to be $p_1$ with $p_2$% being bat population (Dontwi and Denteh, 2011).

Assume the rate of animals entering the forest by being born or coming from other places to settle there to be in the neighbourhood of $p_3$ animals per year with $p_4$% of the new arrivals of bats. Assume the rate of departure of the number of animals by extinction, either by dying, hunting, poaching, or leaving to other places to be approximately $p_5$ animals per year. Assume the departing number has the same structure as the number of species at the time of departure in the forest.

### 4.13.3 The Model

An important and applicable fact useful to any venture where input and output are imperative could be used:

$$Total\ Rate = Rate\ in - Rate\ out \tag{4.40}$$

In situations of this nature $n$ could be chosen to denote the number of bats in the forest and also use $t$ to stand for time in years from the current time, then the total rate will be the rate of change of the number of bats with respect to time given as

$$Total\ Rate = \frac{dn}{dt} \tag{4.41}$$

It is important to calculate the rate in as well. This implies that

$$rate\ in = p_3 p_4 \% \tag{4.42}$$

The population in the forest would be the initial number in addition to an added population per year multiplied by the number of years. It needs no telling that there are $p_3$ animals and $p_5$ leaving, the number at time t could be given by

$$Y(t) = p_1 + (p_3 - p_6)t = p_1 + q_1 t \tag{4.43}$$

where $q_1 = p_3 - p_5$.

The rate out could be given by

$$Rate\ out\ =\ \frac{Number\ of\ bats}{Number\ of\ animals\ in\ the\ forest} \times Rate \tag{4.44}$$

of all leaving the forest, this gives

$$\frac{p_5 n}{p_1 + q_1 t}. \tag{4.45}$$

Using the rate in and rate out conditions that would yield

$$\frac{dn}{dt} = \frac{p_3 p_4}{100} - \frac{p_5}{p_1 + q_1 t} n \tag{4.46}$$

which simplifies to

$$\frac{dn}{dt} + \frac{p_5}{p_1 + q_1 t} n = \frac{p_3 p_4}{100} \tag{4.47}$$

That gives a first order differential equation which has the following integrating factor

$$\mu = e^{\int \frac{p_5}{p_1 + q_1 t} dt} = e^{\frac{p_5}{q_1} \ln(p_1 + q_1 t)} = (p_1 + q_1 t)^{\frac{p_5}{q_1}} \tag{4.48}$$

The result gives rise to

$$n = (p_1 + q_1 t)^{-\frac{p_5}{q_1}} \int (p_1 + q_1 t)^{\frac{p_5}{q_1}} \frac{p_3 p_5}{100} dt \tag{4.49}$$

$$n = \frac{p_3 p_5}{100} (p_1 + q_1 t)^{-\frac{p_5}{q_1}} \left( \frac{q_1}{p_5 + q_1} \right) \left[ (p_1 + q_1 t)^{\frac{p_5 + q_1}{q_1}} + C \right] \tag{4.50}$$

$$n = \frac{p_3 p_4 q_1}{100(p_5 + q_1)} \left[ (p_1 + q_1 t) + C (p_1 + q_1 t)^{-\frac{p_5}{q_1}} \right]$$

At $t = 0$ there would be $p_2$% of $p_1$ which implies that $n = \frac{p_1 p_2}{100}$ at $t = 0$

$$\frac{p_1p_2}{100} = \frac{p_4p_4q_1}{100}\left[(p_1 + C(p_1)^{\frac{p_5}{q_1}}\right] \tag{4.51}$$

$$\frac{p_1p_2}{p_3p_4q_1} - p_1 = C(p_1)^{-\frac{p_5}{q_1}} \tag{4.52}$$

$$c = \left[\frac{p_1p_2}{p_3p_4q_1} - p_1\right] p_1^{\frac{p_5}{q_1}} \tag{4.53}$$

$$\frac{p_1p_2}{p_3p_4q_1} - p_1 = C(p_1)^{-\frac{p_5}{q_1}} \tag{4.54}$$

$$n(t) = \frac{p_3p_4q_1}{100(p_5 + q_1)}[p_1 + q_1t] + p_1^{\frac{p_5}{q_1}}\left[\frac{p_1p_2 - p_1p_3p_4q_1}{p_3p_4q_1}\right](p_1 + q_1t)^{-\frac{p_5}{q_1}} \tag{4.55}$$

### 4.13.4 Illustrative Example

In a forest it was recorded that the current population of animals was 40000 and 38% were bats. The rate of animals entering was around 1000 every year with 8% been new arrivals. The rate of departure of the total population was 0.4 thousand every year. We wish to analyze the topological (qualitative) behaviour of the population of bats.

Let the rate of change of the number of bats with respect to time be given by $\frac{dn}{dt}$

Rate in=0.8(1) = 0.8

The total number of animals in the forest $y(t) = 40 + (1 - 0.4)t = 40 + 0.6t$

Rate out=$\frac{n}{40 + 0.6t} \times 0.4$

Therefore $\dfrac{dn}{dt} = 0.8 - \dfrac{0.4n}{40 + 0.6t}$

$$\frac{dn}{dt} + \frac{0.4n}{40 + 0.6t} = 0.8$$



Using integrating factor gives

$$\mu = e^{\int \frac{0.4}{40+0.6t} dt} = e^{\frac{0.4}{0.6}} \ln(40 + 0.6t)$$

$$= (40 + 0.6t)^{\frac{2}{3}}$$

$$n = (40 + 0.6t)^{-\frac{2}{3}} \int (40 + 0.6t) 0.8 dt$$

$$n = 0.8(40 + 0.6t)^{-\frac{2}{3}} \times \frac{1}{0.6} \times \frac{3}{5} \left[ (40 + 0.6t)^{\frac{5}{3}} + c \right]$$

$$n = 0.8 \left[ (40 + 0.6t) + c(40 + 0.6t)^{-\frac{2}{3}} \right]$$

At $t$=0, $n$=15.2 thousand.

$$15.2 = 0.8 \left[ 40 + c(40)^{-\frac{2}{3}} \right]$$

Therefore c=-246

$$n = 0.8 \left[ (40 + 0.6t) - 246(40 + 0.6t)^{-\frac{2}{3}} \right]$$

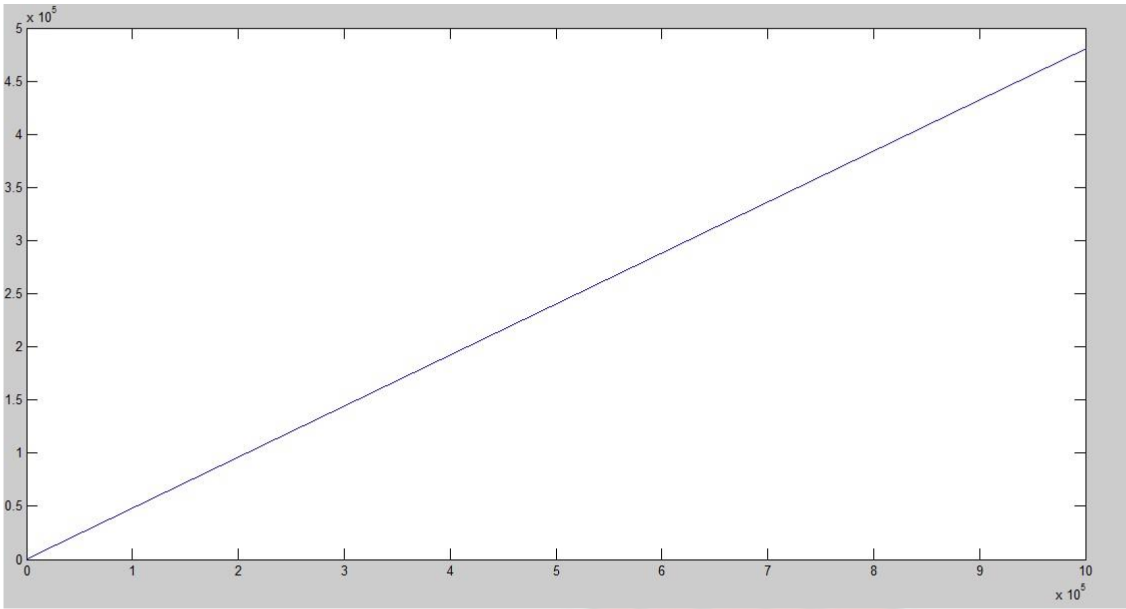


Figure 4.6: Plot of $n$ against $t$

The qualitative or topological behaviour of the solution pack is studied as $t$ approaches infinity. From the Matlab sketch of the graph it is obvious that as $t$ approaches infinity, $n$ approaches infinity. Bats are strong species and can withstand harsh weather conditions and survive. If poaching and killing of the bats are curtailed, then the trend revealed will represent the ideal situation. With all things being equal the number of bats will increase with the passage of time. Considering it for a long period of time, the limit would be obtained as follows:

$$\begin{aligned}\lim_{t\to\infty} n(t) &= \lim_{t\to\infty} 0.8\left[(40+0.6t) - 246(40+0.6t)^{-\frac{2}{3}}\right] \\ &= \infty\end{aligned}$$

which means that for a long period of time, the number of bats will diverge to infinity.

# Chapter 5

# CONCLUSION AND RECOMMENDATION

## 5.1 Conclusion

This research work hinged on the Integral Equation that resulted in Dontwi (2005). Topological Dynamical techniques were used to analyse it and confirmed the results. In doing that, a lot of other ideas were visited to lend credence to the concept under study. In the end it was shown that the efficacy of the concept has been well-grafted into the repertoire of already existing knowledge as the new contribution to knowledge. Sell developed methods which allowed one to apply the theory of topological dynamics to a very general class of nonautonomous ordinary differential equations. This was extended to non-linear Volterra's Integral Equations. This research took off from there and applied the techniques of topological dynamics to an integral equation. The usage of limiting equations which were used by Sell in his application to integral equations were extended to recurrent motions. In our bid to lend new innovations to our system we then went on further to apply recurrence motions to our systems and then studied the solution path. It thus confirmed the existence of contraction and the stationary point in the Dontwi (2005).

## 5.2 Recommendation

It is recommended that Topological Dynamics could be introduced as a core course into post-graduate programmes for Pure Mathematics students.

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