KWAME NKRUMAH UNIVERSITY OF SCIENCE AND TECHNOLOGY, KUMASI

USING THE WHITTAKER HENDERSON GRADUATION METHOD TO ESTIMATE MORTALITY RATES FOR A PENSION SCHEME

By

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Declaration

I hereby declare that this submission is my own work towards the award of the M. Phil degree and that, to the best of my knowledge, it contains no material previously published by another person nor material which had been accepted for the award of any other degree of the university, except where due acknowledgement had been made in the text.

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Dedication

This work is dedicated to all who always support, encourage and motivate me to aim higher in life. Much love!

Cheers!

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Abstract

Mortality rates are an essential input to actuaries. This is because it is a key requirement in estimating life expectancies, which can further be used to determine the cost of the long-term obligations of a pension fund. However, relatively fewer studies on mortality rates have been conducted in most developing countries such as Ghana. To this end, this study explored the mortality trend and estimated mortality rates for actuarial use using data from Ghana Universities Sta Superannuation Scheme (GUSSS) for a 10-year period. The actuarial estimator was used to compute the crude mortality rates. It was found that observed mortality trend of the members of GUSSS increased with some random uctuations as the age of the member increased. To be used in pension plans, these uctuations were further smoothed using the Whittaker Henderson method of graduation because this method simultaneously considers smoothness whiles adhering to some degree of t to the initial mortality rates. Thus, the estimates of mortality rates for the members of GUSSS were the sequence of values obtained using the Type B Whittaker Henderson graduation method with smoothing parameter, $h = 10$ and order of di erencing, $z = 4$ since those values minimized the composite measure M. The estimates of mortality rates were then used to computed the life expectancies of the members at each age spanning from ages 30 to 85.

Acknowledgements

All thanks be to God and to all who supported me in one way or the other to complete this work especially my dad (Prof. K. A. Danso).

Thank you and God bless you all.

Contents

List of Abbreviation

DB .... De ned Bene t

DC .... De ned Contribution GUSSS ....

Ghana Universities Sta Supernnuation Scheme

NPRA ....National Pensions Regulations Authority

SSNIT ....Social Security and National Insurance Trust

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Chapter 1

Introduction

1.1 Background of study

Over the years, the idea of mortality rate has played a major role in determining population growth. Mortality rate is generally de ned as the measure at which death occurs in a given population due to a speci c cause or not. In actuarial literature, a mortality rate is de ned as the probability, according to a given mortality table that a life age say x, dies before the next age. A tabular array with numerical values of mortality rates presented for certain selected values of x, where x is mostly an integeris called a mortality table, a life table or an actuarial table (London (1997)).

There are di erent types of mortality tables for di erent purposes in existence. However, depending on the length of age interval in which data are presented, Anderson (1999) classi es mortality tables into complete and abridged mortality tables. The complete tables contains data for every single age whiles the abridged table contains data by 5- or 10-year interval. Mortality tables can also be grouped into select or ultimate mortality tables and period based tables or cohort based table.

The existence of mortality table dates back to the sixteen century with the rst mortality table developed in the year 1662 by John Graunt based on data from London’s Bills of mortality. Several years later, Edmund Halley also developed another mortality table using data from Breslau in the year 1693 (Ciecka (2008)). Since then, the study and use of mortality rates have been of signi cant importance in many elds of science such as medicine and demography for implementation and evaluation of health programmes as well as for population projections. For actuaries, the use of mortality rates is very signi cant in building models for estimating premiums, reserves, annuity factors, determining insurance and pension obligations, and for sound actuarial valuations. An example of a published mortality table used by actuaries is "The English Life Tables" (ELT) which based on the mortality experience in England and Wales segregated by age and gender. However, the available tables for Ghana according to Abaitey and Oduro (2017), include the World Health Organization tables for the year 1960, 2000, 2010 and 2012 computed from census data.

It is known that mortality experience has improved with time and hence people live longer resulting in an increase in general life expectancy. This has been attributed to improved health care, improved standards of living and technological advancement. Other factors such as age, sex, occupation, geographical and location may also in uence mortality in various ways.

As a result of this, the estimation and application of mortality rates is key to the actuary in planning. Also, most employees plan for life after work while in service because there is no regular ow of income after retirement. This led to the establishment of pension schemes initially in highly developed countries and has gradually become global. According to the The International Labour Organization (2011), the main reason for adapting pension schemes is that income insecurity or poverty is the major risk faced during old age as a result of the person's inability to earn income fully or partially. Thus ensuring retirement income security is a major concern of every individual, the employer and the country as a whole.

Many a time, the pension plan sponsor is the employer and the retirement bene t may be provided by the government or a private entity in a form of a lump sum or annuity bene ts or both. These schemes are managed and monitored by regulatory authorities set up in every country. For instance, in Ghana, all pension plans are monitored and regulated by the National Pensions Regulations Authority (NPRA) which was established in the year 2008 under Act 766 of the constitution. The pension scheme design may vary from country to country yet the basic principles underlying them are similar. Employer sponsored plans are usually classi ed into two namely; De ned Bene t (DB) and De ned Contribution (DC). Under the DC plan, the bene t level depends on the total contributions in addition to the investment earnings. Hence, this type of plan is similar to a savings account. However, the DB plan is predetermined by a formula in relation to the salary near retirement or salary throughout the employment. The employer bears the risk because he is obliged to pay the bene t in future and therefore needs to invest towards the target bene t. In the event of adverse investment or demographic experience including mortality, the level of contributions may require adjustments to be able to meet the target bene t.

1.2 Problem Statement

A major task of an actuary according to Batten (1978) and da Rocha Neves and Migon (2007), is the selection of the table that re ect the mortality pattern of the population under study and is also suitable for the purpose at hand. Brouhns et al. (2002) suggests that inasmuch as improvement in mortality is viewed as a social achievement and a positive change to individuals, they pose a challenge to private life insurances and for the planning of public retirement systems.

Also, even though the health of any population is revealed in its mortality statistics (kause), Abaitey and Oduro (2017) stated that in Ghana, actuaries usually rely on mortality tables from other countries such as South Africa, to estimate their own mortality rates to be used to compute premiums, sum assureds, annuities and other actuarial values for insurance companies. This is challenging since the underlying population and its characteristics may vary from that of Ghana. They argued further that if researchers were to obtain reliable life tables for the country, it would solve the issue of whether or not the insurance companies are over charging premiums or under paying actual bene ts or vice versa. Therefore there is the need to obtain a table using local data.

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1.3 Objectives of the study

The main objective of this study is to estimate a mortality table for the pension plan. The speci c objectives are to;

- Explore the mortality trend of the population under study
- Estimate mortality rates for actuarial use.
- Compute life expectancy at various ages

1.4 Scope of the study

A pension plan is a nancial contract established between a pension provider and the member(s) of the plan for the purpose of providing an income during the member(s) retirement period. These plans/schemes could be employer, government or privately sponsored.

There are two basic types of employer sponsored pension plans in existence namely;

- De ned Bene t (DB)

  This type of plan speci es a level of bene t by a predetermined formular. This formula mostly depends on the number of years of service and member(s) salary at retirement. The contributions made into the plan may vary due to the demographic or investment experience hence the pension plan actuary monitors the plan fund on a regular basis to assess if contributions need to be changed. Investment risk is assumed by the employer, because it has to invest now for pension payments that will occur in the future therefore the employer bears the risk.

- De ned Contribution (DC)

  The DC plan speci es how much the employer will contribute, as a percentage of salary into a plan. The contributions are accumulated and made available to the plan member upon retirement. Here, the employee bears the risk because the level of bene t to be received depends on the investment experience of the pension fund.

There are several pension plans for various groups of people in Ghana. Among them are the Social Security and National Insurance Trust (SSNIT) which a statutory public trust charged under the National Pensions Act 2008 Act 766, pension plan for military and the Ghana Universities Sta Superannuation Scheme (GUSSS). The GUSSS was established on January 1, 1976 for all existing members as at this date, University teachers and research fellows, University administrative, library and professional sta of the status comparable with that of University teachers. This plan operates on the De ned bene t plan and hence contributions may varies due to demographic experience including mortality.

1.5 Justi cation

The estimation of mortality rates and its use is very crucial for projections such as bene ts and contributions for insurance companies and pension plans. Actuaries take into consideration the mortality rate and the life expectancy of the plan member in determining the cost of long-term obligation of a pension fund, Beard (1971). Hence, they seek to ensure that future liabilities are covered with a high probability to ensure sustainability. This procedure is necessary because the e ect of mortality cannot also be completely overlooked in bene t valuation. Ignoring the e ect of mortality completely may overstate or understate the liabilities of the pension scheme resulting in charging inadequate premiums to fund future liabilities.

1.6 Organization of thesis

The study is organized into ve(5) chapters.

Chapter 1 takes into account the background of the study, problem statement, research objectives, justi cation and organization of the thesis.

Chapter 2 reviews various relevant literature on pension plans, mortality table and its estimation.

Chapter 3 discusses the various methodology employed in the analysis of this work.

Chapter 4 deals with the data analysis and discussion of major ndings.

Finally, the concluding comments is contained in Chapter 5.

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Chapter 2

Literature Review

2.1 Introduction

This chapter of the study discusses some basic concepts and theories in relation to mortality rates and its estimation. The chapter also reviews relevant literature and related works in these areas.

2.2 Mortality models

When mortality probabilities are given by a mathematical model, it is referred to be in parametric form (London (1997)). Mathematical models that represent mortality functions are based on laws of mortality such as the Uniform, exponential, Gompertz, Makeham and Weibull distributions.

The Uniform distribution is a two-parameter distribution with a constant Probability Density Function (PDF) where the distribution parameters are the limits of the interval on the real number axis over which it is de ned. This distribution is mostly used as a model for human mortality over short ranges of time due to its inappropriateness over a broad range of time. The Gompertz distribution was also suggested in 1825. This model is simple to use as well as providing a fairly good t to mortality data over some age ranges from middle age to early old age but it rarely represents mortality over the whole span of human ages. Makeham in the year 1860 modi ed Gompertz model by adding a constant term independent of age to the model allowing for accidental deaths. The xed term improves mortality data at younger ages.

Mathematical symbols mostly used by actuaries include but are not limited to the expected number of deaths in a unit age interval. That is $[x,x+1)$ denoted as d_x the probability that a life age x dies before his next birthday or age $x+1$. That is $P(x < X \le x+1|X > x)$ denoted as q_x

the force of mortality at age x, μ_x

These symbols according to Batten (1978) are annualized measures. Due to this, actuaries have found it necessary to make assumptions about mortality over fractional ages to account for migration in and out of the population under study for life insurance applications. There has been several approaches and laws developed but Batten and many other writers discuss three frequently used assumptions in actuarial practice namely;

1. The Uniform distribution of Death, UDD
2. The Balducci Assumption
3. Constant force of mortality

2.2.1 Uniform Distribution of Death, (UDD)

In the 1800s, Abraham De Moivre attempted to describe the mortality pattern of human lives algebraically by suggesting that survivorship curve (l_x) of a mortality table could be expressed in a linear form. Thus, the UDD assumption can be expressed mathematically as

$$ {}_tq_x = tq_x \qquad (0 \le t \le 1) \tag{2.1}$$

Even though much of his work was proven to be unrealistic, the assumption of uniform deaths occurring between any two ages which one unit apart is still in use.

2.2.2 Balducci Hypothesis

In 1920, an Italian actuary called Gaetano Balducci also described the mortality pattern over one unit interval by the equation shown below

$$_{1-t}q_{x+t} = (1-t)q_x \qquad (0 \le t \le 1) \tag{2.2}$$

2.2.3 Constant Force of Mortality

The third assumption states that within a given unit age interval, force of mortality is constant. That is

$$\mu_{x+t} = \mu \qquad (0 \le t \le 1) \tag{2.3}$$

2.2.4 Study Design

The estimation of mortality rates and its use is very important for decision making. Thus practitioners such as actuaries, demographers, and clinical statisticians have adopted several study designs to estimate the unknown future lifetime of an individual. According to London (1997), the study designs can be divided into;

1. Longitudinal Studies

   The study selects an initial group called cohort time say t = 0 and follows the study population extensively into the future until they have all died. However, this approach is feasible when the failure time is short and is normally adopted for clinical studies. This type of study is also called cohort study and the initial group of study units is called a cohort group

2. Cross-sectional Studies

   This method is normally used by actuaries and demographers because it is convenient for large sample studies. Here, an observational period is chosen for the population whose mortality pattern is to be studied. Migration is allowed hence members are classi ed according to how they join or leave the study period. Batten (1978) describes them as follows;

Starters (s) - members who are active at the beginning of the observation period

New entrants (n) - members who joined the group during the observation period

Withdrawal (w) - members who ceased to participate during the observation

Deaths (d) - members who died during the observation period

Enders (e)- members who are still active at the end of the observation period

The advantageous part of this method is that, it is feasible when time until failure is lengthy since the failure data are observed within the interval and the approach also allows for migration in and out of the study period.

2.3 Construction of mortality table

Sutawanir (2015) de ned a mortality table as a tabular array of numerical values of

$$S(x) = P(X > x) = 1 - F(x)$$

such that x is the age of the person, $S(x)$ is the survival function and $F(x)$ is the distribution function.

These tables are often used to describe the mortality pattern of the underlying population. London categorized the process of constructing of mortality tables into two steps namely; producing a sequence of age-speci c mortality rates based on the experience of the group under study a systematic revision of the initial estimates to produce a "better" representation of the unknown population pattern. This the process of revising the rates is called graduation.

2.3.1 Estimating crude mortality Rates

Mortality rate q_x, in actuarial literature is de ned as the probability that a life age x dies before attaining the next birthday according to a given mortality table. In reality, the true mortality rates are unknown hence the need for estimation. Classical statistical estimation methods including the Maximum Likelihood estimate (MLE) and the method of moments can be used to derive the estimates of mortality but the selection of a model also depends on whether the interest lies in continuous or discrete rates that is, force of mortality or probability of death respectively.

The poisson model can be used to estimate the force of mortality with parameter $\tilde{\mu}E_x^c$. The force of mortality is thereby computed by

$$\tilde{\mu} = \frac{d_x}{E_x^c} \tag{2.4}$$

Another popular model often used is the binomial model. The binomial and poisson models have been in use for the analysis of mortality data but the binomial model is often cited due it its direct connection with mortality table quantities l_x and d_x (Macdonald (1996)). Using the binomial model, the number of death is a random variable $D\ Binom(N,q_x)$ where N is the number of observations. The estimate of the probability of death is given by

$$\hat{q}_x = \frac{d}{N} \tag{2.5}$$

This is also a maximum likelihood estimate with an unbiased mean of q_x and variance $q_x(1 - q_x)$. Sutawanir (2015) discussed the basic statistical estimation methods and concluded that the MLE is preferred over the method of moments due to its asymptotic normality properties. The binomial model also assumes independent and identical distribution.

Several researchers including KOLOS (2001) and Macdonald (1996) have criticized this model because a generalization of the model leads to problems for more realistic observations for instance, all lives may not be observed for the same interval of time.

An alternate model which have been employed by actuaries is the actuarial estimate of mortality. This is similar to the binomial model with the denominator adjusted to mean n years instead of n persons (KOLOS (2001)) The estimate of the probability of death is then given by

$$q_x = \frac{d_x}{E_x} \tag{2.6}$$

where d_x and E_x represent the number of death and initial exposure at age x respectively. Exposure is de ned as the annual number of units of human life which are subject to death, disability, or some other decrement, within a de ned period of observation

In an experience study, each life may contribute to exposure either by already being a member of the observed group when the study began or by entering the observed group between the starting and ending dates of the study. Exposure may also terminate either by withdrawal (voluntarily or otherwise) or by ending the observation period.

Exposure can be grouped into

Exact exposure; this method calculates the exact period of time for which a group of live has been exposed to the risk of death for a particular age.

The actuarial exposure; This approach is similar to the exact exposure method except that for each life that dies, the period of time until the next birthday is added.

The latter approach to calculated exposures was strongly criticized by Hoem (1984) but was defended by Dorrington and Slawski (1993) indicating that the conventional approach of computing the exposed to risk is logically consistent the the aggregate expected number of deaths. Exposure can be calculated using the seriatim method where each exposed life is considered one by one or the grouped individual record method.

2.3.2 Graduation of mortality rates

The development of graduation methods dates back to the nineteenth century. Miller (1946) de ned graduation as the process of securing, from an irregular series of observed values of a continuous variable, a smooth regular series of values consistent in a general way with the observed series of values. He also suggested that the graduated values should not deviate too much from the initial values. Similarly, London (1985) described the initial mortality rates calculated from the raw data by the actuary as an irregular series of values which can be smoothed by a process called graduation.

There are a number of methods in existence through which graduation can be obtained. These methods are grouped into two main types namely

Parametric methods

Non-parametric methods

Parametric methods of graduation expresses the prior opinion of the underlying phenomena by a particular functional form such as the Gompertz Form, Makeham's Form and Weibull's Form. The revised estimates are also expressed as a mathematical function in terms of x.

However, non-parametric graduation of mortality data estimates mortality rates by smoothing of the crude rates obtained directly from original data. Examples of non parametric methods of graduation include; The use of splines, graphical methods, the Generalized Additive Models (GAM), weighted moving average, kernel method, graduation with reference to standard mortality rate and Whittaker Henderson method of graduation. Cubic polynomials are tted over subranges of the data for graduation using splines, giving special attention to the manner in which adjacent tted function meets each other. Then the method of least squares is used to determine the parameters of the spline function. But the Smooth junction interpolation is used when a limited number of initial estimates is known and a di erent interpolating arc is being tted in each subrange of the data.

Also, the method of graduation with reference to standard table is based on the assumption that the graduated rates will exhibit a pattern similar to that of a selected standard table. Therefore, the standard table expresses prior knowledge about the pattern of the true rates we are trying to estimate and so the choice of the standard table used in this case should be one derived from a population with similar characteristics to the one we are trying to estimate.

The moving weighted average is another non parametric method which is simple to use but it is limited by the inability to smooth the near ends of the data in addition to the xed degree of smoothing for any particular lter (Weinert (2007)). Whittaker sought to solve the de ciencies of the moving weighted average(Weinert (2007)) and is credited by this invention. Following the works of Whittaker, Henderson (1924) & Henderson (1925) also made several contributions to the theory. In fact, in his most important publication, he solved the di erence equations by ignoring the boundary conditions; after which he approximately compensated for them. Since then the method has been called the Whittaker Henderson Method. According to Nocon and Scott (2012), this method is one of the frequently used smoothing techniques in existence for the construction of mortality tables.. This method derives its estimates by minimizing a linear function which combines smoothness and tness to data. Henderson was the rst to use matrix approach to solve normal equations and this approach has proved very e cient. The ndings of Henderson were corroborated by Aitken (1927).

Kitagawa and Gersch (1984) also solved the problem via formulating the problem as a stochastic estimation, where the smoothing parameter h is de ned as the signal-noise ratio and e ected a xed interval smoothing algorithm but they observed that their state space algorithm were comparatively ine cient.

While other earlier researchers in the eld failed to apply automatic approach of selecting smoother parameter h, Brooks et al. (1988) documented for the rst time the measurement-based-generated-cross-validated method, which had already been introduced, that is almost a decade by Craven and Wahba (1979) for continuous smoothing.

Above all these, (Papaioannou and Sachlas, 2004) concluded that there is no best method in performing graduation. Thus, nature of the problem, its constraints and the purpose in uences the choice of graduation method. However, the nonparametric methods are advantageous because it is based on the assumption that the age-dependent function is unnecessary hence the problem of choosing an inappropriate model is eliminated when the information behind the model is unknown.

Observing the change in mortality over time, Dickson et al. (2013) groups these changes into three categories namely;

Trend
The trend is explained as the gradual decrease or increase in mortality rates over time.

Shock
This describes a short-term jump in mortality rates resulting from a hazard.

Idiosyncratic The idiosyncratic is explained as the random variations resulting from neither shock nor trend each year.

2.4 Review of related works

The rst mortality table was developed by John Graunt in the year 1662 using the London's Bills of mortality. However, his work was critised by Ciecka (2008) based on the use of inaccurate data and credit was given to Halley as the rst to develop a mortality table based on sound demographic data from Breslau, a city in Silesia which is now the Polish city, Wroclaw. Breslau kept detailed records of births, deaths, and the ages of people when they died. Since then, researchers have estimated several mortality tables for di erent groups and population via di erent methods.

Sutawanir (2015) in his study discussed the statistical procedures used in estimating the mortality parameters from sample data and concluded that the statistical procedures can be used for the estimation process but a graduation process should be used in smoothing the raw mortality table for actuarial functions.

Haberman et al. (1983) also compared some methods of graduation. The various methods were then applied to a standard data set and the results were compared with those available from graphical and parametric graduations of the same data. It was concluded that the Kernel method was simple to use and favorable in terms of adherence to data, exibility and smoothness

A similar study was performed by Deb n et al. (2006) who also compared non parametric methods used in the graduation of mortality. The study reviewed various methods including kernel smoothing, splines, locally-weighted regression (LOESS) and Generalized Linear Models (GAM), and applied mortality data from Valencia Region(Spain) for a three-year period from 1999-2001. The ndings from the study led to the conclusion that Generalized Linear Models(GAM) provide a better t for both gender and also allows the use of real distribution of data.

da Rocha Neves and Migon (2007) also proposed dynamic Bayesian models(local and global) to smooth mortality rates. The authors employed the Markov Chain Monte Carlo (MCMC) techniques with the assumption that the number of deaths at each age follow the poisson distribtion and the probabilities for future number of deaths were estimated. A comparison between the two models were done and the global model proved best for constructing mortality tables for both sexes.

A recent study by Chanco (2016) also estimated mortality rates using the Whittaker Henderson method. Data from the Philipine Social Security for a ten year period was used for the study. The author compared the mortality rates obtained to published mortality rates from previous SSS Actuarial Valuations and observed that the mortality trend observed contradicted the natural increasing pattern of mortality at older ages. To resolve this, the Partial Credibility theory was applied to adjust the revised mortality rates at older ages.

To this end, this study implements the Whittaker Henderson method in the smoothing process due to its advantage of simultaneously smoothing data and adhering to t. Di erent assumptions of the Whittaker graduation method will also be considered in the smoothing process to make informed decisions.

Chapter 3

Methodology

3.1 Introduction

This chapter discusses the methods, models and concepts employed in the analysis of the work. The discussions also include the process of estimating the mortality rates and computing life expectancy as well as the tests to validate the work.

3.2 Study Design

This study employs the Cross-sectional study design. An observational period of ten (10) years is chosen for the population whose mortality pattern is to be studied and the classi cation of the members is according to how they join or leave the study period. Only events observed within the interval are used in the analysis. This method is feasible when time until failure is lengthy and also allows for migration in and out of the study period. A. W. JOSEPH (1952)

3.3 Estimation of q_x, from data

The time of death of an individual is unknown in advance hence in order to estimate the time at which a death bene t is payable, a model of human mortality is needed. From this model, the probabilities of death at a particular age can be computed.

Let (x) denote a life aged x, where $x \geq 0$. The future lifetime of (x) can be modeled by a continuous random variable T_x or a discrete random variable K_x. Hence, $x + T_x$ represents the exact age at death random variable for (x). The distribution function of the future lifetime is given by

$$F_x(t) = P[T_x \leq t]$$

This is interpreted as the probability that a person who is currently age x years dies at or before age $x+t$. The complement of the distribution function is called the survival function and is also given by

$$S_x(t) = 1 - F_x(t) = P[T_x > t]$$

The notations $S_x(t)$ and $F_x(t)$ as shown above are standard statistical notations. However, according to the International Actuarial notations, the equivalent survival and mortality notations are given by

$$_tp_x = S_x(t)\ _tq_x$$

$$= F_x(t)$$

In the case of $t = 1$, the subscript t is normally dropped. Hence p_x represents the probability that (x) survives to at least age $x + 1$ whiles the probability that (x) does not survive to age $x + 1$ is represented by q_x. However, q_x is referred to as the mortality rate at age x in actuarial terminology

The mortality rate q_x, is de ned in actuarial literature as the probability that a life age x dies before attaining the next birthday according to a given mortality table.

In practice, the unknown true values of q_x are estimated from raw data. Several methods for estimating q_x are in existence but the common methods used by actuaries are the actuarial estimate, Kaplan Meier estimator and the maximum likelihood estimate. Classical statistical estimation methods including the Maximum Likelihood estimate (MLE) and the method of moments can be used to derive the estimate of mortality.

For the purpose of this study, the Actuarial estimate of q_x will be used.

3.4 Actuarial Estimator

The Actuarial Estimator is de ned by

$$\hat{q}_x = \frac{d_x}{E_x} \tag{3.1}$$

where d_x number of deaths at age x and E_x is the initial exposed to risk de ned as

$$E_x = E_x^c + \sum_{i=1}^{d}(1 - t_i) \tag{3.2}$$

In this equation, E_x^c is the Central exposed to risk denoting the total observed waiting time and $x + t_i$ is the exact age at death of the i^{th} death in the year of age [x, x + 1] and

t_i records the time of death during the year of death for the i^{th}. It is only de ned for the lives that die during the year of age we are considering and can only take values between 0 and 1. For instance, if we are considering the year of age [40,41] and one person dies when they are aged 40 years and 3 months, then the value of t_x for that person will be 0.25.

The calculation of exposures by this approach uses the Seriatim method. This method involves the consideration of each exposed life one by one. The ages at which exposure contribution began and ended are determined for each life independently, and the results are combined to produce the numbers of life-years of exposure for each unit age interval as de ned by the study.

According to London (1997) all estimators of the form $\frac{death}{exposure}$ such as equation 3.1 are approximately binomial proportions. This is because E is usually large compared to d and the degree of error in assuming q_x is a binomial proportion is small. Therefore, the actuarial estimator is also unbiased with

$$\text{Mean} = q_x \tag{3.3}$$

and

$$\text{Variance} = \frac{p_x \cdot q_x}{Exposure} \tag{3.4}$$

and the 95% con dence interval is given by

$$q_x \pm 1.96\left(\sqrt{\frac{p_x \cdot q_x}{Exposure}}\right) \tag{3.5}$$

3.5 Graduation of Initial Mortality Rates

The initial mortality rates calculated from the raw data normally exhibit some random variations and hence will not be the nal rates used for actuarial valuations because

Miller (1946) suggests that irregularities in the mortality tables from age to age will disturb the orderly progression of premiums gures and so on and inconsistent with the normal view that these gures should be reasonably regular. This description suggests that the underlying law of the true rates is smooth, regular and continuous. He further recommended a process called graduation which he de ned as the process of securing, from an irregular series of observed values of a continuous variable, a smooth regular series of values consistent in a general way with the observed series of values. He also suggested that the graduated values should not deviate too much from the initial values.

Haberman et al. (1983) also stated several years later that the primary objective of graduating a set of initial/crude mortality rates is to produce a set of rates that progress smoothly with age whiles simultaneously re ecting the underlying mortality pattern.

There are a number of methods in existence through which graduation can be obtained. These methods are grouped into two main types namely Parametric methods and Non-parametric methods. The main di erence between the two groups according to Deb n et al. (2006) is that, the non parametric models is based on the assumption that the age-dependent function is unnecessary. This is advantageous because it eliminates the problem of choosing an inappropriate model when the information behind the model is unknown.

For the purpose of this study, a non parametric approach speci cally the Whittaker Henderson approach will be used.

3.5.1 Whittaker Henderson Graduation Method

The Whittaker Henderson method is the most frequently used method in actuarial practice for smoothing mortality rates. This method developed by Whittaker in 1923. Henderson also made a signi cant contribution to the theory in 1924 and 1925. Since then this method come to be known as Whittaker Henderson method.

This method simultaneously considers smoothness of the initial mortality rates whiles adhering to some degree of t to the initial rates.

De ne the t measure as

$$F = \sum_{x=1}^{n} w_i(\tilde{q}_x - \hat{q}_x)^2 \tag{3.6}$$

and the smoothness measure as

$$S = h\sum_{i=x}^{n-z}(\Delta^z \tilde{q}_x)^2 \tag{3.7}$$

A linear combination of the tness and smoothness measures gives the equation below

$$M = F + hS = \sum_{x=1}^{n} w_i(\tilde{q}_x - \hat{q}_x)^2 + h\sum_{i=x}^{n-z}(\Delta^z \tilde{q}_x)^2 \tag{3.8}$$

where

h is a positive real number that controls the relative emphasis given to F and S in the minimization of M. As $h \to 0$, the graduated values converge to the original data and as $h \to \infty$, the graduated values converge to a polynomial of degree $z - 1$.

w_x is a sequence of positive weights. The graduation method is referred to as Whittaker Type A graduation when $w_x = 1$ for all ages, x and type B Whittaker graduation when $w_x = \frac{E_x}{\bar{E}}$. Where E_x and $\bar{E}$ are de ned as the exposure at age x and the arithmetic average of E_x over all x respectively.

Δ is the di erence operator de ned by $\Delta \tilde{q}_x = \tilde{q}_x - \tilde{q}_{(x-1)}$ z is a parameter that determines the order of the di erence equation ($z < n$). The values of z commonly employed are $z = 2,3$ and 3.

This method has its source from the works of Bohlmann (1899), who provided a complete solution for normal equations treated as a di erence equation of order $2z$ with boundary conditions of z at the ends. An approximate solution valid for $z = 3$ a and a small value of the parameter h was provided by Whittaker(1923) with six boundary conditions. With this approach, it is easy to derive graduated values corresponding to different values of h and hence choose the most suitable h. Whittaker made the following observations;

1. The choice of the parameter h depends on the relative emphasis placed on smoothness and the goodness of fit.
2. There is no problem of tails to be dealt with compared to the case of the adjusted-average graduation
3. The theorem of conservation of moments was proved for $z = 3$

3.5.2 The Minimization of M

The sequence of values $\tilde{q}_x$ for $x = 1,2,3,...,n$ that minimizes M in equation 3.8 are then taken as the solution. This can be obtained by solving n equations resulting from taking the partial derivatives of M with respect to $\tilde{q}_x$ and equating to zero.

The set of graduated values $\tilde{q}_x$, for $x = 1,2,3,...,n$ is unique for every h chosen and the solution to the system of equations from 3.8 equals to zero. The matrix - vector approach is used to solve the minimization problem.

Let u and v denote the n^{th} - element column vector of the initial and revised mortality estimates respectively and w denote an $n \times n$ diagonal matrix of weights such that

$$\mathrm{w} = \begin{pmatrix} w_1 & 0 & 0 & \dots & 0 \\ 0 & w_2 & 0 & \dots & 0 \\ \dots & \dots & \ddots & \dots & \dots \\ 0 & 0 & 0 & \dots & w_n \end{pmatrix} \quad (3.9)$$

Let k_z also be another matrix whose elements are the binomial coe cients of order z.

For example;

With due regard for sign changes, the matrix product $k_z v$ is the vector containing the values of $\delta^z v_x$. For n values of v_x, v is an $n \times 1$ vector. Hence k_z will have dimension $(n - z) \times n$ and $k_z v$ will be a vector with dimension $(n - z) \times 1$

From the above de nitions, 3.7 can be rewritten as

$$
\begin{aligned}
M &= (v - u)^T w(v - u) + h(k_z v)^T k_z v \\
&= (v - u)^T w(v - u) + hv^T (k_z^T k_z) v
\end{aligned}
\tag{3.10}
$$

Hence, taking the partial derivatives of M with respect to v_x and equating to zero, the vector v that minimizes M in 3.7 and 3.10 is given by

$$
c\text{v} = \text{wu} \tag{3.11}
$$

where

$$
c = (\mathbf{w} + hk_z^T k_z)
$$

and c is a symmetric, non-singular and positive de nite matrix.

3.6 Test for Goodness of Fit

The Chi-square test for goodness of t is used to test whether or not the revised rates deviate from the initial/crude mortality rates at 95% con dence interval.

The test statistic is computed by

$$
z = \sum_{i=1}^{p} n_i \times \frac{(\hat{q}_i - q_i)}{q_i(1 - q_i)} \tag{3.12}
$$

H_0 ; The graduated values is consistent with the observed mortality

H_α ; The graduated values is not consistent with the observed mortality rates at

$\alpha = 5\%$.

3.7 Computing Life expectancy

For a life age x, the curtate future lifetime random variable K_x, is de ned as the integer part of the future lifetime. That is

$$K_x = \lfloor T_x \rfloor \tag{3.13}$$

When $K_x = k$, such that $k = 0,1,2,\ldots$, then the probability function is given as

$$\begin{aligned} P[K_x = k] &= P[k \le T_x < k + 1] \\ &= {}_{k|}q_x \\ &= {}_kp_x - {}_{k+1}p_x \\ &= {}_kp_x - p_{x+k} \times {}_kp_x \\ &= {}_kp_x \times q_{x+k} \end{aligned} \tag{3.14}$$

The expected value of K_x is called the curtate expectation of life and is denoted by

$$\begin{aligned} E[K_x] &= e_x \\ &= \sum_{k=0}^{\infty}(k \times P[K_x = k]) \\ &= \sum_{k=0}^{\infty} k \times ({}_kp_x - {}_{k+1}p_x) \\ &= \sum_{k=1}^{\infty} {}_kp_x \end{aligned} \tag{3.15}$$

Chapter 4

Data Analysis and Results

4.1 Introduction

This chapter describes the data employed in the analysis of this work and also discusses the results obtained from the analysis of the data collected.

4.2 Presentation of Data

The observed group is made up of the set of members (both active and retired) of the GUSSS pension plan in Ghana for a ten(10) year period. The data comprise the date of birth, date of entry into the plan and date of death or withdrawal of each plan member.

Where the date of entry into the scheme is unknown, it was assumed that the date of employment is equal to the date of entry into the scheme.

4.3 Estimation of initial mortality rates

The actuarial exposure method was used to determine each plan member(s) actual contribution to every unit age in the study period. Thus for each death, the period of time from the date of death to till the end of year of age is added. Exposure is measured in life years.

In the study, the minimum and maximum age of the members who were involved in the study are 30 and 85 respectively. The minimum and maximum exposure recorded were 6.507 and 569.034 life years respectively as shown in the table 4.1.

Table 4.1: Summary Statistics of data

	Age	d_x	Exposure	q_x

Min.	30.00	0.00	6.507	0.000000
1st Qu.	43.75	1.00	102.789	0.003258
Median	57.50	4.50	260.145	0.036329
Mean	57.50	5.75	284.608	0.056263
3rd Qu.	71.25	8.25	478.430	0.089771
Max.	85.00	21.00	569.034	0.307352

Having determined the amount of exposure for each member and for each age, the number of deaths recorded during the study period was also determined. The age last birthday de nition was used to tabulate the number of deaths at each age. A total of 322 deaths were recorded during the period of study with the rst death recorded at age 42. Summary statistics of the data is given in table
4.1.

From Figure 4.1, it can be observed that age 66 recorded the highest number of

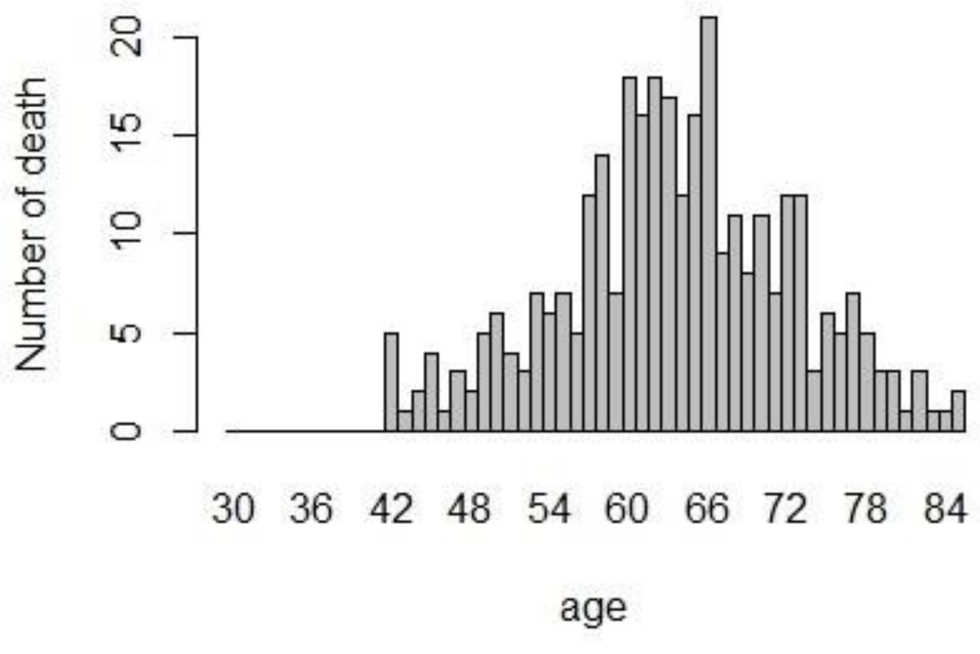


Figure 4.1: Plot of Number of Deaths recorded

death that is 21 deaths and ages 30 to 41 recorded no deaths.

Further calculations were done to determine the initial (crude) mortality rates at the various ages. A plot of the initial rates computed using the actuarial estimator is as shown in Figure 4.2. It can be observed that the trend of

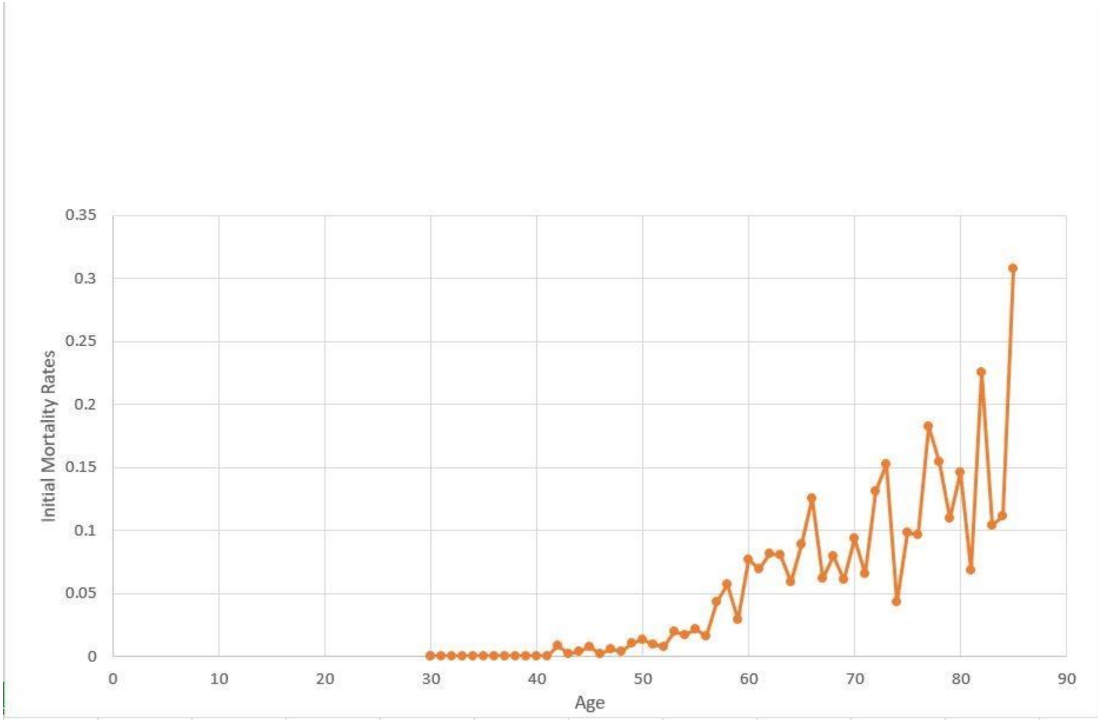


Figure 4.2: Plot of Initial Mortality Rates

mortality rates remained constant at zero(0) from ages 30 to 41. This is because no deaths were recorded at these ages. The rates gradually increased from 0 at age 41 to 0.152139 at age 73 with some slight uctuations after which there was a reduction in the rate to 0.043167 at age 74. The mortality rates then increased to 0.307352 at age 85 with some random uctuations. Refer to table 5.1 at the appendix for the table showing the numeric values of the initial mortality rates and the 95% con dence interval of these mortality estimates at each age.

4.4 Graduation of Initial Mortality rates

It can also be observed from gure 4.2 that there are some irregularities in general increasing pattern of the curve. These irregularities were smoothed using the

Whittaker Henderson technique of graduation. This technique considers smoothness and tness simultaneously. Recall also that there is "Type A" and "Type B" Whittaker Graduation methods where the choice of weights, w_x is the distinguishing factor and the formula minimizes M to obtain the best values revised mortality rates with respect to arbitrarily assigned values for h and z.

The smoothing parameter and order of di erencing was varied (h = 10,50 and 100 and z = 3,4 respectively) for the two types of graduation. Tables 4.2 and 4.3 show the results of M.

Table 4.2: Table of Minimized M for Whittaker Type A h

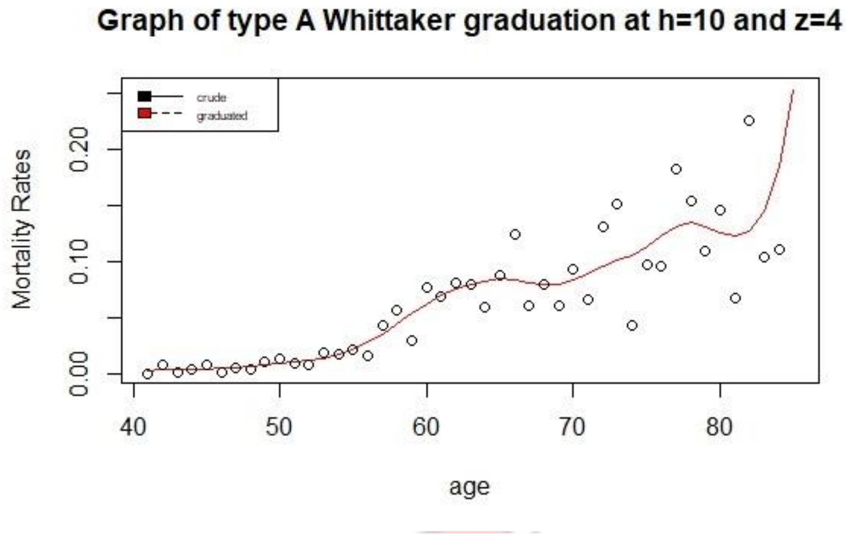
	Z = 3	Z = 4
10	0.043075	0.03909
50	0.047041	0.041765
100	0.04856	0.042964
1000	0.052035	0.047441

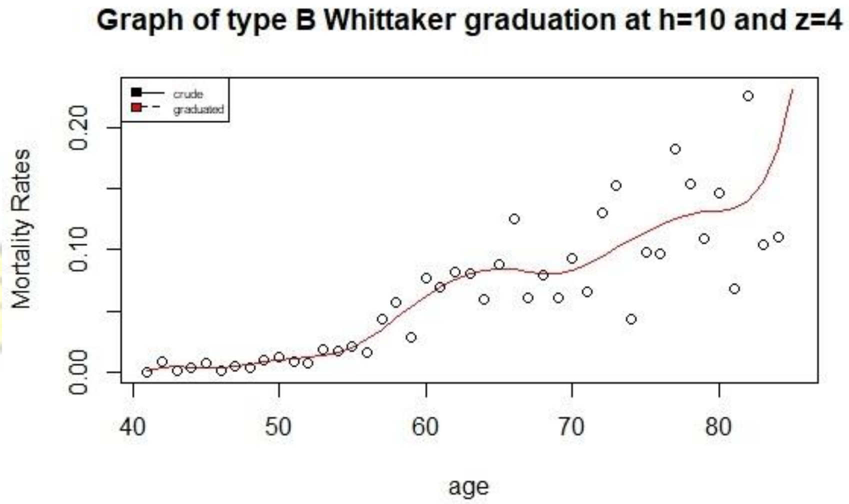
Table 4.3: Table of Minimized M for Whittaker Type B h

	Z = 3	Z = 4
10	0.008801	0.008614
50	0.009085	0.008829
100	0.009474	0.009125
1000	0.009897	0.00928

The graduated values $\tilde{q}_x$, are then taken as those which minimize the composite measure M. The results shown in tables 4.2 and 4.3 indicate that the Type B Whittaker graduation produce smaller values for the composite measure M compared to the results of Type A when h and z were varied for both cases. This could be as a result of the unequal weights, w_x applied in the case Type B graduation. Thus, the set of graduated values that minimized the composite measure M were obtained at h = 10 and z = 4 for both cases of Type A and B Whittaker Henderson graduation that is, 0.03909 and 0.008614 respectively. A plot of the graph of the graduated values for the two cases of Whittaker graduation is shown in gure 4.3.

Figure 4.3: Plot of Smooth Mortality rates





Based on the assumption that mortality increases with age and for the purpose for which the rates will be used, irregularities in the rates from age to age will disturb the smooth progression of premiums, etc.(Miller (1946)).Hence the graduated values at the parameters of $h = 10$ and $z = 4$ using Type B case of Whittaker graduation were chosen.

This parameter choice is similar to the work on estimating mortality rates using Whittaker graduation method by Chanco (2016). The author used data from the Philippine Social Security System settled on the assumption of the smoothing parameter, $h = 10$ and order of di erencing, $z = 4$.

Following choice of the graduated values, a chi square test for goodness of t was then performed to con rm whether or not the graduated rates at h = 10 and z = 4 t the data at α = 5%. The results of the test are shown below with degrees of freedom n − 1.

Chi-squared test for given probabilities

X-squared = 31.849, df = 44, p-value = 0.9139

The p-value indicate that there is not enough evidence to reject the null hypothesis the graduated values at h = 10 and z = 4 t the observed mortality rates at α = 5%. Hence, it is concluded that the graduated rates is consistent with the observed rates.

4.5 Computing life Expectancies

Further computations were done to determine the average number of years more a life aged say x, is expected to live given mortality follows the pattern experienced in the study that is the life expectancy, e_x. The results are shown in Table 4.4. The life expectancy decreased from approximately 36 years at age 30 to about 1 year at age 85.

Table 4.4: Life expectancy at age x

Age	e_x	Age	e_x
30	35.78	61	9.42
31	34.78	62	9.13
32	33.78	63	8.88
33	32.78	64	8.66
34	31.78	65	8.45
35	30.78	66	8.23
36	29.78	67	7.98
37	28.78	68	7.70
38	27.78	69	7.37
39	26.78	70	7.02
40	25.78	71	6.66
41	24.78	72	6.30
42	23.82	73	5.97

43	22.93	74	5.64
44	22.05	75	5.32
45	21.16	76	5.00
46	20.24	77	4.68
47	19.33	78	4.35
48	18.43	79	4.00
49	17.55	80	3.60
50	16.70	81	3.15
51	15.87	82	2.64
52	15.06	83	2.07
53	14.25	84	1.44
54	13.45	85	0.77
55	12.67		
56	11.94		
57	11.28		
58	10.69		
59	10.18		
60	9.76		

Chapter 5

Summary of Findings, Conclusion and Recommendations

5.1 Summary Findings

1. The results of the study on GUSSS indicated that the more people that is, 118 out of the 322 deaths recorded were within the rst six years from the normal retirement age (that is, age sixty (60)).

2. It was also observed that initial mortality rates exhibited a general increasing pattern. This increase is consistent with the assumption by Takis and Athanasios (2004) that the true but unknown underlying mortality pattern

increases with age x (monotone) and more steeply increasing in higher ages (convex).

3. The average life expectancy for a member of GUSSS is about 36 years given that the minimum age for the study group was 30 years. That is the member has survived to age 30.

5.2 Conclusion

In conclusion, the study observed that;

The observed mortality trend of the members of GUSSS increased at a decreasing rate with some random uctuations as the age of the member increased.

The estimates of mortality rates for the members of GUSSS were the sequence of values obtained using the Type B Whittaker Henderson graduation method with parameters $h = 10$ and $z = 4$.

The life expectancies of the members at each age was also computed using the graduated rates from the graduation process and the results are shown in table 5.2 spanning from ages 30 to 85.

5.3 Study Limitation

Eventhough the study conducted has achieved the targeted objectives, like all other studies, it also had some limitation.

The ndings of this study are all based of the general mortality of members since data obtained were not segregated by gender. Therefore, the mortality of males and females could not be explored independently.

5.4 Recommendations

Further studies could be done taking gender into consideration and also using decrements other than death such as withdrawal.

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Appendix A

Table 5.1: Con dence Interval for q_x

Age	d_x	Exposure	$\hat{q}_x$	sd($\hat{q}_x$)	$\hat{q}_x - 1.96$Se	$\hat{q}_x + 1.96$Se
30	0	322.8392	0.0000	0.0000	0.0000	0.0000
31	0	383.2930	0.0000	0.0000	0.0000	0.0000
32	0	414.6318	0.0000	0.0000	0.0000	0.0000
33	0	434.9384	0.0000	0.0000	0.0000	0.0000
34	0	450.0198	0.0000	0.0000	0.0000	0.0000
35	0	502.5702	0.0000	0.0000	0.0000	0.0000
36	0	534.2923	0.0000	0.0000	0.0000	0.0000
37	0	536.5838	0.0000	0.0000	0.0000	0.0000
38	0	527.2902	0.0000	0.0000	0.0000	0.0000
39	0	542.9897	0.0000	0.0000	0.0000	0.0000
40	0	553.4750	0.0000	0.0000	0.0000	0.0000
41	0	562.4278	0.0000	0.0000	0.0000	0.0000
42	5	569.0342	0.0088	0.0039	0.0011	0.0165
43	1	539.7132	0.0019	0.0019	0.0000	0.0055
44	2	538.0479	0.0037	0.0026	0.0000	0.0089
45	4	529.7467	0.0076	0.0038	0.0002	0.0149
46	1	532.0411	0.0019	0.0019	0.0000	0.0056
47	3	515.1697	0.0058	0.0034	0.0000	0.0124
48	2	496.4771	0.0040	0.0028	0.0000	0.0096
49	5	472.4141	0.0106	0.0047	0.0014	0.0198
50	6	452.2101	0.0133	0.0054	0.0027	0.0238
51	4	419.9528	0.0095	0.0047	0.0002	0.0188
52	3	387.4791	0.0077	0.0045	0.0000	0.0165
53	7	358.4928	0.0195	0.0073	0.0052	0.0338
54	6	348.7803	0.0172	0.0070	0.0036	0.0308

55	7	328.7666	0.0213	0.0080	0.0057	0.0369
56	5	313.3224	0.0160	0.0071	0.0021	0.0298
57	12	276.5975	0.0434	0.0122	0.0194	0.0674
58	14	243.6927	0.0574	0.0149	0.0282	0.0867
59	7	237.3641	0.0295	0.0110	0.0080	0.0510
60	18	234.4935	0.0768	0.0174	0.0427	0.1108
61	16	231.3765	0.0692	0.0167	0.0365	0.1018
62	18	220.4750	0.0816	0.0184	0.0455	0.1178
63	17	211.4257	0.0804	0.0187	0.0438	0.1171
64	12	202.1198	0.0594	0.0166	0.0268	0.0920
65	16	180.5421	0.0886	0.0212	0.0472	0.1301
66	21	167.7159	0.1252	0.0256	0.0751	0.1753
67	9	146.3491	0.0615	0.0199	0.0226	0.1004
68	11	138.0678	0.0797	0.0230	0.0345	0.1248
69	8	131.6167	0.0608	0.0208	0.0200	0.1016
70	11	118.0021	0.0932	0.0268	0.0408	0.1457
71	7	106.4805	0.0657	0.0240	0.0187	0.1128
72	12	91.7139	0.1308	0.0352	0.0618	0.1999
73	12	78.8754	0.1521	0.0404	0.0729	0.2314
74	3	69.4983	0.0432	0.0244	0.0000	0.0909
75	6	61.1061	0.0982	0.0381	0.0236	0.1728
76	5	51.7673	0.0966	0.0411	0.0161	0.1771
77	7	38.3155	0.1827	0.0624	0.0603	0.3050
78	5	32.4305	0.1542	0.0634	0.0299	0.2785
79	3	27.3819	0.1096	0.0597	0.0000	0.2266
80	3	20.5900	0.1457	0.0778	0.0000	0.2981
81	1	14.6831	0.0681	0.0657	0.0000	0.1970
82	3	13.2854	0.2258	0.1147	0.0010	0.4506
83	1	9.5941	0.1042	0.0986	0.0000	0.2976
84	1	9.0068	0.1110	0.1047	0.0000	0.3162
85	2	6.5072	0.3074	0.1809	0.0000	0.6619
TOTAL	322	16646.3669				

Table 5.2: ESTIMATED MORTALITY TABLE

Age	E_x	d_x	$\hat{q}_x$	$\tilde{q}_x$	p_x	e_x
30	322.8392	0	0	0	1.0000	35.78
31	383.293	0	0	0	1.0000	34.78
32	414.6318	0	0	0	1.0000	33.78
33	434.9384	0	0	0	1.0000	32.78
34	450.0198	0	0	0	1.0000	31.78
35	502.5702	0	0	0	1.0000	30.78
36	534.2923	0	0	0	1.0000	29.78
37	536.5838	0	0	0	1.0000	28.78
38	527.2902	0	0	0	1.0000	27.78
39	542.9897	0	0	0	1.0000	26.78
40	553.475	0	0	0	1.0000	25.78
41	562.4278	0	0	0.001509093	0.9985	24.78
42	569.0342	5	0.008787	0.004620055	0.9954	23.82
43	539.7132	1	0.001853	0.005213508	0.9948	22.93
44	538.0479	2	0.003717	0.004703658	0.9953	22.05
45	529.7467	4	0.007551	0.004148926	0.9959	21.16
46	532.0411	1	0.00188	0.004178492	0.9958	20.24
47	515.1697	3	0.005823	0.005078963	0.9949	19.33
48	496.4771	2	0.004028	0.006696466	0.9933	18.43
49	472.4141	5	0.010584	0.008564882	0.9914	17.55
50	452.2101	6	0.013268	0.010158221	0.9898	16.70
51	419.9528	4	0.009525	0.011325884	0.9887	15.87
52	387.4791	3	0.007742	0.012414291	0.9876	15.06
53	358.4928	7	0.019526	0.013978183	0.9860	14.25
54	348.7803	6	0.017203	0.016574424	0.9834	13.45
55	328.7666	7	0.021292	0.020814082	0.9792	12.67

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56	313.3224	5	0.015958	0.027089738	0.9729	11.94
57	276.5975	12	0.043384	0.035228685	0.9648	11.28
58	243.6927	14	0.057449	0.044414558	0.9556	10.69
59	237.3641	7	0.029491	0.05378575	0.9462	10.18
60	234.4935	18	0.076761	0.062656948	0.9373	9.76
61	231.3765	16	0.069151	0.070255708	0.9297	9.42
62	220.475	18	0.081642	0.076197542	0.9238	9.13
63	211.4257	17	0.080406	0.080505635	0.9195	8.88
64	202.1198	12	0.059371	0.083367682	0.9166	8.66
65	180.5421	16	0.088622	0.084715265	0.9153	8.45
66	167.7159	21	0.125212	0.084214303	0.9158	8.23
67	146.3491	9	0.061497	0.082246362	0.9178	7.98
68	138.0678	11	0.079671	0.080427457	0.9196	7.70
69	131.6167	8	0.060783	0.080506608	0.9195	7.37
70	118.0021	11	0.093219	0.08336874	0.9166	7.02
71	106.4805	7	0.06574	0.088593532	0.9114	6.66
72	91.71389	12	0.130842	0.094978086	0.9050	6.30
73	78.87543	12	0.152139	0.10134306	0.8987	5.97
74	69.49829	3	0.043167	0.107429126	0.8926	5.64
75	61.10609	6	0.09819	0.113670566	0.8863	5.32
76	51.76728	5	0.096586	0.120011561	0.8800	5.00
77	38.31554	7	0.182694	0.125610216	0.8744	4.68
78	32.43053	5	0.154176	0.129403235	0.8706	4.35
79	27.38193	3	0.109561	0.1311076	0.8689	4.00
80	20.59001	3	0.145702	0.13172732	0.8683	3.60
81	14.68309	1	0.068106	0.133550094	0.8664	3.15
82	13.28542	3	0.225811	0.139955542	0.8600	2.64
83	9.594114	1	0.104231	0.155109317	0.8449	2.07
84	9.006845	1	0.111027	0.183738049	0.8163	1.44
85	6.507187	2	0.307352	0.230777244	0.7692	0.77