Kwame Nkrumah University of Science and Technology, Kumasi.

Revised Mathematical Morphological Concepts: Dilation, Erosion, Opening and Closing.

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Declaration

We hereby declare that this submission is our own work towards the degree of MPhil Pure in Mathematics and that, to the best of our knowledge, it contains no material(s) previously published by another person(s) or material(s), which have been accepted for the award of any other degree of the University, except where the acknowledgement has been made in the text.

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Abstract

Mathematical morphology is the theory and technique for the analysis and processing of geometrical structures, based on set theory, lattice theory, topology, and random functions. Mathematical Morphology is most commonly applied to digital images, but it can be employed as well on graphs, surface meshes, solids, and many other spatial structures. Mathematical Morphology has a lots of operators but the most basic and important ones are Dilation and Erosion. Since it development, Morphological operators have been governed by algebraic properties, which we seek to improve in this study. Mathematical proofs are outlined for propositions which were discovered during the investigation of what happens to Dilation and Erosion when the set or structural element in a morphological operation is partitioned before the operation is taken. It turns out that some of the operators distribute over union and intersection with a few exceptions and it is also possible to partitioned the set or structural element before carrying out the morphological operation.

Contents

Chapter 1

GENERAL INTRODUCTION

1.1 Introduction

The objective of this chapter is to give the reader an introduction of Mathematical Morphology, and also to provide some useful background information about Mathematical Morphology in order to give the reader a full understanding of the subject this thesis will explore. Furthermore, this chapter includes a problem discussion, the research questions and the aim of the study. Finally, the delimitations, targeted audience and the disposition of the thesis are presented.

1.2 Background of Study

Mathematical morphology is the theory and technique for the analysis and processing of geometrical structures, based on set theory, lattice theory, topology, and random functions. Mathematical Morphology is most commonly applied to digital images, but it can be employed as well on graphs, surface meshes, solids, and many other spatial structures. Topological and geometrical continuous-space concepts such as size, shape, convexity, connectivity, and geodesic distance, were introduced by Mathematical Morphology on both continuous and discrete spaces. Mathematical Morphology is also the foundation of morphological image processing, which consists of a set of operators that transform images according to the above characterizations.

Mathematical morphology mostly deals with the mathematical theory of describing shapes using sets. In image processing, mathematical morphology is used to investigate the interaction between an image and a certain chosen structuring element using the basic operations of erosion and dilation. Mathematical morphology can also be described as the science of transforming images. Perhaps one could say that it serves for images as Fourier analysis serves for sounds. Using Fourier analysis one can analyze and manipulate sounds, e.g., remove noise. Using mathematical morphology one can in a similar way analyze and manipulate images. Mathematical morphology stands somewhat apart from traditional linear image processing, since the basic operations of morphology are non-linear in nature, and thus make use of a totally different type of algebra than the linear algebra.

Mathematical morphology is a collection of algorithmic tools that can be executed by a digital computer and, when applied to an image, yield a transformed image. Transforming images is referred to as *image processing*. The goal of applying such algorithms to images might be improving the appearance of these images, creating art, performing measurements, or understanding what is imaged. In these last two cases we speak of *image* *analysis*. Mathematical morphology was born in the mid 1960s from work by Georges Matheron and Jean Serra. At that time they heavily stressed the mathematical formalism's (probably because computers took a long time to compute the complex transforms they were describing, and they recognized the importance of a strong mathematical base). Many authors since have extended this set of tools, mostly working on the mathematical base (definitions, propositions and theorems). Nonetheless, mathematical morphology is a relatively simple and powerful tool to solve a wide variety of problems in image processing and analysis. Mathematical morphology can also be applied to other things besides images.

Looking at all this uses and importance of mathematical morphology, it prudent that one understand the basic theories and operators of mathematical morphology and hence the topic at hand.

1.3 Statement of Problem

The field that has become known as mathematical morphology is quite old in a sense. It is about operations on sets and functions that have been around for a long time, but which are now being systematized and studied under a new angle, precisely because it is possible to actually perform operations on the computer and see on the screen what happens. Morphology can be viewed as having its origin in our trying to understand a complicated world. The world is so complex that the human mind and the human eye cannot perceive all its minute details, but needs a simplified image, a simplified structure. The need to simplify a complicated object is, in this view of things, the basic impulse behind mathematical morphology, and this is what mathematical morphology does. Related to this is the fact that an image may contain a lot of disturbances, or rather, it almost always does. Therefore, most images need to be tidied up. Hence another need to process images. However, this research will be purely concern with the operations used in simplifying the images.

1.4 Research Questions

- What are the two basic Mathematical Morphology Operators?
- What is a structuring element in Mathematical Morphology?
- What are the operators in Mathematical Morphology?
- What are the algebraic structures of Morphological Operators?
- Can the intersection or union of 2 morphological operators be found if they have the same structural element but different sets and vice versa?

1.5 Objectives of Study

- The main aim of this thesis is to analyze in details the mathematical morphology operators and their algebraic structure when they are linked with union or intersection.
- We will also investigate what happens to Dilation, Erosion, Opening and closing when the set or structural element in a morphological operation is partitioned before the operation is taken.

- Furthermore, we will also investigate the distributive property of morphological operators over set union and intersection.

1.6 Justification of Study

Mathematical Morphology is a theory which provides a number of useful tools for image analysis. Tools such as a way to shrink or dilate, a way to make blur images clearer and so on. It is also seen by some as a self-contained approach to handling images and by others as complementary to the other methods. Hence it is of importance to know the operators of mathematical morphology which goes into these tools.

Mathematical morphology mostly deals with the mathematical theory of describing shapes using sets. In image processing, mathematical morphology is used to investigate the interaction between an image and a certain chosen structuring element using the basic operations of erosion and dilation. Mathematical morphology stands somewhat apart from traditional linear image processing, since the basic operations of morphology are non-linear in nature, and thus make use of a totally different type of algebra than the linear algebra.

Mathematical Morphology is also a well-founded non-linear theory for Image Processing. Its geometry-oriented nature provides a strong framework for addressing shape characteristics such as size, connectivity, and others, which are not easily accessed by the traditional linear approach. Morphology has been used in applications such as nonlinear filtering, sharpening, compression, shape analysis, segmentation, and others.

In modern society, huge amounts of images are collected and stored in computers so that useful information can be later extracted. In a concrete example, the online image hosting Flickr reported in August 2011, that it was hosting more than 6 billion images and 80 million unique visitors. The growing complexity and volume of digitized sensor measurements, the requirements for their sophisticated real time exploitation, the limitations of human attention, and increasing reliance on automated adaptive systems all drive a trend towards heavily automated computational processing in order to refine out essential information and permit effective exploitation.

The science of extracting useful information from images is usually referred to as image processing. From the mathematical point of view, image processing is any form of signal processing for which the input is an image and the output may be either an image or a set of characteristics or features related to the input image. In essence, image processing is concerned with efficient algorithms for acquiring and extracting information from images. In order to design such algorithms for a particular problem, we must have realistic models for the images of interest. In general, models are useful for incorporating a priori knowledge to help to distinguish interesting images, from uninteresting, which can help us to improve the methods for acquisition, analysis, and transmission. Therefore the need to research into mathematical morphology keeps growing everyday.

1.7 Limitations of Study

The research area of mathematical morphology is immense. The study of mathematical morphology in the present dissertation will concentrate on a particular discipline to obtain a focused result, namely the mathematical aspect only especially the set theory aspect of the operators. Therefore applications of mathematical morphology and specific application of the morphological operators would not be a focus of this thesis.

1.8 Organization of Study

This thesis is divided into five chapters each with a number of subcategories. The first chapter covers seven subcategories, the first one aims at defining the key concepts and terms, which will be employed throughout the thesis. It also deals with the introduction, which encompasses a short preview of the thesis. Furthermore, chapter 1 addresses the rationale and motivation which led the author to choose the presented topic. The problem formulation and specifically the core questions intended to be answered throughout the thesis will also be dealt with in the first chapter. Finally, attention will be focused on objective, justification, scope of the study and limitations.

Chapter two is the theoretical approach, discussing and describing the theories employed throughout the thesis, and the rationale for choosing them. The birth of mathematical morphology and the developments in morphological operators will be comprehensively analyzed in this section along with the views and the definition of mathematical morphology and its applications.

Chapter three describes the research methodology, how images are transformed using morphological operators. It also defines key operations of mathematical morphology such as dilation, erosion, opening and closing in binary and gray scale morphology. An analysis of the morphological operators and their algebraic structure when linked with union or intersection is covered in chapter four. The last part is the conclusion of the thesis, and is intended to sum up the analysis undertaken throughout this paper, and its implications.

Chapter 2

Literature Review

2.1 Introduction

This chapter deals with the introduction of mathematical morphology, the birth of this branch of mathematics and its history. It also gives the development of mathematical morphology over the past years and it related research areas. Furthermore, the basic principles and fundamental properties of mathematical morphology would be considered in this section.

2.2 Introduction of Mathematical Morphology

Mathematical Morphology is the analysis of signals in terms of shape. This simply means that morphology works by changing the shape of objects contained within the signal. In the processing and analysis of images it is important to be able to extract features, describe shapes and recognize patterns.

Such tasks refer to geometrical concepts such as size, shape, and orientation. Mathematical morphology uses concepts from set theory, geometry and topology to analyze geometrical structures in an image.

Mathematical morphology is about operations on sets and functions. It is systematized and studied under a new angle, precisely because it is possible to actually perform operations on the computer and see on the screen what happens. The need to simplify a complicated object is the basic impulse behind mathematical morphology. Related to this is the fact that an image may contain a lot of disturbances. Therefore, most images need to be tidied up. Hence another need to process images; it is related to the first, for the border line between dirt and of other kind disturbances is not too clear.

Consider Euclidean geometry, and consider cardinalities. The set $\mathbb{Z}^+$ of non-negative integers is infinite, and its cardinality is denoted by $card(\mathbb{Z}^+) = \aleph_0$ (Aleph null). The set of real numbers $\mathbb{R}$ has the same cardinality as the set of all subsets of $\mathbb{N}$, thus $card(\mathbb{R}) = 2^{\aleph_0}$. The points in the Euclidean plane have the same cardinality:

$$card(\mathbb{R}^2) = card(\mathbb{R}).$$

But the set of all subsets of the line or the plane has the larger cardinality. There are too many sets in the plane. Consider a large subclass of this huge class, a subclass consisting of nice sets. For instance, the set of all disks has a much smaller cardinality, because three numbers suffice to determine a disk in the plane: its radius and the two coordinates of its center. Similarly, four numbers suffice to specify a rectangle $[a_1, b_1] \times [a_2, b_2]$ with sides parallel to the axes; a fifth is needed to rotate it. This leads to the idea of simplifying a general, all too wild set, to some reasonable, better-behaved set. Euclidean line containing denumerably many points. Consider a line as the set of solutions in $\mathbb{Q}^2$ of an equation $a_1x_1 + a_2x_2 + a_3 = 0$ with integer coefficients. Then two lines which are not parallel intersect in a point with rational coordinates. The cardinality of the set of all subsets of $\mathbb{Q}^2$ is $2^{\aleph_0}$, so there are fewer sets to keep track of than in the real case. So there are too many subsets in the plane. Consider digital geometry. On a computer screen with, say, $1,024$ pixels in a horizontal row and 768 pixels in a vertical column there are $1,024 \times 768 = 786,432$ pixels. On such a screen a rectangle with sides parallel to the axes is the Cartesian product $\mathbb{R}(a, b) = [a_1, b_1]_{\mathbb{Z}} \times [a_2, b_2]_{\mathbb{Z}}$ of two intervals.

There are only finitely many binary images. But the number of binary images must be compared with other finite numbers. Thus, although the number of binary images on a computer screen is finite, it is so huge that the conclusion must be the same as in the case of the infinite cardinal: there are too many and it is not possible to search through the whole set and hence must be simplified. This leads to image processing and mathematical morphology, with subsets of $\mathbb{Z}^2$, or, generally, of $\mathbb{Z}^n$, the set of all n-tuples of integers.

Mathematical Morphology was first introduced by Matheron and Serra in 1967. Serra (1982) lists "the four principles of quantification", which are ways to gather information about the external world. They apply also, but not exclusively, to image analysis.

Serra's first principle is "compatibility under translation". For a mapping, this means that $f(A + b) = f(A) + b$, which is expressed as $f \circ T_b = T_b \circ f$, where $\circ$ denotes composition of mappings defined by $(f \circ T_b)(x) = f(T_b(x))$, thus a kind of commutativity, writing T_b for the translation $T_b(A) = A + b$. It means that f commutes with translations. On a finite screen like $\{x \in \mathbb{Z}^2; 0 < x_1 < 1,024,\ 0 < x_2 < 768\}$, almost nothing can commute with translations. Therefore consider the ideal, infinite, computer screen with sets of addresses equal to $\mathbb{Z}^2$. The principle is equally useful in $\mathbb{R}^n$ and $\mathbb{Z}^n$.

Serra's second principle is "compatibility under change of scale". For a mapping this means that it commutes with homotheties (or dilatations).

The third principle is that of "local knowledge". This principle says that in order to know some bounded part of $f(A)$, there is no need to know all of A, only some bounded part of A. Mathematically speaking: for every bounded set Y, there exists a bounded set Z such that $f(A \cap Z) \cap Y = f(A) \cap Y$.

Serra's fourth principle of quantification is that of "semi continuity". It means that if a decreasing sequence (A_j) of closed sets tends to a limit A, thus $A = \cap A_j$, then $f(A_j)$ tends to $f(A)$. Thus if A_j is close to A in some sense and A_j contains A, then $f(A_j)$ must be close to $f(A)$. To express this property as semi continuity, one must define a topology. In this thesis an attempt is made to derive some meaningful results by introducing some topological properties to the theory of morphological operators.

Over the last 10-15 years, the tools of mathematical morphology have become part of the mainstream of image analysis and image processing technologies. The growth of popularity is due to the development of powerful techniques, like granulometries and the pattern spectrum analysis, that provide insights into shapes, and tools like the watershed or connected operators that segment an image. But part of the acceptance in industrial applications is also due to the discovery of fast algorithms that make mathematical morphology competitive with linear operations in terms of computational speed. A breakthrough in the use of mathematical morphology was reached, in 1995, when morphological operators were adopted for the production of segmentation maps in MPEG-4.

J.Serra and George Matheron worked on image analysis. Their work lead to the development of the theory of Mathematical Morphology. Later Petros Maragos contributed to enrich the theory by introducing theory of lattices. Firstly the theory is purely based on set theory and operators are defined for binary cases only. Later, the theory extended to Gray scale images also. He also gave a representation theory for image processing. Heink J.Heijmans gave an algebraic basis for the theory. Heink J.Heijmans extended the theory to Signal processing also.

He also defined the operators for convex structuring elements. Rein Van Den Boomgaard introduced Morphological Scale space operators. In this thesis, an attempt to link some topological concepts to operators is made.

2.3 Birth of Mathematical Morphology

Mathematical morphology (MM) originates from the study of the geometry of binary porous media such as sandstones. It can be considered as binary in the sense it is made up of two phases: the pores embedded in a matrix. This led Matheron and Serra to introduce in 1967 a set formalism for analyzing binary images.

Mathematical morphology is a non-linear theory of image processing. Its geometry oriented nature provides an efficient method for analyzing object shape characteristics such as size and connectivity, which are not easily accessed by linear approaches. Mathematical Morphology (MM) is associated with the names of Georges Matheron and Jean Serra, who developed its main concepts and tools. (Matheron, 1975; Serra, 1982; Serra, 1988).

They created a team at the Paris School of Mines. Mathematical Morphology is heavily mathematized. In this respect, it contrasts with different experimental approaches to image processing . MM stands also as an alternative to another strongly mathematized branch of image processing, the one that bases itself on signal processing and information theory. Main contributors in this area are Wiener, Shannon, Gabor, etc. These classical approaches has a lot of applications in telecommunications. Analysis of the information of an image is not similar to transmitting a signal on a channel. An image should not be considered as a combination of sinusoidal frequencies, nor as the result of a Markov process on individual points .The purpose of image analysis is to find spatial objects. Hence images consist of geometrical shapes with luminance (or colour) profiles. This can be analyzed by their interactions with other shapes and luminance profiles. In this sense the morphological approach is more relevant.

MM has taken concepts and tools from different branches of mathematics like algebra (lattice theory), topology, discrete geometry, integral geometry, geometrical probability, partial differential equations, etc.

2.4 Overview of Developments in Morphological Operators and Related Areas

Before we outline our scope of coverage on morphological operators, we provide a brief historic tour of developments in the corresponding field of morphological image analysis. Classic mathematical morphology, as a field of nonlinear geometric image analysis, was developed initially by Matheron (1975), Serra (1982) and their collaborators and was applied successfully to geological and biomedical problems of image analysis. In this first period, i.e. the late 1960s and throughout the 1970s, the basic morphological operators were developed first for binary images based on set theory (Matheron, 1975; Serra, 1982) inspired by the work of Minkowski (1903) and Hadwiger (1957), second for graylevel images based on local min/max operators and level sets (Meyer, 1978; Serra, 1982) or on fuzzy sets (Nakagawa and Rosenfeld, 1978; Goetcherian, 1980), and third for gray level images but with weighted min/max operators using a geometric interpretation based on the umbra approach of Sternberg (1980, 1986) which is algebraically equivalent to maxplus convolutions. All these operators were translation-invariant and their set generators were Minkowski set addition and subtraction; thus, we shall refer to them either as Minkowski operators or as Euclidean morphological operators since their most common domain is the Euclidean plane ($\mathbb{R}^2$) or its discretized version ($\mathbb{Z}^2$) and they commute with Euclidean translations.

In the 1980s, extensions of classic mathematical morphology and connections to other fields were developed by several research groups worldwide along various directions including: applications to pattern recognition and computer vision problems; unified analysis and representation of large classes of nonlinear filters, including morphological, rank and stack filters (Maragos and Schafer, 1987a,b); multiscale image processing and shape and texture analysis; statistical analysis and optimal design of morphological filters. Accounts and references at varying degrees of detail can be found in books by (Serra, 1982, 1988; Heijmans, 1994; Haralick and Shapiro, 1992; Dougherty and Astola, 1994) or tutorial chapters and papers by (Sternberg, 1986; Haralick et al., 1987; Maragos and Schafer, 1990; Serra and Vincent, 1992; Goutsias, 1992; Maragos, 1998, 2005a) that deal with mathematical morphology. Overall, during the first two decades (late 1960s until late 1980s), this whole methodology was essentially a Euclidean morphology where the basic operators could be understood geometrically as translation-invariant set operations based on Minkowski-type set operations and implemented algebraically as nonlinear signal operations, i.e. Boolean or min/max superpositions and max-plus convolutions. Its image analysis applications were mainly in denoising, nonlinear multiscale filtering, feature extraction, simple object detection, shape and texture analysis, and watershed-based segmentation.

In the late 1980s and early 1990s a new and more general formalization of morphological operators was introduced: Lattice morphology. Specifically, the need to unify its analysis tools for both binary and gray images as well as to use it for more abstract data types such as graphs led researchers in mathematical morphology to extend its theory by generalizing the image space to a complete lattice and viewing all image transformations as lattice operators. The theoretical foundations of morphology on complete lattices were developed by Serra and Matheron, presented as chapters in Serra (1988), and further extended by Heijmans and Ronse (1990); Ronse and Heijmans (1991), Heijmans (1994), and Roerdink (1993). Later another algebraic approach to morphology was developed by Keshet (Keshet, 2000) based not on complete lattices but on a related weaker algebraic structure, inf-semilattices. The basic advance of lattice morphology was to develop more general operators that shared with the standard dilation, erosion, opening and closing only a few algebraic properties. One such fundamental algebraic structure is a pair of erosion and dilation operators that form an adjunction. This guarantees the formation of openings and closings via composition of the adjunction constituents. Other new concepts include the group invariant operators (Heijmans and Ronse, 1990; Roerdink, 2000); connected operators (Serra and Salembier, 1993; Salembier and Serra, 1995) and connectivity-based segmentation (Serra, 2000), graph morphology (Vincent, 1989; Heijmans et al., 1992); and the slope transform (Dorst and van den Boomgaard, 1994; Maragos, 1994) defined and studied in Maragos (1994, 1995, 1996) and Heijmans and Maragos (1997) from the viewpoint of lattice morphology. Overall, the lattice framework allows us to unify the concepts and analysis of large classes of operators that share a few fundamental properties, independently of whether they are defined for sets (shapes), binary signals (binary images), multilevel signals (gray level images), or even more abstract image data types such as graphs. The lattice operators have found many applications in important image analysis computer vision tasks, such as segmentation, shape analysis, motion analysis, and object detection.

During the 1990s, in parallel to the development of lattice morphology, another new development was that of differential morphology (Maragos, 1996).

This term contains two subareas, both related to nonlinear dynamical systems:

- The first subarea combined ideas from linear (Gaussian) scale-space analysis in computer vision based on the linear isotropic heat diffusion partial differential equation (PDE) and from multiscale morphology (Maragos, 1989b) to develop nonlinear PDEs that generate continuous-scale morphological filters (mainly Minkowski-type dilation and erosion). The main three independent contributions in morphological PDEs are (Alvarez et al., 1993), Brockett and Maragos (1994) and van den Boomgaard and Smeulders (1994). For overviews, we refer the interested reader to two tutorial chapters of Guichard et al. (2005) and Maragos (2005c). Connections between the morphological PDEs and the slope transform were developed in Dorst and van den Boomgaard (1994) and Heijmans and Maragos (1997).

- The second subarea deals with 2D difference equations modeling distance transforms and their analysis using slope transforms.

In this Thesis we shall not pursue the analysis of these aspects of morphological operators.

The scientific field of convex analysis and optimization (Rockafellar, 1970; Lucet, 2010), was initially unrelated to mathematical morphology, but it has been using extensively some of the main mathematical tools that morphology has also been using such as max-plus convolution and its dual, called supremal and infimal convolution respectively in convex analysis (Bellman and Karush, 1963; Rockafellar, 1970), and the hypograph of a function which is called umbra in morphology. At the end of 1990s the strong connections between convex analysis and lattice morphology were realized and studied in (Heijmans and Maragos, 1997). Examples include

- The distance transform, which is expressed via infimal convolution in convex analysis and via max-plus difference equations in digital image analysis Borgefors (1986).
- The Legendre-Fenchel conjugate (transform) of convex analysis, which is very closely related to the lattice-based slope transform (Maragos, 1995, 1996; Heijmans and Maragos, 1997).

Felzenszwalb and Huttenlocher (2004a) used the connection between distance transform and slope transform to develop a fast distance transform that has found application in computer vision problems such as distance computation and optimization in belief networks (Felzenszwalb and Huttenlocher, 2004b). There is a recent detailed review by Lucet (2010) of convex analysis, slope transforms and related optimization where the cross fertilization between these areas and mathematical morphology was explained from many different aspects. Returning to the area of differential morphology (Maragos, 1996, 2001), this refers to the intersection between image processing with max-plus convolutions, differential calculus, max/min dynamical systems and convex analysis (distance and slope transforms). Another field that combines ideas from mathematical morphology and convex analysis is digital geometry (Kiselman, 2003). In this Thesis we shall not consider the analysis of these aspects of morphological operators.

In the 1980s an effort started to unify all digital image operations under a common image algebra amenable to computation. The term image algebra was first coined by Sternberg (1980) but it referred only to the algebraic structure of mathematical morphology. Obviously, classic (Minkowski) erosions and dilations by finite structuring elements are insufficient by themselves to represent all possible image operations. In parallel to the development of mathematical morphology, there has been another independent effort in the 1980s by Ritter and his collaborators (Ritter and Gader, 1987; Ritter and Wilson, 1987) to develop a more complete image algebra that represents all digital finite-extent image-to-image operations as a finite composition of a few basic operations, which include Minkowski-type erosions and dilations; a subalgebra of their full image algebra covers the classic part of mathematical morphology. The goal of the image algebra by Ritter and his coworkers was to unify all digital image operations (linear and nonlinear) using traditional algebraic structures, e.g. groups, rings, fields, vector spaces, monoids, semirings. A fusion of image algebra and lattice structures was done in Davidson (1993). The culmination of all these efforts can be found in the book by Ritter and Wilson (2001).

For problems in fields totally separate from image processing, e.g. scheduling and operations research, Cuninghame-Green (1979, 1994) has developed a nonlinear matrix algebra called minimax algebra, also known as max-plus algebra, where he has exploited the interaction of the max/min idempotent algebraic structure with the group structure of real addition + and has developed analogies with the product-of-sums structure of linear algebra. Minimax algebra was not originally developed for image analysis. Its traditional applications areas were and still are in scheduling (e.g. material flow in automated manufacturing, traffic flow in transportation or communication networks) and operations research (Cuninghame-Green, 1979), shortest path problems in graphs (Peteanu, 1967), as well as in algebraic modeling of discrete event control systems (DEDS) (Cohen et al., 1989). Minimax algebra is a nonlinear matrix oriented algebra, where the underlying archetypal structure for scalars is the set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ of extended reals equipped with max or min operations and addition. Its basic operators are max-plus or dual min-plus generalized products of matrices with vectors where the standard multiplication of a row vector with a column vector is done via a max-of-additions instead of the standard sum of products of linear algebra.

Thus, in addition to mathematical morphology (i.e. its Euclidean and lattice-based versions), we have mentioned so far two other related algebraic systems, image algebra and minimax algebra. All three systems have had some theoretical missing parts for completion. Both the image algebra and the minimax algebra use min-max superpositions, max-plus arithmetic and some concepts from lattice theory. A fusion of concepts from image algebra and minimax algebra was also done by Davidson (1993). However, the above efforts have not exploited the complete lattice structure to the level that mathematical morphology has done and have not focused on the concept of lattice operators and especially adjunctions (Galois connections). Furthermore, both have remained in the finite-dimensional case. Minimax algebra is a matrix algebra over finite-dimensional vector spaces. Similarly, image algebra (Ritter and Wilson, 2001) deals with finite-extent digital images either by processing them with finite templates in the spatial domain or via finite discrete transforms (e.g. the discrete Fourier transform) in the frequency domain. From both approaches there seems to be missing the case of working over infinite-dimensional spaces; e.g. morphological transformations with infinite-extent structuring functions either on a continuous or a discrete domain. Missing also is the complete lattice structure which allows infinite signal superpositions based on supremum and infimum operations. From the other side, of importance for the weighted lattice operators discussed in this chapter is to note that, both Euclidean and lattice-based mathematical morphology have focused on and exploited mainly the standard lattice structure, i.e. supremum and infimum superpositions which become maximum and minimum in the finite case. Although some useful operations in mathematical morphology combine the sup/inf with max-plus arithmetic (e.g., Minkowski operations with gray structuring elements, chamfer distance transforms), such cases have always remained a minority in mainstream morphological image analysis.

Maragos (2005a) bridged the above gaps and fused lattice-based mathematical morphology with minimax algebra, by allowing for both finite as well as infinite-dimensional spaces and for sup/inf superpositions over infinite signal collections. Toward this goal, a more general algebraic structure was introduced, called clodum (complete lattice-ordered double monoid), that combines the sup/inf lattice structure with a scalar semi-ring arithmetic that possesses generalized additions and multiplications. This clodum structure enabled him to develop a unified analysis for:

(i) representations of translation-invariant operators compatible with these generalized algebraic structures as nonlinear (sup/inf) convolutions, and

(ii) representations of all increasing translation-invariant operators as suprema of such nonlinear convolutions with functions from a special collection that characterizes the operator.

Special cases of this unification include generalized Minkowski operators and lattice fuzzy image operators. Applications of this nonlinear signal algebra have appeared in (Maragos and Tzafestas, 1999; Maragos et al., 2000) for max-plus nonlinear control and in (Maragos et al., 2001, 2003) for image analysis based on fuzzy logic.

2.5 Image Processing using Mathematical Morphology

Mathematical morphology is theoretically based on set theory. It contributes a wide range of operators to image processing, based on a few simple mathematical concepts. MM started by considering binary images and usually referred to as standard mathematical morphology. It also used set-theoretical operations like the relation of inclusion and the operations of union and intersection. In order to apply it to other types of images, for example grey-level ones (numerical functions), it was necessary to generalize set-theoretical notions.

Using the lattice-theory it is generalized. The notions are, the partial order relation between images, for which the operations of supremum (least upper bound) and infimum (greatest lower bound) are defined. Therefore the main structure in MM is that of a complete lattice. All the basic morphological operators are defined by using this framework. Nowadays, most morphological techniques combine lattice-theoretical and topological methods.

The computer processing of pictures led to digital models of geometry. Azriel Rosenfeld has contributed in this field after having contributed to digital geometry and image processing for 40 years. Mathematical morphology is perfectly adapted to the digital framework.

The operators are particularly useful for the analysis of binary images, boundary detection ,noise removal, image enhancement, shape extraction, skeleton transforms and image segmentation. The advantages of morphological approaches over linear approaches are

1. Direct geometric interpretation

2. Simplicity and

3. Efficiency in hardware implementation.

An image can be represented by a set of pixels. A morphological operation uses two sets of pixels, i.e., two images: the original data image to be analyzed and a structuring element which is a set of pixels constituting a specific shape such as a line, a disk, or a square. A structuring element is characterized by a well-defined shape (such as line, segment, or ball), size, and origin. Its shape can be regarded as a parameter to a morphological operation.

2.6 Mathematical Morphology

From a general scientific perspective, the word morphology refers to the study of forms and structures. In image processing, morphology is the name of a specific methodology for analyzing the geometric structure inherent within an image. The morphological filter, which can be constructed on the basis of the underlying morphological operations, are more suitable for shape analysis than the standard linear filters since the latter sometimes distort the underlying geometric form of the image. Some of the salient points regarding the morphological approach are as follows:

1. Morphological operations provide for the systematic alteration of the geometric content of an image while maintaining the stability of the important geometric characteristics.

2. There exists a well-developed morphological algebra that can be employed for representation and optimization.

3. It is possible to express digital algorithms in terms of a very small class of primitive morphological operations.

4. There exist rigorous representations theorems by means of which one can obtain the expression of morphological filters in terms of the primitive morphological operations.

Generally speaking, the morphological operators transform the original image into another image through the interaction with the other image of certain shape and size, which is known as the structure element. Geometric feature of the image that are similar in shape and size to the structure element are preserved, while other features are suppressed. Therefore, morphological operations can simplify the image data, preserving their shape characteristics and eliminate irrelevancies. In view of applications, morphological operations can be employed for many purposes including edge detection, segmentation, and enhancement of images and so on.

The next chapter begins with binary morphology that is based on set theory. The following grayscale morphology can be regarded as the extension of binary morphology to the three-dimensional space since a grayscale image can be considered as a set of points in 3D space. While binary morphology and grayscale morphology are well-developed and widely used, it is not straightforward to extend mathematical morphology to color images. The part of color morphology in our study comes mainly from and simply introduces the approaches for extension of mathematical morphology to color images. The basic geometric characteristics of the primitive morphology operators and the detail of morphological algebra are introduced in the next chapter. A systematic introduction of the theoretical foundations of mathematical morphology, its main image operations, and their applications. Mathematical morphology defined on Euclidean setting is called Euclidean morphology and that defined on digital setting is called digital morphology. In general, their relationship is akin to that between continuous signal processing and digital signal processing. Although the actual implementation of morphological operators will be in the digital setting, the Euclidean model is essential to the development of an understanding of and intuitive feel of how the operators function in theory and application.

2.6.1 Fundamental Properties of Mathematical Morphology

In mathematical morphology images are represented by mathematical sets in a Euclidean space $\mathbb{E}$. The Euclidean space can be either the continuous space $\mathbb{R}^2$ or the discrete space $\mathbb{Z}^2$. A binary image is consequently a subset of

$\mathbb{Z}^2$, where the image pixels are represented by integer pairs of $\mathbb{Z}^2$. This set of integer pairs can be viewed as coordinates of a set of corresponding vectors with respect to two basis unit vectors. The basis unit vectors both have a length of one pixel width, and, using a rectangular coordinate system, have an angle of 90% between them. Each image is assumed to contain its boundary, and is therefore represented by a closed subset of $\mathbb{Z}^2$. Furthermore, the structuring element is represented by a compact subset of $\mathbb{Z}^2$. This implies that morphological transformations are upper semi-continuous. The four simplest transformations are dilation, erosion, opening and closing. Through combination of these transforms more advanced morphological operations can be realized, such as the skeleton transformation.

2.6.2 Fundamental Definitions

Definition (Translation) 2.6.1 *Let X be a closed subset of the discrete space $\mathbb{Z}^2$: $X \subset \mathbb{Z}^2$, and let a be a vector belonging to X: $a \in X$. The translate of X by a is defined as*

$$X_a = \{x + a \in \mathbb{Z}^2 : \forall(x \in X)\}.$$

Definition (Complement) 2.6.2 *Let the closed set X be a subset of the discrete space $\mathbb{Z}^2$: $X \subset \mathbb{Z}^2$. The complement of X, denoted by X^c is the set of all element not in X i.e*

$$X^c = \{x \in \mathbb{Z}^2 : x \notin X)\}.$$

The complement X^c of a set X represents the image background of X.

Definition (Reflection) 2.6.3 *Let the closed set X be a subset of the discrete space $\mathbb{Z}^2$: $X \subset \mathbb{Z}^2$. The Reflection $\check{X}$, is the symmetrical set of X with* *respect to the origin i.e*

$$\check{X} = \{-x \in \mathbb{Z}^2 : x \in X)\}.$$

Definition (Duality) 2.6.4 *An operation is the dual of another operation if it can be written in terms of the other operation.*

Definition (Identity) 2.6.5 *Let the identity set I be a subset of the discrete space $\mathbb{Z}^2$: $I \subset \mathbb{Z}^2$, consisting of the element in the origin $(0,0)$ of I i.e*

$$I = \{(0,0)\}.$$

Definition (Idempotence) 2.6.6 *An operation is said to be idempotent if applying the operation more than once does not further alter the image.*

Chapter 3

Mathematical Morphological Concepts

3.1 Introduction

This chapter outlines the morphological concepts adopted in this study. The chapter begins with an introduction to the basic methods of mathematical morphology, that is erosion and dilation. The chapter continues with binary morphology, gray scale morphology and color morphology. Finally, the chapter also gives details of all the morphology operators that will be considered throughout the study and their algebraic structure.

3.2 Binary Morphology

The theoretical foundation of binary mathematical morphology is *set theory*. In binary images, those points in the set are called the foreground and those in the complement set are called the background. Besides dealing with the usual set-theoretic operations of union and intersection, morphology depends extensively on the *translation* operation. For convenience, $\cup$ denotes the set-union, $\cap$ denotes set-intersection and $+$ inside the set notation refers to vector addition.

3.2.1 Binary Dilation

The set which is formed by adding all the vectors of two sets is denoted the dilated set. Let the closed set X be that of the image, and set B that of the structuring element. If vectors x and b belong to the sets X and B, respectively, then the dilated set will consist of all vectors $c = x + b \in Z^2$.

Definition (Dilation) 3.2.1 *Let X and B be subsets of the discrete space Z^2: $X \subset Z^2$, $B \subset Z^2$. The dilation of X by B is defined as*

$$X \oplus B = \{c = x + b \in Z^2 : x \in X; b \in B\}. \tag{3.1}$$

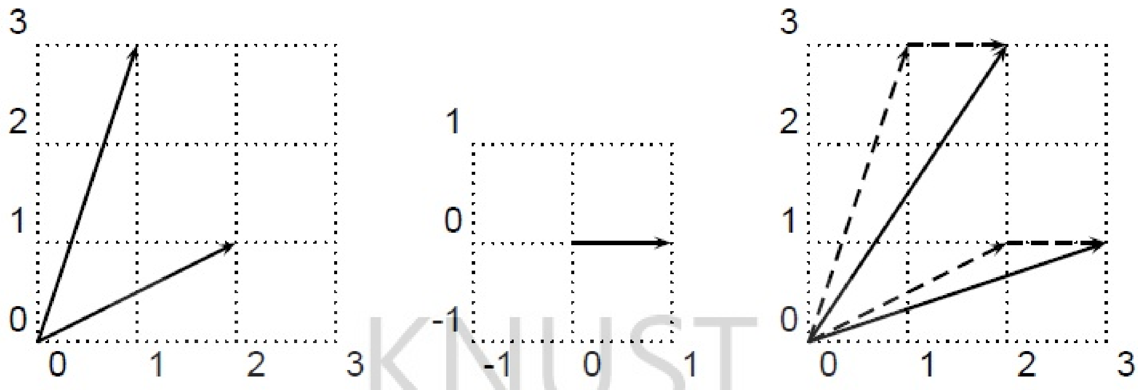
$$OR$$

Dilation of a binary image A by structure element B, denoted by $A \oplus B$, is defined as

$$A \oplus B = \{a + b \mid for\ a \in A\ and\ b \in B\}. \tag{3.2}$$

The dilation transform generally causes image objects to grow in size, or dilate - hence the name.

This is best illustrated with an example. In Fig 2.1 vectors $(1, 3)$ and $(2, 1)$ are part of set X, and vector $(1, 0)$ is part of set B. The dilation of X by B is formed by adding the vectors of the two sets as shown in Fig. 2.1(c).



(a) $X = \{(1,3),(2,1)\}$ (b) $B = \{(1,0)\}$ (c) $X \oplus B = \{(2,3),(3,1)\}$

Figure 3.1: Dilation By Vector Addition

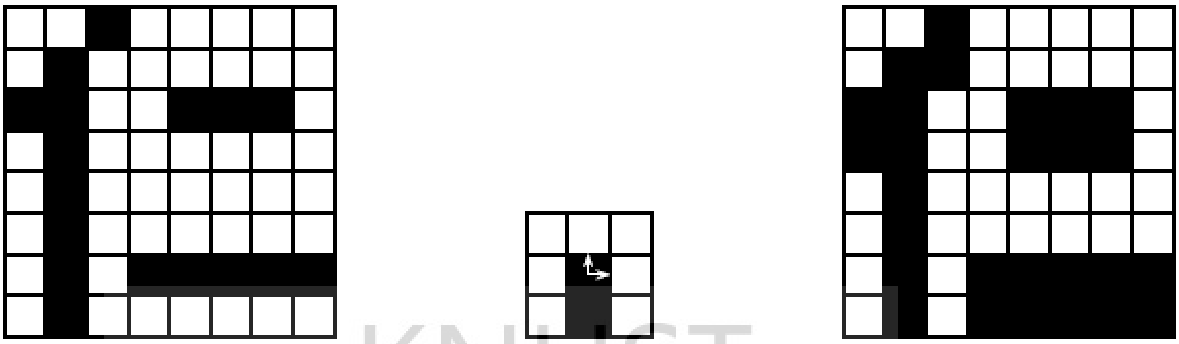
From equation (2.2) above, dilation is equivalent to a union of translates of the original image with respect to the structure element:

$$A \oplus B = \bigcup_{b \in B} A_b. \tag{3.3}$$

Intuitively, the structure element plays the role of the template. Dilation is found by placing the center of the template over each of the foreground pixels of the original image and then taking the union of all the resulting copies of the structure element, produced by using the translation operation. In general, dilation has the effect of "expanding" an image, and hence, the small hole can be eliminated.

Dilation applied to an image is illustrated in Fig. 2.2. X is the image to be dilated, and B is the structuring element. The two small arrows in B mark the origin of the structuring element, pointing in the x and y-axis directions, respectively.

As can be seen in Fig. 2.2(c), the dilated image $X \oplus B$ has grown downwards compared to image X. It is the structuring element B that has dic-



(a) X (b) B (c) $X \oplus B$

Figure 3.2: Dilation of X by the structuring element B

tated the downwards direction of the growth, because the lower pixel of the structuring element has been set. The center pixel of the structuring element results in a copy of the original image X at the same location. Had not the center pixel of B been set, the dilated image would $X \oplus B$ only have been shifted one step downwards, i.e. no growth would have occurred.

3.2.2 Binary Erosion

Erosion is the morphological dual to dilation. Whereas the dilated set is formed by vector addition of two sets, the eroded set is formed by vector subtraction. As before the closed set X denotes the image, and set B the structuring element. The eroded set is the set of vectors $c = x - b \in Z^2$, for which all of the vectors $b \in B$ there exists an $x \in X$

Definition (Erosion) 3.2.1 *Let X and B be subsets of the discrete space Z^2: $X \subset Z^2$, $B \subset Z^2$. The erosion of X by B is defined as*

$$X \ominus B = \{c \in Z^2 : \forall(b \in B)\ \exists(x \in X)\ c = x - b\}.$$

OR

Erosion of a binary image A by structure element B, denoted by $A \ominus B$, is defined as

$$A \ominus B = \{p \mid p + b \in A \ \forall b \in B\}.$$

Or to put it in other words: the eroded set is the set of vectors c for which $c + b \in X$ for all of the vectors $b \in B$. Thus for a vector c to be part of the eroded set, all the vectors $c + b$ (where $b \in B$) must be part of the image set X.

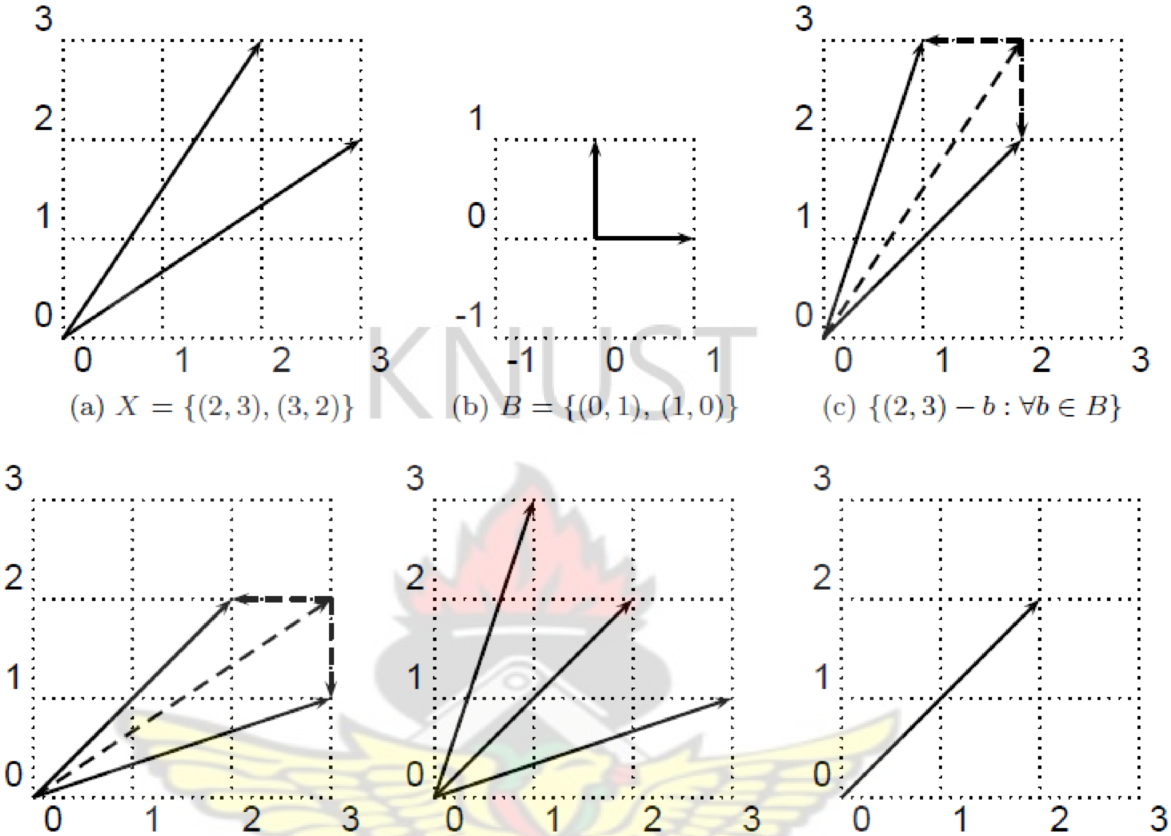
The erosion transform generally causes image objects to shrink in size, or erode - hence the name.

This is best illustrated with an example. In Fig. 2.3 vectors $(2, 3)$ and $(3, 2)$ are part of set X, and vectors $(0, 1)$ and $(1, 0)$ are part of set B. The erosion of X by B is formed by subtracting vectors $b \in B$ from each vector $x \in X$, and selecting only those vectors $c+b$ which can be found in image X. By subtracting the structuring element B from each vector of the image X, three different vectors are obtained: $(1, 3)$, $(2, 2)$, and $(3, 1)$. (Fig. 2.3(e).) However, as can be seen in Fig. 2.4(f) only vector $(2, 2)$ becomes part of the eroded image $X \ominus B$, whereas vectors $(1, 3)$ and $(3, 1)$ do not. This is because after adding set B (i.e. vectors $(0, 1)$ and $(1, 0)$) to each of these three vectors, only the result from vector $(2, 2)$ will be inside image X.

Whereas dilation can be represented as a union translates, erosion can be represented as an intersection of the negative translates:

$$A \ominus B = \bigcap_{b \in B} A_{-b}.$$

where $-b$ is the scalar multiple of the vector b by -1.

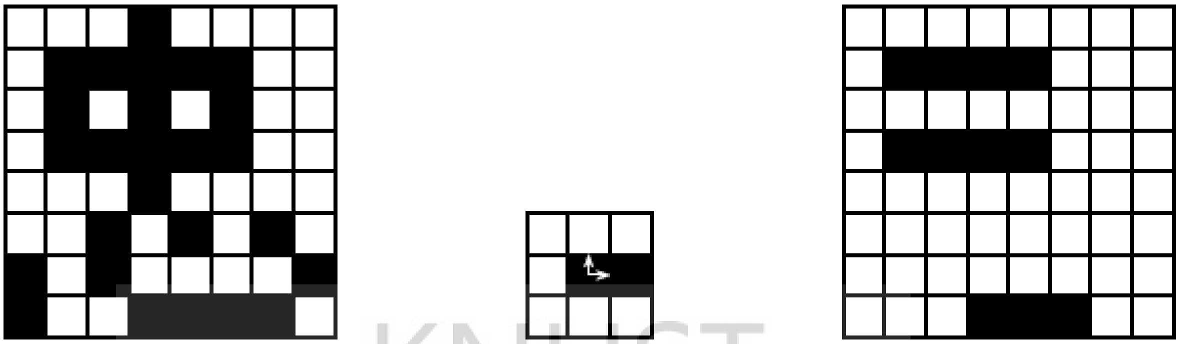


(a) $X = \{(2,3),(3,2)\}$ (b) $B = \{(0,1),(1,0)\}$ (c) $\{(2,3) - b : \forall b \in B\}$

(d) $\{(3,2) - b : \forall b \in B\}$ (e) $\{x - b : \forall x, b \in X, B\}$ (f) $X \ominus B = \{(2,2)\}$

Figure 3.3: Erosion by vector subtraction

Like dilation, the erosion of the original image by the structure element can be described intuitively by template translation. The template is moved across the original image. For a given foreground pixel, put the center of the template onto it, i.e. translate the template to that pixel. If the translation of the template is a sub-image of the original image, that pixel is activated in erosion; otherwise, it is not activated. Figure 2.3 shows that erosion "shrinks" the original image and the small peak is eliminated.



(a) X (b) B (c) $X \ominus B$

Figure 3.4: Erosion of X by the structuring element B

Erosion applied to an image is illustrated in Fig. 2.4. As usual X is the image to be eroded, and B is the structuring element. In the eroded image $X \ominus B$ only elements which are at least two pixels wide have been kept from image X. This selection has been dictated by the structuring element B. With the center and right pixels of B set, a pixel will exist in $X \ominus B$ only if both the center and right pixels of X also exist. Had not the center pixel of B been set, the eroded image would have been shifted one step to the left instead of being reduced. Would only the center pixel have been set, there would be no difference from the original image.

3.2.3 Algebraic structure of Dilation and Erosion

The Dilation transform has the following properties:

1. **Commutative** - Dilation is commutative, since addition is commutative

$$A \oplus B = B \oplus A.$$

2. **Associative** - Dilation is associative, since addition is associative

$$(A \oplus B) \oplus C = A \oplus (B \oplus C).$$

The Erosion transform has the following properties:

1. **Non-Commutative** - Erosion is non-commutative, since subtraction is non-commutative

$$A \ominus B \neq B \ominus A.$$

2. **Non-Associative** - Erosion is non-associative, since subtraction is non-associative

$$(A \ominus B) \ominus C \neq A \ominus (B \ominus C).$$

Furthermore, the Dilation and Erosion transforms also have the following common properties:

1. **Translation In-variance** - Let x be a vector belonging to A and B: $x \in A$, $x \in B$. Both Dilation and Erosion are translation invariant by x

$$A \oplus B_x = A_x \oplus B = (A \oplus B)_x$$

$$A \ominus B_x = A_x \ominus B = (A \ominus B)_x$$

2. **Increasing in** A - If an image set A_1 is a subset of A_2, $A_1 \subset A_2$, both Dilation and Erosion are increasing in A

$$A_1 \oplus B \subset A_2 \oplus B$$

$$A_1 \ominus B \subset A_2 \ominus B$$

3. **Decreasing in** B - If a structuring element B_1 is a subset of B_2, $B_1 \subset B_2$, Erosion is decreasing in B

$$A \ominus B_1 \supset A \ominus B_2$$

4. **Duality** - The Dilation and Erosion transforms are duals

$$(A \oplus B)^c = A^c \ominus \breve{B}$$

$$(A \ominus B)^c = A^c \oplus \breve{B}$$

5. **Non-Inverse** - Dilation and Erosion are not the inverse of each other

$$(A \oplus B) \ominus B \neq A$$

$$(A \ominus B) \oplus B \neq A$$

Both dilation and erosion are nonlinear operations, and are generally non-invertible.

6. **Identity Set** - Both the dilation and erosion transforms have an identity set I

$$A \oplus I = A$$

$$A \ominus I = A$$

7. **Empty Set** - The Dilation transform has an empty set

$$A \oplus \emptyset = \emptyset$$

3.2.4 Binary Opening

By repeatedly dilating and eroding an image, it is possible to eliminate specific details, while at the same time preserving the structure and shape of the objects contained within the image. Two such morphological operations are *Opening* and *Closing*. These operators play an important part in mathematical morphology, as they are often used as building blocks for other, more complicated morphological operators.

Opening is formed by first eroding a set X, after which this eroded set, $X \ominus B$, is dilated. Note that the structuring element B used is the same for the two operations.

Definition (Opening) 3.2.1 *Let X and B be subsets of the discrete space Z^2: $X \subset Z^2$, $B \subset Z^2$. The Opening of X by B is defined as*

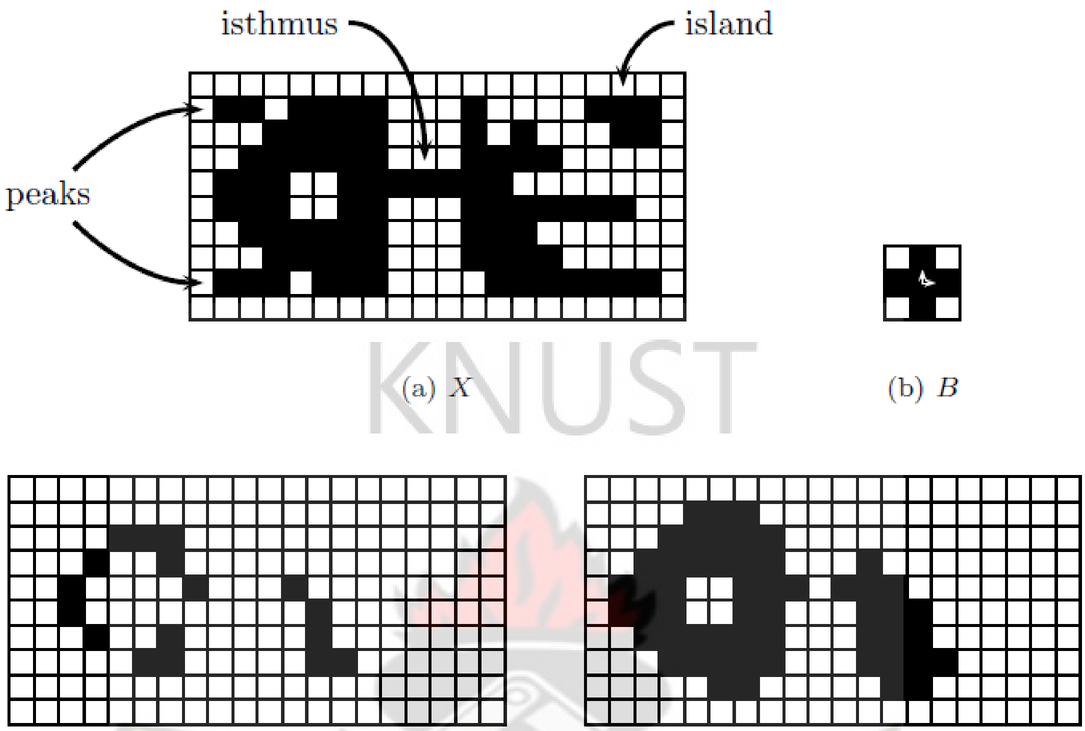
$$X \circ B = (X \ominus B) \oplus B.$$

$$OR$$

Opening of a binary image A by structure element B, denoted by $A \circ B$, is defined as

$$A \circ B = (A \ominus B) \oplus B.$$

With the opening of an image, contours of objects are smoothed, narrow isthmuses are cut, and small islands and sharp peaks are removed. From the definition, the original image is eroded first and then dilated. Therefore, it can intuitively be thought as "rolling the structure element about the inside boundary of the image". The following definition gives a rigorous set-theoretic characterization of this "fitting" property. It states that the



(a) X (b) B

(c) $X \ominus B$ (d) $X \circ B = (X \ominus B) \oplus B$

Figure 3.5: Opening of X by the structuring element B

opening of A by B is obtained by taking the union of all the translates of B which fit into A.

An example of the opening transformation can be seen in Fig. 2.5. Here image X is opened using a structuring element B the shape of rhomboid. In Fig. 2.5(a) two protruding peaks are marked, as well as an isthmus connecting the two major bodies, and a small island in the upper right corner. In Fig. 2.5(d) is shown the result from the opening of X. Note how the narrow isthmus has been cut, and both the small island and the two sharp peaks have been removed. Note also how the contours have been smoothed around the original object.

Definition (Opening(2)) 3.2.4.1

$$A \circ B = \bigcup \{B + x \mid B + x \subset A\}.$$

Figure 2.4 shows how this original image is smoothed and the spot-like noise is removed because the disk can't fit into them. It is worth noticing that smoothing effect of the object boundary highly depends on the shape of the structure element.

3.2.5 Binary Closing

Analogous to relationship between dilation and erosion, closing is the morphological dual to opening. The difference between opening and closing lies in the order in which the erosion and dilation transforms are applied. Closing is formed by first dilating a set X, after which this dilated set, $X \oplus B$, is eroded.

Definition (Closing) 3.2.1 *Let X and B be subsets of the discrete space Z^2: $X \subset Z^2$, $B \subset Z^2$. The Closing of X by B is defined as*

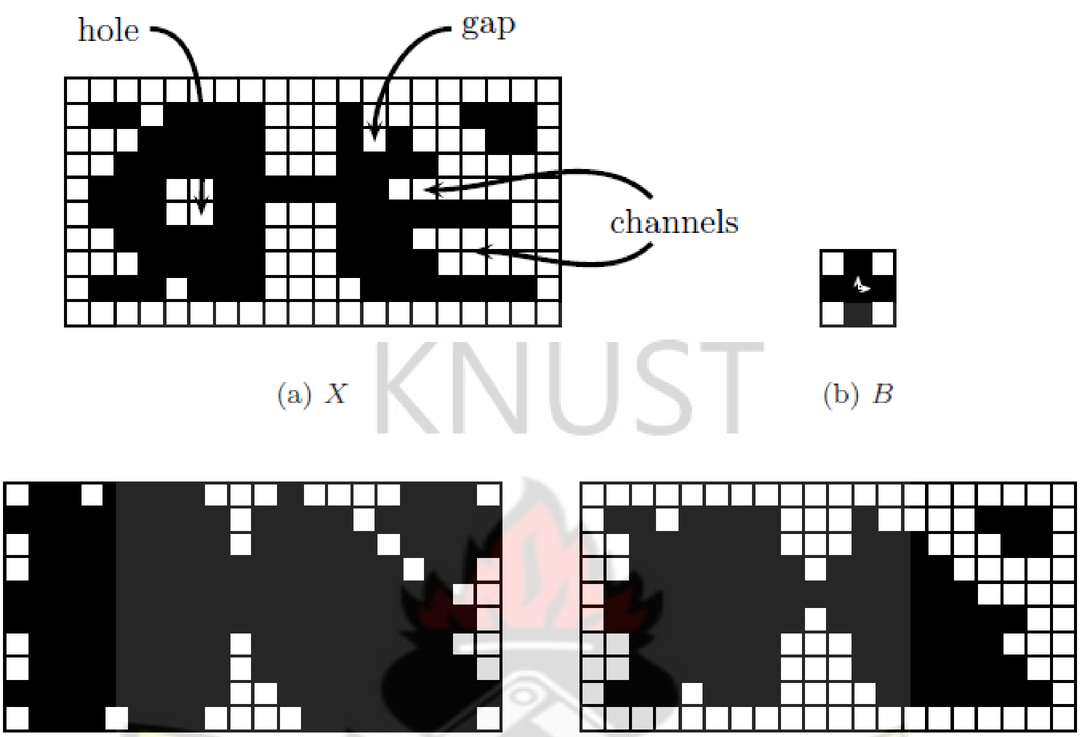
$$X \bullet B = (X \oplus B) \ominus B.$$

$$OR$$

Closing of a binary image A by structure element B, denoted by $A \bullet B$, is defined as

$$A \bullet B = (A \oplus B) \ominus B.$$

By closing an image, contour of objects are smoothed (just as with opening), narrow channels are blocked up, and small holes and gaps on the contours are filled up.



(a) X (b) B

(c) $X \oplus B$ (d) $X \bullet B = (X \oplus B) \ominus B$

Figure 3.6: Closing of X by the structuring element B

Though closing does remove various specific *image details*, just as opening does, it does not remove any *image objects*. Rather, the closing transformation *expands* the image objects, thereby eliminating image details. This is in contrast to opening, where reduction of detail is achieved through reduction of objects. The closing of the original image includes all points satisfying the condition that anytime the point can be covered by a translation of the structure element, there must be some point in common between the translated structure element and the original image.

An example of closing can be seen in Fig. 2.6. The image and the structuring element are the same as used for opening (Fig. 2.5). In Fig.

2.6(a) two channels of different width are marked, as well as a small hole and a gap in the contour. In Fig. 2.6(d) is shown the result from the opening of X. Note here how the two narrow channels have been blocked up, and both the small hole and the gap have been filled up. Note also how the contours have been smoothed along the isthmus.

Definition (Closing(2)) 3.2.5.1 *z is the element of $A \bullet B$ if and only if $(B + y) \cap A \neq \emptyset$, for any translate $(B + y)$ containing z.*

As stated in Eqn. (2.8), closing is done by first dilating the image and then eroding it. Hence, instead of eliminating the small peaks, it will "fill" the holes, as shown in Fig. 2.5. In other words, it has the effect of "clustering" each spatial point to a connected set.

3.2.6 Algebraic structure of opening and closing

The opening and closing transforms have the following common properties:

1. **Duality** - The opening and closing transforms are duals

$$(A \circ B)^c = A^c \bullet \breve{B}$$

$$(A \bullet B)^c = A^c \circ \breve{B}$$

2. **Non-Inverse** - Opening and Closing are not the inverse of each other

$$(A \circ B) \bullet B \neq A$$

$$(A \bullet B) \circ B \neq A$$

3. **Translation In-variance** - Let x be a vector belonging to A and B: $x \in A$, $x \in B$. Both Opening and Closing are translation invariant by x

$$A \circ B_x = A_x \circ B = (A \circ B)_x$$

$$A \bullet B_x = A_x \bullet B = (A \bullet B)_x$$

4. **Anti-Extensivity** - The opening transform is anti-extensive, i.e. the Opening of A by a structuring element B is always contained in A, regardless of B

$$A \circ B \subseteq A$$

5. **Extensivity** - The Closing transform is extensive, i.e. the Closing of A by a structuring element B always contains A, regardless of B

$$A \subseteq A \bullet B$$

6. **Increasing Monotonicity** - If an image set A_1 is a subset of or equal to A_2, $A_1 \subseteq A_2$, both Opening and Closing are increasing

$$A_1 \circ B \subseteq A_2 \circ B$$

$$A_1 \bullet B \subseteq A_2 \bullet B$$

7. **Decreasing in** B - If a structuring element B_1 is a subset of B_2, $B_1 \subset B_2$, Opening is decreasing in B

$$A \circ B_1 \supset A \circ B_2$$

8. **Idempotence** - The opening and closing transforms are idempotent

$$A \circ B \circ B = A \circ B$$

$$A \bullet B \bullet B = A \bullet B$$

If X is unchanged by opening with B, X is said to be open, whereas if X is unchanged by closing with B, X is said to be closed.

3.3 Gray-Scale Morphology

In intuition, a 2D grayscale image can be thought of a set of points in 3D space, $p = (x; y; f(x; y)); (f(x, y)$ is the function to represent the gray-scale image). By applying the *umbra transform* **U**, a 2D gray-scale image can be transformed as a 3D binary image. Therefore, gray-scale morphological operators may be regarded as the extension of binary morphological operators to three-dimensional space.

Definition (Umbra Transform) 3.3.1 *Given a signal f, the umbra transform of f, denoted as $\boldsymbol{U}[f]$, is defined as:*

$$\boldsymbol{U}[f] = \{(x, y) : x \in D_f \text{ and } y \leq f(x)\}.$$

Definition (Top Surface) 3.3.2 *Given an umbra A, we define the top surface of A, denoted as $\boldsymbol{S}[A]$, to be the set of all points (x, y) such that x is in the domain of A and $y = sup\{y : (x; y) \in A\}$.*

Theoretically, given two signals f and g, dilation of f by g can be computed as $S[U[f] \oplus U[g]]$ where $\oplus$ is the binary dilation operator.

However, although the umbra transform tells us conceptually how to compute gray-scale morphology from binary morphology, the mathematical expression is still necessary for implementation. Two operators are employed for defining the gray-scale mathematical morphology operators, EXTSUP and INF.

Definition (EXTSUP) 3.3.3 *Given a collection of signals $\{f_k\}$, we define*

$$[EXTSUP(f_k)](t) = \begin{cases} sup[f_k(t)] & \textit{if there exists at least one } k \textit{ such that} \\ & f_k \textit{ is defined at } t, \textit{ and where the} \\ & \textit{supremum is over all such } k \\ undefined & \textit{if } f_k(t) \textit{ is undefined for all } k. \end{cases}$$

Definition (INF) 3.3.4 *Given a collection of signals $\{f_k\}$, we define*

$$[INF(f_k)](t) = \begin{cases} inf[f_k(t)] & \textit{if } f_k(t) \textit{ is defined for all } k \\ undefined & \textit{otherwise.} \end{cases}$$

For any signal f, with domain D_f , and one point x, we define the translation f_x by $f_x(t) = f(t-x)$. That is, f_x is f translated x units to the right. Hence, $D_{f_x} = D_f + x$. In addition, $f(t)+y$ translates f up y units and possesses the same domain. Furthermore, given two signals f and g, $f \ll g$ means that D_f is the subset of D_g, and for any t in D_f, $f(t) \leq g(t)$.

3.3.1 Grayscale Dilation

Definition (Grayscale Dilation) 3.3.5 *For signals f and g, with respective domain D_f and D_g, we define the dilation of f by g as*

$$\boldsymbol{D}(f,g) = EXTSUP_{x \in D_f}[g_x + f(x)]$$

Geometrically, the dilation is obtained by taking an extended supremum of all copies of g that have been translated over x units and up $f(x)$ units. As binary dilation, g plays the role of template. In gray-scale dilation, for each point of f, shift g so that its center coincides with $(x, f(x))$ and $EXTSUP$ is applied to the resulting copies of g.

In implementation, the "supremum" operation in Eqn. (2.10) is replaced by "maximum", and furthermore, Eqn. (2.10) can be re-written as follows.

Definition (Grayscale Dilation (2)) 3.3.6 *Dilation of a gray-scale image $f(r,c)$ by a gray-scale structure element $g(r,c)$ is defined as*

$$\boldsymbol{D}(f,g)(r,c) = max_{(i,j)}[f(r-i,c-j)+g(i,j)]$$

3.3.2 Grayscale Erosion

Definition (Grayscale Erosion) 3.3.7 *For signals f and g, with respective domain D_f and D_g, we define the erosion of f by g as*

$$[\boldsymbol{E}(f,g)](x) = Sup\{y : g_x + y \ll\}$$

To find the value of the erosion of f by g at the point x, we shift g so that it is centered at x and then find out the largest vertical translation y that will leave $g_x + y$ beneath f. It is analogous to the "fitting" property in binary erosion. Eqn.(2.12) can be expressed in terms of INF operation as:

$$\mathbf{E}(f,g) = INF_{x \in D_g}[f_{-x} - g(x)]$$

As gray-scale dilation, by replacing "infimum" with "minimum", Eqn. (2.13) can be re-written as follows:

Definition (Grayscale Erosion (2)) 3.3.8 *Erosion of a gray-scale image $f(r,c)$ by a gray-scale structure element $g(r,c)$ is defined as*

$$\boldsymbol{E}(f,g)(r,c) = min_{(i,j)}[f(r+i,c+j)-g(i,j)]$$

3.3.3 Grayscale Opening

Definition (Grayscale Opening) 3.3.9 *For signals f and g, we define the opening of f by g as*

$$\boldsymbol{O}(f, g) = \boldsymbol{D}[\boldsymbol{E}(f, g), g]$$

3.3.4 Grayscale Closing

Definition (Grayscale Closing) 3.3.10 *For signals f and g, we define the closing of f by g as*

$$\boldsymbol{C}(f, g) = \boldsymbol{E}[\boldsymbol{D}(f, g), g]$$

3.4 Color Morphology

After introducing binary and gray-scale morphology, the task turns on dealing with the color image morphology. However, the extension of mathematical morphology to color images is not straightforward. Two approaches for color morphology would be investigated: a vector approach and a component-wise approach.

In component-wise approach, the gray-scale morphological operator is applied to each channel of the color image. For example, component-wise color dilation of $f(x;y) = [f_R(x;y); f_G(x;y); f_B(x;y)]^T$ by the structure element $h(x;y) = [h_R(x;y); h_G(x;y); h_B(x;y)]^T$ in RGB color space is defined as:

$$(f \oplus_c h)(x, y) = [(f_R \oplus h_R)(x, y), (f_G \oplus h_G)(x, y); (f_B \oplus h_B)(x, y)]^T$$

where the symbol $\oplus_c$ represents component-wise dilation and *oplus* on the right-hand side is gray-scale dilation. Component-wise color erosion, opening, and closing can be defined in the same way. Because the component images are filtered separately with the component-wise filter, there is a possibility of altering the spectral composition of the image, e.g., the color balance and object boundary.

A different way to examining the color morphology is to treat the color at each pixel as a vector. Furthermore, vector-based color morphology makes use of the multivariate ranking concept. First, reduced ordering is performed. Each multivariate sample is mapped to a scalar value based on reduced ordering function and then the samples are ordered according to the mapped scalar value. Given the reduced ordering function d and the set H, the value of vector color dilation of f by H at the point (x, y) is defined as:

$$(f \oplus_v h)(x, y) = a \ (\oplus_v \texttt{ denotes vector dilation})$$

where

$$a \in \{f(r, s) : (r, s) \in H\}$$

and

$$d(a) \geq d(f(r, s)) \ \ \forall (r, s) \in H_(x; y)$$

Similarly, the vector erosion of f by H at the point (x, y) is defined as:

$$(f \ominus_v h)(x, y) = a \ (\ominus_v \texttt{ denotes vector dilation})$$

where

$$a \in \{f(r, s) : (r, s) \in H\}$$

and

$$d(a) \leq d(f(r, s)) \ \ \forall (r, s) \in H_(x; y)$$

Chapter 4

Analysis of Morphological Operators

4.1 Introduction

This chapter mainly involves the analysis of morphological operators in connection with set union and intersection. We would consider the morphological results of two sets in connection with a single structural element and vice versa. This is to make inference about partitioning sets before using morphological operators on them.

Furthermore, we would also look at the distribution properties of morphological operators over set union and intersection.

4.2 The Morphological Operators with 2 distinct sets

Proposition (The Union of Dilation with 2 different sets) 4.2.1

$$(A_1 \oplus B) \cup (A_2 \oplus B) = (A_1 \cup A_2) \oplus B \tag{4.1}$$

Proof:

$$\begin{aligned}
(A_1 \oplus B) \cup (A_2 \oplus B) &= \bigcup_{b\in B} A_{1_b} \bigcup \bigcup_{b\in B} A_{2_b} \\
&= \bigcup_{b\in B} \{A_{1_b} \cup A_{2_b}\} \\
&= \bigcup_{b\in B} (A_1 \cup A_2)_b \\
&= (A_1 \cup A_2) \oplus B
\end{aligned}$$

■

From equation 4.1 above, it implies that when you have to take the dilation of 2 different sets with the same structural element and take the union of the results after, then it is the same as taking the union of the sets (images) and dilating it with the structural element. This also leads to an implication that if you have to take the dilation of any set with a certain structural element, you can always partition the set into 2 sets and take the union of their respective dilations. We can also make a generalized implication from the above analyses. This leads to the proposition below:

If any set can be partition into n distinct parts then the union of the each of the partitions dilation with the structural element is equal to the set's (the one which was partitioned) dilation with the structural element.

Proposition 4.2.2

$$\textit{If } A = A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n \textit{ for } n \geq 2$$
$$\textit{Then } A \oplus B = (A_1 \oplus B) \cup (A_2 \oplus B) \cup (A_3 \oplus B) \cup \cdots \cup (A_n \oplus B)$$

Proof:

$$\begin{aligned}
\text{If } A &= A_1 \cup A_2 \\
\text{Then } (A_1 \cup A_2) \oplus B &= (A_1 \oplus B) \cup (A_2 \oplus B) \\
\text{Let assume that if } A &= A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_k \text{ for } k \geq 2 \\
\text{Then } A \oplus B &= (A_1 \oplus B) \cup (A_2 \oplus B) \cup (A_3 \oplus B) \cup \cdots \cup (A_k \oplus B) \\
\text{Now we show that if } A &= A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_k \cup A_{k+1} \\
\text{Then} & \\
A \oplus B &= (A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_k \cup A_{k+1}) \oplus B \\
&= [(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_k) \cup A_{k+1}] \oplus B \\
&= [(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_k) \oplus B] \cup (A_{k+1} \oplus B) \\
&= (A_1 \oplus B) \cup (A_2 \oplus B) \cup (A_3 \oplus B) \cup \cdots \\
&\cup (A_k \oplus B) \cup (A_{k+1} \oplus B)
\end{aligned}$$

■

Proposition (The Intersection of Dilation with 2 different sets) 4.2.3

$$(A_1 \oplus B) \cap (A_2 \oplus B) = (A_1 \cap A_2) \oplus B \tag{4.2}$$

Proof:

$$
\begin{aligned}
(A_1 \oplus B) \cap (A_2 \oplus B) &= \bigcup_{b \in B} A_{1_b} \bigcap \bigcup_{b \in B} A_{2_b} \\
&= \bigcup_{b \in B} \{A_{1_b} \cap A_{2_b}\} \\
&= \bigcup_{b \in B} (A_1 \cap A_2)_b \\
&= (A_1 \cap A_2) \oplus B
\end{aligned}
$$

■

It can be analyzed from the above that when you have to take the dilation of 2 different sets with the same structural element and take the intersection of the results after, then it is the same as taking the intersection of the sets (images) and dilating it with the structural element. This also leads to an implication that if you have to take the dilation of any set with a certain structural element, you can always find 2 sets containing the set you wish to dilate and take the intersection of their respective dilations. We can also make a generalized implication from the above analyses. This leads to the proposition below:
If any set can be found in n distinct sets then the intersection of the each of the n distinct set's dilation with the structural element is equal to the set's (the one that can be found in the n distinct sets) dilation with the structural element.

Proposition 4.2.4

$$\textit{If } A = A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n \textit{ for } n \geq 2$$

$$\textit{Then } A \oplus B = (A_1 \oplus B) \cap (A_2 \oplus B) \cap (A_3 \oplus B) \cap \cdots \cap (A_n \oplus B)$$

Proof:

$$
\begin{aligned}
\text{If } A &= A_1 \cap A_2 \\
\text{Then } (A_1 \cap A_2) \oplus B &= (A_1 \oplus B) \cap (A_2 \oplus B) \\
\text{Let assume that if } A &= A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_k \text{ for } k \geq 2 \\
\text{Then } A \oplus B &= (A_1 \oplus B) \cap (A_2 \oplus B) \cap (A_3 \oplus B) \cap \cdots \cap (A_k \oplus B) \\
\text{Now we show that if } A &= A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_k \cap A_{k+1} \\
\text{Then} & \\
A \oplus B &= (A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_k \cap A_{k+1}) \oplus B \\
&= [(A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_k) \cap A_{k+1}] \oplus B \\
&= [(A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_k) \oplus B] \cap (A_{k+1} \oplus B) \\
&= (A_1 \oplus B) \cap (A_2 \oplus B) \cap (A_3 \oplus B) \cap \cdots \\
&\cap \; (A_k \oplus B) \cap (A_{k+1} \oplus B)
\end{aligned}
$$

■

Proposition (The Union of Erosion with 2 different sets) 4.2.5

$$(A_1 \ominus B) \cup (A_2 \ominus B) = (A_1 \cup A_2) \ominus B \tag{4.3}$$

Proof:

$$
\begin{aligned}
(A_1 \ominus B) \cup (A_2 \ominus B) &= \bigcap_{b \in B} A_{1_{-b}} \bigcup \bigcap_{b \in B} A_{2_{-b}} \\
&= \bigcap_{b \in B} \left\{ A_{1_{-b}} \cup A_{2_{-b}} \right\} \\
&= \bigcap_{b \in B} (A_1 \cup A_2)_{-b} \\
&= (A_1 \cup A_2) \ominus B
\end{aligned}
$$

■

When you have to take the erosion of 2 different sets with the same structural element and take the union of the results after, then it is the same as taking the union of the sets (images) and taking the erosion of the result with the structural element. This assertion is as a result of the proof above. It is clear from these that morphological erosion behaves the same way as it morphological dual with respect to union. Therefore, we can generalized the same way we did with dilation which leads to the proposition below:

Provided a set can be partition into n distinct parts then the union of the each of the partitions erosion with the structural element is equal to the set's erosion with the structural element.

Proposition 4.2.6

$$\textit{If } A = A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n \textit{ for } n \geq 2$$

$$\textit{Then } A \ominus B = (A_1 \ominus B) \cup (A_2 \ominus B) \cup (A_3 \ominus B) \cup \cdots \cup (A_n \ominus B)$$

Proof:

$$
\begin{aligned}
\text{If } A &= A_1 \cup A_2 \\
\text{Then } (A_1 \cup A_2) \ominus B &= (A_1 \ominus B) \cup (A_2 \ominus B) \\
\text{Let assume that if } A &= A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_k \text{ for } k \geq 2 \\
\text{Then } A \ominus B &= (A_1 \ominus B) \cup (A_2 \ominus B) \cup (A_3 \ominus B) \cup \cdots \cup (A_k \ominus B) \\
\text{Now we show that if } A &= A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_k \cup A_{k+1} \\
\text{Then} & \\
A \ominus B &= (A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_k \cup A_{k+1}) \ominus B \\
&= [(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_k) \cup A_{k+1}] \ominus B \\
&= [(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_k) \ominus B] \cup (A_{k+1} \ominus B) \\
&= (A_1 \ominus B) \cup (A_2 \ominus B) \cup (A_3 \ominus B) \cup \cdots \\
&\cup \; (A_k \ominus B) \cup (A_{k+1} \ominus B)
\end{aligned}
$$

■

Proposition (The Intersection of Erosion with 2 different sets) 4.2.7

$$(A_1 \ominus B) \cap (A_2 \ominus B) = (A_1 \cap A_2) \ominus B \tag{4.4}$$

Proof:

$$
\begin{aligned}
(A_1 \ominus B) \cap (A_2 \ominus B) &= \bigcap_{b \in B} A_{1_{-b}} \bigcap \bigcap_{b \in B} A_{2_{-b}} \\
&= \bigcap_{b \in B} \left\{A_{1_{-b}} \cap A_{2_{-b}}\right\} \\
&= \bigcap_{b \in B} (A_1 \cap A_2)_{-b} \\
&= (A_1 \cup A_2) \ominus B
\end{aligned}
$$

■

As it can be seen, from the above proof, morphological erosion and intersection behaves the same way as it dual, dilation and union, does. The implication is that assuming you want to take the erosion of 2 different sets with the same structural element and take the intersection of the results after, then it is the same as taking the intersection of the sets(images) and taking the erosion with the structural element. Hence it can be generalized from the above analyses that:
Whenever any set can be found in n distinct sets then the intersection of each of the n distinct sets erosion with the structural element is equal to the set's erosion with the structural element.

Proposition 4.2.8

$$\textit{If } A = A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n \textit{ for } n \geq 2$$

$$\textit{Then } A \ominus B = (A_1 \ominus B) \cap (A_2 \ominus B) \cap (A_3 \ominus B) \cap \cdots \cap (A_n \ominus B)$$

Proof:

$$\begin{aligned}
\text{If } A &= A_1 \cap A_2 \\
\text{Then } (A_1 \cap A_2) \ominus B &= (A_1 \ominus B) \cap (A_2 \ominus B) \\
\text{Let assume that if } A &= A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_k \text{ for } k \geq 2 \\
\text{Then } A \ominus B &= (A_1 \ominus B) \cap (A_2 \ominus B) \cap (A_3 \ominus B) \cap \cdots \cap (A_k \ominus B) \\
\text{Now we show that if } A &= A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_k \cap A_{k+1} \\
\text{Then} & \\
A \ominus B &= (A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_k \cap A_{k+1}) \ominus B \\
&= [(A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_k) \cap A_{k+1}] \ominus B \\
&= [(A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_k) \ominus B] \cap (A_{k+1} \ominus B) \\
&= (A_1 \ominus B) \cap (A_2 \ominus B) \cap (A_3 \ominus B) \cap \cdots \\
&\cap \; (A_k \ominus B) \cap (A_{k+1} \ominus B)
\end{aligned}$$

■

Proposition (The Union of Opening with 2 different sets) 4.2.9

$$(A_1 \circ B) \cup (A_2 \circ B) = (A_1 \cup A_2) \circ B \tag{4.5}$$

Proof:

$$\begin{aligned}
(A_1 \circ B) \cup (A_2 \circ B) &= (A_1 \ominus B) \oplus B \cup (A_2 \ominus B) \oplus B \\
&= [(A_1 \ominus B) \cup (A_2 \ominus B)] \oplus B \\
&= [(A_1 \cup A_2) \ominus B] \oplus B \\
&= (A_1 \cup A_2) \circ B
\end{aligned}$$

■

From equation 4.5 above, it implies that when you have to take the opening of 2 different sets with the same structural element and take the union of the results after, then it is the same as taking the union of the sets (images) and then opening it with the structural element. This also leads to an implication that if you have to take the opening of any set with a certain structural element, you can always partition the set into 2 sets and take the union of their respective openings. We can also make a generalized implication from the above analyses. This leads to the proposition below:
If any set can be partition into n distinct parts then the union of the each of the partitions opening with the structural element is equal to the set's (the one which was partitioned) opening with the structural element.

Proposition 4.2.10

$$\text{If } A = A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n \text{ for } n \geq 2$$

$$\text{Then } A \circ B = (A_1 \circ B) \cup (A_2 \circ B) \cup (A_3 \circ B) \cup \cdots \cup (A_n \circ B)$$

Proof:

$$\begin{aligned}
\text{If } A &= A_1 \cup A_2 \\
\text{Then } (A_1 \cup A_2) \circ B &= (A_1 \circ B) \cup (A_2 \circ B) \\
\text{Let assume that if } A &= A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_k \text{ for } k \geq 2 \\
\text{Then } A \circ B &= (A_1 \circ B) \cup (A_2 \circ B) \cup (A_3 \circ B) \cup \cdots \cup (A_k \circ B) \\
\text{Now we show that if } A &= A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_k \cup A_{k+1} \\
\text{Then} & \\
A \circ B &= (A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_k \cup A_{k+1}) \circ B \\
&= [(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_k) \cup A_{k+1}] \circ B \\
&= [(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_k) \circ B] \cup (A_{k+1} \circ B) \\
&= (A_1 \circ B) \cup (A_2 \circ B) \cup (A_3 \circ B) \cup \cdots \\
&\cup (A_k \circ B) \cup (A_{k+1} \circ B)
\end{aligned}$$

■

Proposition (The Intersection of Opening with 2 different sets) 4.2.11

$$(A_1 \circ B) \cap (A_2 \circ B) = (A_1 \cap A_2) \circ B \tag{4.6}$$

Proof:

$$\begin{aligned}
(A_1 \circ B) \cap (A_2 \circ B) &= (A_1 \ominus B) \oplus B \cap (A_2 \ominus B) \oplus B \\
&= [(A_1 \ominus B) \cap (A_2 \ominus B)] \oplus B \\
&= [(A_1 \cap A_2) \ominus B] \oplus B \\
&= (A_1 \cap A_2) \circ B
\end{aligned}$$

$\blacksquare$

Taking the opening of 2 different sets with the same structural element before taking the intersection of the results after can be done by first taking the intersection of the sets (images) and then opening it with the structural element. This leads to an implication that if you have to take the opening of any set with a certain structural element, you can always find 2 sets containing the set you wish to open and take the intersection of their respective openings. We can also make a generalized implication from the above analyses which leads to the proposition below:
If any set can be found in n distinct sets then the intersection of the each of the n distinct set's opening with the structural element is equal to the set's (the one that can be found in the n distinct sets) opening with the structural element.

Proposition 4.2.12

$$\textit{If } A = A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n \textit{ for } n \geq 2$$
$$\textit{Then } A \circ B = (A_1 \circ B) \cap (A_2 \circ B) \cap (A_3 \circ B) \cap \cdots \cap (A_n \circ B)$$

Proof:

$$
\begin{aligned}
\text{If } A &= A_1 \cap A_2 \\
\text{Then } (A_1 \cap A_2) \circ B &= (A_1 \circ B) \cap (A_2 \circ B) \\
\text{Let assume that if } A &= A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_k \text{ for } k \geq 2 \\
\text{Then } A \circ B &= (A_1 \circ B) \cap (A_2 \circ B) \cap (A_3 \circ B) \cap \cdots \cap (A_k \circ B) \\
\text{Now we show that if } A &= A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_k \cap A_{k+1} \\
\text{Then} & \\
A \circ B &= (A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_k \cap A_{k+1}) \circ B \\
&= [(A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_k) \cap A_{k+1}] \circ B \\
&= [(A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_k) \circ B] \cap (A_{k+1} \circ B) \\
&= (A_1 \circ B) \cap (A_2 \circ B) \cap (A_3 \circ B) \cap \cdots \\
&\cap\ (A_k \circ B) \cap (A_{k+1} \circ B)
\end{aligned}
$$

∎

Proposition (The Union of Closing with 2 different sets) 4.2.13

$$(A_1 \bullet B) \cup (A_2 \bullet B) = (A_1 \cup A_2) \bullet B \tag{4.7}$$

Proof:

$$
\begin{aligned}
(A_1 \bullet B) \cup (A_2 \bullet B) &= (A_1 \oplus B) \ominus B \cup (A_2 \oplus B) \ominus B \\
&= [(A_1 \oplus B) \cup (A_2 \oplus B)] \ominus B \\
&= [(A_1 \cup A_2) \oplus B] \ominus B \\
&= (A_1 \cup A_2) \bullet B
\end{aligned}
$$

∎

It can be analyzed from the above that when you have to take the closing of 2 different sets with the same structural element and take the union of the results after, then it is the same as taking the union of the sets (images) and then closing it with the structural element. This assertion also leads to an implication that if you have to take the closing of any set with a certain structural element, you can always partition the set into 2 sets and take the union of their respective closings. Hence the generalization that:
Provided any set can be partition into n distinct parts then the union of the each of the partitions closing with the structural element is equal to the set's (the one which was partitioned) closing with the structural element.

Proposition 4.2.14

$$\textit{If } A = A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n \textit{ for } n \geq 2$$
$$\textit{Then } A \bullet B = (A_1 \bullet B) \cup (A_2 \bullet B) \cup (A_3 \bullet B) \cup \cdots \cup (A_n \bullet B)$$

Proof:

$$
\begin{aligned}
\text{If } A &= A_1 \cup A_2 \\
\text{Then } (A_1 \cup A_2) \bullet B &= (A_1 \bullet B) \cup (A_2 \bullet B) \\
\text{Let assume that if } A &= A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_k \text{ for } k \geq 2 \\
\text{Then } A \bullet B &= (A_1 \bullet B) \cup (A_2 \bullet B) \cup (A_3 \bullet B) \cup \cdots \cup (A_k \bullet B) \\
\text{Now we show that if } A &= A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_k \cup A_{k+1} \\
\text{Then} & \\
A \bullet B &= (A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_k \cup A_{k+1}) \bullet B \\
&= [(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_k) \cup A_{k+1}] \bullet B \\
&= [(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_k) \bullet B] \cup (A_{k+1} \bullet B) \\
&= (A_1 \bullet B) \cup (A_2 \bullet B) \cup (A_3 \bullet B) \cup \cdots \\
&\cup \; (A_k \bullet B) \cup (A_{k+1} \bullet B)
\end{aligned}
$$

■

It is quiet clear from the proof below that, the morphological dual of opening and union, behaves the same way as it counterpart, closing and intersection. It therefore implies that when you have to take the closing of 2 different sets with the same structural element and take the intersection of the results after, then it is the same as taking the intersection of the sets (images) and then closing it with the structural element.

Proposition (The Intersection of Closing with 2 different sets) 4.2.15

$$(A_1 \bullet B) \cap (A_2 \bullet B) = (A_1 \cap A_2) \bullet B \tag{4.8}$$

Proof:

$$\begin{aligned}(A_1 \bullet B) \cap (A_2 \bullet B) &= (A_1 \oplus B) \ominus B \cap (A_2 \oplus B) \ominus B \\ &= [(A_1 \oplus B) \cap (A_2 \oplus B)] \ominus B \\ &= [(A_1 \cap A_2) \oplus B] \ominus B \\ &= (A_1 \cap A_2) \bullet B\end{aligned}$$

■

We can also make a generalized implication from the above analyses which leads to the proposition below:
Whenever any set can be found in n distinct sets then the intersection of the each of the n distinct set's closing with the structural element is equal to the set's (the one that can be found in the n distinct sets) closing with the structural element.

Proposition 4.2.16

$$\textit{If } A = A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n \textit{ for } n \geq 2$$
$$\textit{Then } A \bullet B = (A_1 \bullet B) \cap (A_2 \bullet B) \cap (A_3 \bullet B) \cap \cdots \cap (A_n \bullet B)$$

Proof:

$$\begin{aligned}
\text{If } A &= A_1 \cap A_2 \\
\text{Then } (A_1 \cap A_2) \bullet B &= (A_1 \bullet B) \cap (A_2 \bullet B) \\
\text{Let assume that if } A &= A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_k \text{ for } k \geq 2 \\
\text{Then } A \bullet B &= (A_1 \bullet B) \cap (A_2 \bullet B) \cap (A_3 \bullet B) \cap \cdots \cap (A_k \bullet B) \\
\text{Now we show that if } A &= A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_k \cap A_{k+1} \\
\text{Then} & \\
A \bullet B &= (A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_k \cap A_{k+1}) \bullet B \\
&= [(A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_k) \cap A_{k+1}] \bullet B \\
&= [(A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_k) \bullet B] \cap (A_{k+1} \bullet B) \\
&= (A_1 \bullet B) \cap (A_2 \bullet B) \cap (A_3 \bullet B) \cap \cdots \\
&\cap \; (A_k \bullet B) \cap (A_{k+1} \bullet B)
\end{aligned}$$

■

4.3 The Distribution of Morphological Operators over Set Union and Intersection

Proposition (The Distribution of Dilation over Union) 4.3.1

$$(A \oplus B_1) \cup (A \oplus B_2) = A \oplus (B_1 \cup B_2) \tag{4.9}$$

Proof:

$$
\begin{aligned}
(A \oplus B_1) \cup (A \oplus B_2) &= \bigcup_{b_1 \in B_1} A_{b_1} \; \bigcup \; \bigcup_{b_2 \in B_2} A_{b_2} \\
&= \bigcup_{b \in B_1 \cup B_2} \{A_b \cap A_{b_1}\} \; \bigcup \; \bigcup_{b \in B_1 \cup B_2} \{A_b \cap A_{b_2}\} \\
&= \bigcup_{b \in B_1 \cup B_2} \{A_b \cap A_{b_1} \cup A_b \cap A_{b_2}\} \\
&= \bigcup_{b \in B_1 \cup B_2} A_b \\
&= A \oplus (B_1 \cup B_2)
\end{aligned}
$$

■

From equation 4.9 above, it implies that when you want to take the dilation of a set with 2 different structural elements and take the union of the results after, then it is the same as taking the union of the structural elements and dilating with the set. This shows that morphological dilation distribute over set union and it also leads to the implication that if you have to take the dilation of any set with a certain structural element, you can always partition the structural elements into 2 sets and take the union of their respective dilations. We can also make a generalized implication from the above analyses. This leads to the proposition below:
If any structural element can be partitioned into n distinct parts then the union of the each of the partitions dilation with the set is equal to the set's dilation with the structural element.

Proposition 4.3.2

$$\textit{If } B = B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_n \textit{ for } n \geq 2$$

$$\textit{Then } A \oplus B = (A \oplus B_1) \cup (A \oplus B_2) \cup (A \oplus B_3) \cup \cdots \cup (A \oplus B_n)$$

Proof:

$$\begin{aligned}
\text{If } B &= B_1 \cup B_2 \\
\text{Then } A \oplus B &= (A \oplus B_1) \cup (A \oplus B_2) \\
\text{Let assume that if } B &= B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k \text{ for } k \geq 2 \\
\text{Then } A \oplus B &= (A \oplus B_1) \cup (A \oplus B_2) \cup (A \oplus B_3) \cup \cdots \cup (A \oplus B_k) \\
\text{Now we show that if } B &= B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k \cup B_{k+1} \\
\text{Then} & \\
A \oplus B &= A \oplus (B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k \cup B_{k+1}) \\
&= A \oplus [(B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k) \cup B_{k+1}] \\
&= [A \oplus (B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k)] \cup (A \oplus B_{k+1}) \\
&= (A \oplus B_1) \cup (A \oplus B_2) \cup (A \oplus B_3) \cup \cdots \\
&\cup \ (A \oplus B_k) \cup (A \oplus B_{k+1})
\end{aligned}$$

■

From the proof below, it implies that when you want to take the erosion of a set with 2 different structural elements and take the union of the results after, then it is the same as taking the union of the structural elements and taking the erosion with the resulting structural element which indicate the distribution of morphological erosion over set union. This also leads to an implication that if you have to take the erosion of any set with a certain structural element, you can always partition the structural element into 2 sets and take the union of their respective erosions.

Proposition (The Distribution of Erosion over Union) 4.3.3

$$(A \ominus B_1) \cup (A \ominus B_2) = A \ominus (B_1 \cup B_2) \tag{4.10}$$

Proof:

$$\begin{aligned}
(A \ominus B_1) \cup (A \ominus B_2) &= \textstyle\bigcap_{b_1 \in B_1} A_{-b_1} \bigcup \bigcap_{b_2 \in B_2} A_{-b_2} \\
&= \textstyle\bigcap_{b \in B_1 \cup B_2} \{A_{-b} \cap A_{-b_1}\} \bigcup \bigcap_{b \in B_1 \cup B_2} \{A_{-b} \cap A_{-b_2}\} \\
&= \textstyle\bigcap_{b \in B_1 \cup B_2} \{A_{-b} \cap A_{-b_1} \cup A_{-b} \cap A_{-b_2}\} \\
&= \textstyle\bigcap_{b \in B_1 \cup B_2} A_{-b} \\
&= A \ominus (B_1 \cup B_2)
\end{aligned}$$

■

We can also make a generalized implication from the above analyses which leads to the proposition below:
Provided any structural element can be partitioned into n distinct parts then the union of the each of the partitions' erosion with the set is equal to the set's erosion with the structural element.

Proposition 4.3.4

$$\textit{If } B = B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_n \textit{ for } n \geq 2$$

$$\textit{Then } A \ominus B = (A \ominus B_1) \cup (A \ominus B_2) \cup (A \ominus B_3) \cup \cdots \cup (A \ominus B_n)$$

Proof:

$$
\begin{aligned}
\text{If } B &= B_1 \cup B_2 \\
\text{Then } A \ominus B &= (A \ominus B_1) \cup (A \ominus B_2) \\
\text{Let assume that if } B &= B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k \text{ for } k \geq 2 \\
\text{Then } A \ominus B &= (A \ominus B_1) \cup (A \ominus B_2) \cup (A \ominus B_3) \cup \cdots \cup (A \ominus B_k) \\
\text{Now we show that if } B &= B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k \cup B_{k+1} \\
\text{Then} & \\
A \ominus B &= A \ominus (B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k \cup B_{k+1}) \\
&= A \ominus [(B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k) \cup B_{k+1}] \\
&= [A \ominus (B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k)] \cup (A \ominus B_{k+1}) \\
&= (A \ominus B_1) \cup (A \ominus B_2) \cup (A \ominus B_3) \cup \cdots \\
&\cup \ (A \ominus B_k) \cup (A \ominus B_{k+1})
\end{aligned}
$$

■

Proposition (The Non Distribution of Erosion over intersection) 4.3.5

$$(A \ominus B_1) \cap (A \ominus B_2) = A \ominus (B_1 \cup B_2) \tag{4.11}$$

Proof:

$$
\begin{aligned}
[(A \ominus B_1) \cap (A \ominus B_2)]^c &= (A \ominus B_1)^c \cup (A \ominus B_2)^c \\
&= (A^c \oplus \breve{B}_1) \cup (A^c \oplus \breve{B}_2) \\
&= A^c \oplus (\breve{B}_1 \cup \breve{B}_2) \\
\Longrightarrow (A \ominus B_1) \cap (A \ominus B_2) &= [A^c \oplus (\breve{B}_1 \cup \breve{B}_2)]^c \\
&= A \ominus (B_1 \cup B_2)
\end{aligned}
$$

■

Taking the erosion of a set with 2 different structural elements and taking the intersection of the results after is the same as taking the union of the structural elements and taking the erosion of the set with the resulting structural element. Since we are suppose to take the union instead of the intersection shows that morphological erosion is non distributive over set intersection. We can analyze that if you have to take the erosion of any set with a certain structural element, you can always partition the structural element into 2 sets and take the intersection of their respective erosions. We can also make a generalized implication from the above analyses. This leads to the proposition below:

Whenever any structural element can be partitioned into n distinct parts then the intersection of the each of the partitions' erosion with the set is equal to the set's erosion with the structural element.

Proposition 4.3.6

$$\textit{If } B = B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_n \textit{ for } n \geq 2$$

$$\textit{Then } A \ominus B = (A \ominus B_1) \cap (A \ominus B_2) \cap (A \ominus B_3) \cap \cdots \cap (A \ominus B_n)$$

Proof:

$$
\begin{aligned}
\text{If } B &= B_1 \cup B_2 \\
\text{Then } A \ominus B &= (A \ominus B_1) \cap (A \ominus B_2) \\
\text{Let assume that if } B &= B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k \text{ for } k \geq 2 \\
\text{Then } A \ominus B &= (A \ominus B_1) \cap (A \ominus B_2) \cap (A \ominus B_3) \cap \cdots \cap (A \ominus B_k) \\
\text{Now we show that if } B &= B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k \cup B_{k+1} \\
\text{Then} & \\
A \ominus B &= A \ominus (B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k \cup B_{k+1}) \\
&= A \ominus [(B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k) \cup B_{k+1}] \\
&= [A \ominus (B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k)] \cap (A \ominus B_{k+1}) \\
&= (A \ominus B_1) \cap (A \ominus B_2) \cap (A \ominus B_3) \cap \cdots \\
&\cap \ (A \ominus B_k) \cap (A \ominus B_{k+1})
\end{aligned}
$$

■

Proposition (The Non Distributive Property of Dilation over Intersection) 4.3.7

$$(A \oplus B_1) \cap (A \oplus B_2) = A \oplus (B_1 \cup B_2) \tag{4.12}$$

Proof:

$$
\begin{aligned}
[(A \oplus B_1) \cap (A \oplus B_2)]^c &= (A \oplus B_1)^c \cup (A \oplus B_2)^c \\
&= (A^c \ominus \breve{B}_1) \cup (A^c \ominus \breve{B}_2) \\
&= A^c \ominus (\breve{B}_1 \cup \breve{B}_2) \\
\Longrightarrow (A \oplus B_1) \cap (A \oplus B_2) &= [A^c \ominus (\breve{B}_1 \cup \breve{B}_2)]^c \\
&= A \oplus (B_1 \cup B_2)
\end{aligned}
$$

■

It can be analyzed from the above that when taking the dilation of a set with 2 different structural elements before taking the intersection of the results after, then it is the same as taking the union of the structural elements and dilating with the set. However, since we took the union instead of the intersection shows the non distributive property of erosion over intersection. This leads to the implication that if you have to take the dilation of any set with a certain structural element, you can always partition the structural elements into 2 sets and take the intersection of their respective dilations. It can be generalized from the above that:
If any structural element can be partitioned into n distinct parts then the intersection of the each of the partitions' dilation with the set is equal to the set's dilation with the structural element.

Proposition 4.3.8

$$
\begin{aligned}
&\textit{If } B = B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_n \textit{ for } n \geq 2 \\
&\textit{Then } A \oplus B = (A \oplus B_1) \cap (A \oplus B_2) \cap (A \oplus B_3) \cap \cdots \cap (A \oplus B_n)
\end{aligned}
$$

Proof:

$$\begin{aligned}
\text{If } B &= B_1 \cup B_2 \\
\text{Then } A \oplus B &= (A \oplus B_1) \cap (A \oplus B_2) \\
\text{Let assume that if } B &= B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k \text{ for } k \geq 2 \\
\text{Then } A \oplus B &= (A \oplus B_1) \cap (A \oplus B_2) \cap (A \oplus B_3) \cap \cdots \cap (A \oplus B_k) \\
\text{Now we show that if } B &= B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k \cup B_{k+1} \\
\text{Then} & \\
A \oplus B &= A \oplus (B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k \cup B_{k+1}) \\
&= A \oplus [(B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k) \cup B_{k+1}] \\
&= [A \oplus (B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k)] \cap (A \oplus B_{k+1}) \\
&= (A \oplus B_1) \cap (A \oplus B_2) \cap (A \oplus B_3) \cap \cdots \\
&\cap \ (A \oplus B_k) \cap (A \oplus B_{k+1})
\end{aligned}$$

■

Chapter 5

CONCLUSIONS AND RECOMMENDATIONS

5.1 Introduction

In this chapter, the author brings to bear the summary of the findings made in the preceding chapter, followed by the conclusions and possible implications to mathematical morphology and recommendations for improvements and further studies in mathematical morphology.

5.2 Summary of findings

In trying to find what happens if the image or structural element in a morphological operation is partitioned, we find the following results:

1

$$(A_1 \oplus B) \cup (A_2 \oplus B) = (A_1 \cup A_2) \oplus B \tag{5.1}$$

2

$$(A_1 \oplus B) \cap (A_2 \oplus B) = (A_1 \cap A_2) \oplus B \tag{5.2}$$

3

$$(A_1 \ominus B) \cup (A_2 \ominus B) = (A_1 \cup A_2) \ominus B \tag{5.3}$$

4

$$(A_1 \ominus B) \cap (A_2 \ominus B) = (A_1 \cap A_2) \ominus B \tag{5.4}$$

5

$$(A \oplus B_1) \cup (A \oplus B_2) = A \oplus (B_1 \cup B_2) \tag{5.5}$$

6

$$(A \ominus B_1) \cup (A \ominus B_2) = A \ominus (B_1 \cup B_2) \tag{5.6}$$

7

$$(A \ominus B_1) \cap (A \ominus B_2) = A \ominus (B_1 \cup B_2) \tag{5.7}$$

8

$$(A \oplus B_1) \cap (A \oplus B_2) = A \oplus (B_1 \cup B_2) \tag{5.8}$$

9

$$(A_1 \circ B) \cup (A_2 \circ B) = (A_1 \cup A_2) \circ B \tag{5.9}$$

10

$$(A_1 \circ B) \cap (A_2 \circ B) = (A_1 \cap A_2) \circ B \tag{5.10}$$

11

$$(A_1 \bullet B) \cup (A_2 \bullet B) = (A_1 \cup A_2) \bullet B \tag{5.11}$$

10

$$(A_1 \bullet B) \cap (A_2 \bullet B) = (A_1 \cap A_2) \bullet B \tag{5.12}$$

5.3 Conclusion

We conclude our research with the fact that, the results above gives us a simplification of morphological operations when dealing with lots of set with the same structural element and vice versa. It also gives us ways of partitioning the structural element in order to carry out the morphological operation with ease and vice versa.

Furthermore, dilation and erosion distribute over set union but does not distribute over set intersection.

5.4 Recommendations

We recommend the following to readers and those in the field of mathematical morphology:

- To test out each of the results with specific examples since the results here were derived with mathematical implications only and no examples carried out.
- Also further research using other morphological operators should be carried out in order to find out the outcome if they behave the same way as the morphological operators used in this research.

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