KWAME NKRUMAH UNIVERSITY OF SCIENCE AND TECHNOLOGY,

KUMASI

CHAOS IN DYNAMICAL SYSTEMS

By

Samuel Effah-Poku

A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS,

KWAME NKRUMAH UNIVERSITY OF SCIENCE AND TECHNOLOGY IN

PARTIAL FUFILLMENT OF THE REQUIREMENT FOR THE DEGREE OF

M.PHIL PURE MATHEMATICS

October 20, 2016

Declaration

I hereby declare that this submission is my own work towards the award of the M. Phil degree and that, to the best of my knowledge, it contains no material previously published by another person or material which had been accepted for the award of any other degree of the university, except where due acknowledgement has been made in the text.

Effah-Poku Samuel(PG 2550314)	....	....
Student	Signature	Date

Certified by:

Prof. I. K. Dontwi	....	....
Supervisor	Signature	Date

Certified by:

Dr. R. K. Avuglah	....	....
Head of Department	Signature	Date

Dedication

I dedicate this work to my family especially my parents Mr and Mrs. Effah and to all my siblings. I love you all and God bless you.

Abstract

The behavior of dynamical system has become an interesting field of endeavor. Periodicity, fixed points and importantly chaos of systems have evolved as an integral part of mathematics and especially in dynamical system. We tend to consider asymptotic behavior of systems especially in the area of chaos. No universally accepted definition exist for chaos but we consider the various routes to chaos including transitivity, expansivity, topological entropy, Lyapunov exponent, dense orbits, period doubling , period three point and sensitive dependence to initial conditions. A combination of each of these guarantees a type of chaos. We study the various distinct routes to chaos and how various kinds of chaos are interrelated. Properties of an unknown map can be associated with that of the known via topological conjugacy, hence properties of unknown maps can always be studied in terms of the unknown. The tent map and logistic maps are two known chaotic maps. We explore how numerical values are used to determine chaos especially in terms of Lyapunov exponents with respect to known maps like the tent map and logistic maps.

’Chaos is when the present determines the future but the approximate present does not approximately determine the future.‘Edward Lorenz’.

Acknowledgements

I thank God for the gift of life, strength and the wisdom to come this far. Thank you Father for my every bit of grace and mercy in this period of my life.

I express my sincere and profound gratitude to Prof I. K. Dontwi, Director, Institute Of Distance Learning, KNUST, who is my supervisor, for his support, trust and the opportunity to work and learn from him.

I also want to appreciate Rev. Dr. W. Obeng-Denteh and Dr. Edward Prempeh, my fathers and mentors in the field of mathematics. I thank all lecturers in the mathematics department. You were always a source of motivation. I thank Mr Parker for his time and effort especially in the early stages of this work.

My sincere thanks goes to my senior course mates in Pure mathematics especially Richard Kwame Ansah for your support and direction. To my fellow Mphil Pure mathematics colleagues, thanks for all the time and motivation , the discussions and various contribution to this work. I say a big thank you whoever helped in producing the LATEX document especially Agyare and

Dorcas and to all friends who played diverse roles

I am grateful to Rev. Joshua Ahiaba, my Pastor and father and to all friends from CASA-KNUST

Lastly to my parents : Mr Stephen Effah and Mrs. Veronica Effah who have always supported me in my education, i am very much grateful.

Contents

List of Figures

Chapter 1

Introduction

This chapter offers a background of the study, which includes a history of chaos theory, the main contributors to chaos over the years and the various phases during which these contributions were made. Also included is the objective of study as well as the general structure of the entire the thesis.

1.1 Background Of The Study

Mathematics generally is considered under two main areas namely Pure mathematics and Applied mathematics. To the mathematician it's either you are dealing with the abstract nature or applying the knowledge to solve a problem in real life. Topology is considered as one of the main areas of Pure mathematics together with algebra and analysis. The areas in topology has received some great attention in recent years and this perhaps has led to the many contributions made in this area over the past few years. Topology in itself is believed to have been born out of geometry.

Science and Mathematics have always been interrelated. Perhaps this could be due the fact that almost every idea in science can actually be presented by mathematical expressions. Science helps to interpret nature whereas mathematics enables us to solve real life problems which are usually difficult to solve or deal with directly.

These expressions normally comes in the form equations and more often differential equations. This is usually done using the concept of modeling. In models, real life science are describe with purely mathematical language.

Most often, these are considered to be adequate and accurate such that solutions to the mathematics model implies the problem in science is solved.

Topology is basically the study of shapes and their corresponding properties. It was born out of a real life challenge some years ago somewhere in the eighteenth century. Once in Russia, a city called Koenigsberg, the river Pregel had overflown its banks and run through the city. There existed some seven bridges that connected the regions in this city. People wanted to find out the possibility of going through the city but crossing each bridge only once. Even Euler believed , it was impossible to walk across the bridges given their position.

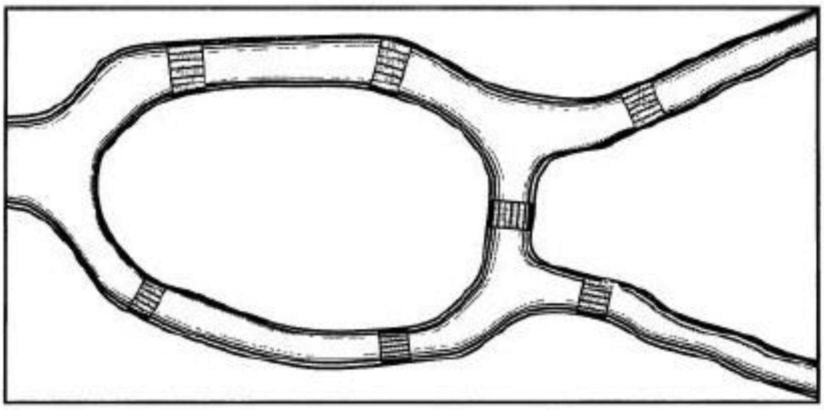

Figure 1.1: The Koenigsberg bridges(Adapted from an illustration in Newman,1983)

Mathematicians believed that, what is now being defined as chaos is not really anything new. It has been with us all these years and we only renaming it to make it look more mathematical. Perhaps they would agree with Henry Poncaire's quote that "Mathematics is the art of giving new names to old thing"

The first experience with what is now called chaos was with Henry Poincaire, the famous French mathematician in the early 1900's. Poncaire is considered as the last of the universalist (people who made major contributions to all major and known areas in mathematics) in mathematics. He studied what was called the three body problem (motion of the solar system) by Newton. In Poncaire's view, there is always a small cause (infinitesimally small) which we are normally not aware of, and irrespective of the fact that we overlook it ,has a big and noticeable effect which we cannot afford to overlook. Often times we attribute this cause to chance.

Most Mathematicians and Physicist like Newton and Laplace made indirect contribution to the field of chaos theory. The pair believed in same cause being equal to same effect. They pointed out the fact that there are always clear rules of life (cause and effect) and that brought about predictability and hence could always be controlled. They believed systems behaved nicely once we keep doing the same thing over and again expecting the same results. A great lesson from their perspective was that once systems could be controlled , then from the mathematical point of view the world is safe. Though Newton and his colleague believed in predictability, it had challenges in predicting systems like the weather. The orbits of the weather or solar system created a gap in what they believed. Basically, all they meant was that, given two bodies in motion from similar points, we should be able to trace on orbit using the other. In the height of all this, his desire was to see the three body problem solved. He discovered in his study that there are orbits of systems which in themselves are not periodic and yet never move closer or converges to any fixed point. Though Poncaire never gave out a solution to Newton's three body problem, he made a great contribution and remarks in that direction. His solution was considered as a partial solution to the problem and was still awarded for it perhaps because other legendary Mathematicians like Euler, Laplace, Lagrange and others could not help out. In Poincare's solution, he did approximate orbits in the form of series. He later realized, he had made a mistake and it was the genuineness of mind in admitting this error that gave rise to what will now be termed as chaos. He realized that little changes had more than just a little effect over time. His idea on chaos almost never fell through because people had lived with Newtonian science for long and maybe because other mathematics like Laplace, Libniz and others still believed in Newtonian science. For them, almost every system is linear and could be predicted to a point unknown and perhaps unseen.

Edward Lorenz from MIT is known and acknowledged as the father (modern) of chaos theory. He was a meteorologist and had so much interest in long term predictions of the weather. This happened during one of his routine computations trying to predict the weather. He did the same computations but with different input values. This was because he continued one of his after break sessions using input values from his computer. The difference in the values was small such that he thought was negligible and insignificant. Edward Lorenz describes chaos in these words "Chaos is when the present determines the future but the approximate present does not approximately determine the future.

He further explained that the unpredictability nature of the weather is because we can only measure the weather approximately

He only realized from his work graphically that though they have almost same starting points, the difference in their final points given the same number of iterations was so wide and unimaginable.

Edward Lorenz after careful consideration and scrutiny realized that the two initial input values differ by decimal points. The output from his machine had three decimal places compare to the six decimals of his original inputs. This small numerical difference has contributed to great difference in his computations. If his computer is not faulty then perhaps, there is something mathematicians are failing to acknowledge; a small change in input produces a very great difference in the end. And what happens next? Chaos has been born and the rest followed.

Chaos theory was discovered in 1963 though Lorenz had observed the phenomenon two years earlier. Quite a number of things could contribute to this irregularity which would later become a giant concept of study years on. Lorenz is believed to owe the idea but he is certainly not the first to associate the term chaos with the phenomenom under study. Alexander Lyapunov also made some contribution in the early stages. His was in the study of the instability of fluids and turbulence in fluids or gases.He tried to measure the transition from order to chaos. Other accademicians who made useful contributions to the course are: G.D Birkhoff, A.N Kolmogorov, M.L Cartwright, J.E Little and Stephen Smale among others. Stephen Smale is the only Mathematician and specifically Pure mathematician to have studied chaos. The others studied it in relation to Physics.

1.2 Useful Definitions and Theorems

1.2.1 Topological Spaces

Definition 1.2.1 *A topology T on a set X is the collection T of subsets of X having the following properties*

1. *$\emptyset$ and X are in T.*
2. *The arbitrary union of the elements of any subcollection of T is in T.*
3. *The finite intersection of the elements of any finite subcollection of T is in T*

Then (X, T) is a topological space

Note 1.2.1 *The empty set $(\emptyset)$ is always a subset of any collection of subset even if not mention.It is usually the first set when considering any power set(a set of all subsets a given set) of a set.*

Definition 1.2.2 *Trivial Topology*

Let $X = \{a, b, c\}$ and Define $T = \{\emptyset, (a, b, c)\}$

1. *$\emptyset \in T$*
2. *$\emptyset \cap (a, b, c) = \emptyset \in T$*
3. *$\emptyset \cup (a, b, c) = \emptyset \in T$*

Now this is a topology and the least or minimal topology we can define on this set T, hence it is a trivial topology on T. This topology sometimes is referred to as indiscrete topology.

Definition 1.2.3 *Discrete topology*

Let $X = \{a, b, c\}$ and Define $T = \{\emptyset, a, b, c, (a, b)(a, c), (b, c), (a, b, c)\}$

(X, T) *is a topology since it satisfies all three needed conditions.*

The empty set is contained in T. Also the intersection and union of any subset of T is a member of T. This is the largest topology that can be defined on the given set X and is called the discrete topology

Remark 1 *The set T is the power set of X which contains all possible subsets of X. Given two topologies T_1 and T_2, both defined on X, if T_1 contains T_2 then, T_1 is a finer topology than T_2. Also if T_2 contains T_1 then, T_2 is a finer topology than T_1.*

Hausdorff Space

Definition 1.2.4 *(Adams and Franzosa,2009) A topological space X is Hausdorff if for every pair of distinct points x and y in X, there exist disjoint neighborhoods U and V of x and y respectively.*

This implies that each point in a pair of points or cluster of points can be kept in its own circle once two points are not the same. Disjointed points ought to have disjointed neighborhoods.

Let $a \in U$, $b \in V$, $U \cap V = \emptyset$

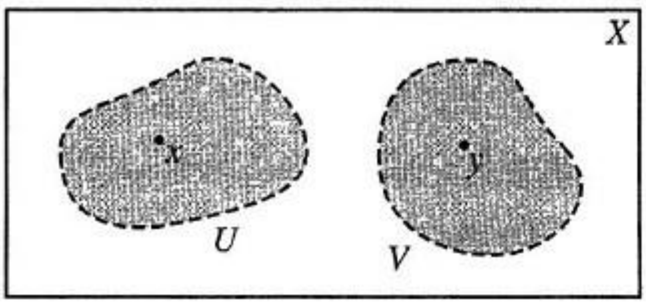

Figure 1.2: Hausdorff space

The real line R with the standard topology is Hausdorff. Give two distinct points a and b, there are disjoint open intervals containing them.It is called normal if it is Hausdorff and for any two closed $X_1, X_2 \subset X$ there exist

$O_1, O_2 \in T$ such that $X_i \subset O_i$ and $O_1 \cap O_2 = \emptyset$

An example is, if $a < b$, then the intervals

$U = (a-1, \frac{a+b}{2})$ and $V = (\frac{a+b}{2}, b+1)$ are disjoint and contain a and b, respectively

1.2.2 Dynamical Systems

A dynamical system on X is defined to be a mapping

$$\pi : X \times T \to X$$

where T is a topological group subject to the conditions

1. $\pi(X, 0) = X$ – identity
2. $\pi(\pi(x, t), s) = \pi(x, t + s)$ – Group property
3. π is continuous.

The triplet (X, T, π), where $\pi : X \times T \longrightarrow X$ continuous mapping satisfying the following conditions:

$\pi(0; x) = x \ (x \in X, 0 \in T)$,

$\pi(T, \pi(t, x)) = \pi(t + \tau, x) \ (x \in X, \tau \in T)$

are called a dynamical system. In that case if $T = RR_+ (RR)$ or $Z_+ (Z)$ then the system (X, T, π) is called a semigroup (group) dynamical system. If $T = R_+ (R)$, the dynamical system is called flow and if $T \in Z$ then (X, T, π)

is called cascade. (Cheban, 2009).

When we talk of a topological dynamical system (X, T) , we mean a compact metric space X together with a continuous map $T : X \longrightarrow X$

Definition 1.2.5 *Let X be a compact metric space and T a continuous map. A dynamical system (X, T) has sensitivity dependence on initial conditions if* $\exists \delta > 0$ *such that for* $x \in X$*, and each* $e > 0$*, there is a* $y \in X$ *with* $d(x, y) < e$ *and* $n \in N$ *such that* $d(T^n x, T^n y) > \delta$

Definition 1.2.6 *A dynamical system (X, T) is called Li-Yorke sensitive if* $\exists$ *some* $\delta > 0$ *such that for any* $x \in X$ *and* $\epsilon > 0$*, there is* $y \in X$*, satisfying* $d(x, y) < \varepsilon$ *such that*

$$\lim_{n\to\infty} \inf d(T^n x, T^n y) = 0 \quad \text{and} \quad \lim_{n\to\infty} \sup d(T^n x, T^n y) > \delta$$

Definition 1.2.7 *Let (X, T) be a dynamical system. A pair* $(x, y) \in X \times X$ *is called scrambled if*

$$\lim_{n\to\infty} \inf d(T^n x, T^n y) = 0 \quad \text{and} \quad \lim_{n\to\infty} \sup d(T^n x, T^n y) > 0$$

Definition 1.2.8 *Let (X, T) be a dynamical system. For a given positive number* $\delta > 0$*, a pair* $(x, y) \in X \times X$ *is called* δ*-scrambled if*

$$\lim_{n\to\infty} \inf d(T^n x, T^n y) = 0 \quad \text{and} \quad \lim_{n\to\infty} \sup d(T^n x, T^n y) > \delta$$

A subset C of X is called δ - scrambled if any two distinct points x and y in C form a δ - scrambled pair.

Definition 1.2.9 *Given a dynamical system (X, T), a pair of points (x, y) in X is*

1. *Asymptotic if*

$$\lim_{n\to\infty} d(T^n x, T^n y) = 0$$

2. *Proximal if*

$$\lim_{n\to\infty} \inf d(T^n x, T^n y) = 0$$

3. *Distal if*

$$\liminf_{n\to\infty} d(T^n x, T^n y) > 0$$

Note 1.2.2 *There exists a difference between transitive systems and minimal systems.*

Definition 1.2.10 *For a given dynamical system $f : S \to S$, $S \subseteq R$, the iterations of the function f is the composition of a function with itself.*

If $f^1(x)$, $f^2(x)$ represent the composition of the function with itself once and twice respectively. The k^{th} iteration of f at a point x represents the k times composition of f with itself. It's written as $f^k(x)$

Definition 1.2.11 *Periodic Points : Assume $m \in Z_+$, x is a periodic point or a period m-orbit if $f^m(x) = x$.*

Under the circumstance the porbit of x is called a periodic orbit or a period-m orbit. The set of all iterations of iterations of a periodic forms a periodic orbit. x is an eventual periodic point if x is not a periodic point but the $f^n(x)$ is a periodic point for some $n \in Z_+$

Definition 1.2.12 *The orbit of a point x in X is the set $Orb(X, T) = \{x, Tx, T^2x, \ldots\}$.*

The individual elements of the set $Orb(X, T)$ represent the path of the iteration for a given function. These represent the trajectory of the function or system

Lyapunov Exponent

Let f be a smooth map of the real line. The Lyapunov number L(x) of the orbit $\{x_1, x_2, x_3, x_4, \ldots\}$ is defined as follows:

$$L(x_1) = \lim_{n\to\infty}(|f'(x_1)| \ldots\ldots |f'(x_n)|)^{1/n}$$

if this limit exist.

The Lyapunov exponent $h(x_1)$ is defined as

$$L(x^1) = \lim_{n\to\infty} \frac{1}{n}[\ln|f'(x_1)|\ldots\ldots \_|f^0(x_n)|]$$

$$= \lim_{n-\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln |f^0(x_i)|$$

$$= \lim_{n-\to\infty} [\ln \frac{1}{n} \prod_{i=1}^{n} f^0(x_1)|],$$

if the limit exist.

1.2.3 Metric Spaces and Functions

Definition 1.2.13 *Given p, q ∈ X, X is a metric space if there exists a distance from p to q given as d(p, q) such that the following conditions are satisfied.*

i. d(p, q) ≥ 0 is p 6= q, d(p, p) = 0 ii. d(p, q) = d(q, p) iii. d(p, q) ≤ d(p, r) + d(r, q) for any r ∈ X

The metric is sometimes called the distance function. A metric is usually denoted *d*, and the set together with the metric is the metric space. It is usually written as (X, d) where *X* is the set and *d* is the metric.

In $\mathbf{R}^2$, $X = \mathbf{R} \times \mathbf{R}$, the metric is defined as $d(x_1, y_1), (x_2, y_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ where x_1, x_2, y_1 and y_2 are all points on the real line.

Definition 1.2.14 *A funtion $f : X -\to Y$ is defined to be injective if for each pair of distinct points of X, the image of X under Y are distinct.*

This implies that to every point in the domain there exist a distinct point in the domain there exist a distinct point int the range. The function is said to be one-to-one.

Definition 1.2.15 *A function $f : X \longrightarrow Y$ is said to be surjective if for every element in the image set has a corresponding element in the domain.*

Here, two distinct points could map to the same image. These maps are called the onto map. A map which is both one-to-one and onto is called a bijective map. In bijective maps, every distinct point in the domain maps unto an image in the range and every image in the image is mapped unto.

Definition 1.2.16 *Given functions $f : A \longrightarrow B$ and $g : C \longrightarrow D$ we define the composite function g of f and g as the function $gof : A \longrightarrow D$ and by the equation $gof(a) = g(f(a))$.*

Example 1.2.1 *Given the function $f : \mathbf{R} \longrightarrow \mathbf{R}$ defined by $2x^2 + 1$ and $g : \mathbf{R} \longrightarrow \mathbf{R}$ defined by $g(x) = 5x + 20$. The composite function*

i. fog is given by

$$\begin{aligned} fog(x) &= f(5x + 20) \\ &= 2(5x + 20)^2 + 1 \end{aligned}$$

ii. gof is given by

$$\begin{aligned} gof(x) &= g(2x^2 + 1) \\ &= 5(2x^2 + 1) + 20 \end{aligned}$$

The composite function *fog* 6= *gof*

Definition 1.2.17 *Let D be a set and T a function that maps D unto another set If $T(D) = D$,then D is an invariant set under T.*

We note that the set D is its own image and remains unchanged even after the function is applied to it. Now , any point in $T(D)$ is the image of at least a point

1.2.4 Set Theory

Definition 1.2.18 *A cantor set is a set of points lying on a single line segment that has remarkable and deep properties. It is usually built by removing the thirds of a given line segment.*

Example 1.2.2 *Given the line segment* $[0, 1]$*, find the cantor set.* $[0,$

$$1] \quad 2^0 = 1 \text{ segment}$$

$$[0, \frac{1}{3}] \cup [\frac{2}{3}, 1] \quad 2^1 = 2 \text{ segment}$$

$$[0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1] \quad 2^2 = 4 \text{ segment}$$

$$[0, \frac{1}{27}] \cup [\frac{2}{27}, \frac{1}{9}] \cup [\frac{2}{9}, \frac{7}{27}] \cup [\frac{8}{27}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{19}{27}] \cup [\frac{20}{27}, \frac{7}{9}] \cup [\frac{8}{9}, \frac{25}{27}] \cup [\frac{26}{27}, 1] \quad 2^3 = 8 \text{ segment}$$

It is generated by 2^n or the number of intervals at each point. The length of the interval (difference between any two points) is given by 3^{-n}. All cantor sets and intervals are closed and bounded (compact). It is sometimes called the middle-third or ternary cantor set. One property of the set could be that, it is totally disconnected.

Definition 1.2.19 *A limit point of a given set A is a number, l such that every deleted δ-neighbourhood of l contains elements of the set A. That is every neighbourhood of the point contains a point either than/apart from P. For every* $\delta > 0$, $\exists\, x \in A$ *s.t* $0 < |x - l| < \delta$.

Example 1.2.3 *Given the set interval* $(0, 1)$*, we want to verify if* 0 *and* 1 *are limit points .*

$$let\ \delta = 0.2$$

$$for\ 0 \Rightarrow a = 0\ a -$$

$$\delta < x < a + \delta$$

$$-0.2 < x < 0.2$$

$x \in (0, 1)$ *s.t x can be* 0.1, 0.15 *and x* 6= 0

$$for \quad 1$$

$$a - \delta < x < a + \delta$$

$$1 - 0.2 < x < 1 + 0.2\ 0.8 <$$

$$x < 1.2$$

$x \in (0, 1)$ *and x* 6= 1. *x can be 0.9*

Example 1.2.4 *Given the set* (1, 2, 3, 4, 5). *In finding any limit point we choose*

$$a = 2$$

$$let\ \delta = 0.2$$

$$2 - 0.2 < x < 2 + 0.2 \Rightarrow 1.8$$

$$< x < 2.2$$

If x 6= *a* → *x* = 2 *then x* ∈/ (1, 2, 3, 4, 5).

We conclude that every finite set has no limit point.

1.3 Objective Of Stdy

The objective of this study is to bring out the basics and fundamentals of chaos theory as a concept in mathematics. I intend to achieve these after everything:

1. To demonstrate the routes to chaos in a dynamical system

2. To asses the different kinds of chaos and their interrelationships
3. To study chaos in dynamical systems in terms of some well known maps
4. Contribute to a few open question on chaos theory in dynamical system

1.4 Scope And Limitation Of Study

1.4.1 Scope Of Study

This research will cover a selected collection of available ideas and works in the field under study. Seasoned textbooks on the subject matter was beneficial as well as the importance of valuable human resource currently working in the field on topology, dynamical systems and chaos theory. The knowledge and use of computer software, MATLAB, is relevant and cannot be overstated.

This work was produced using LATEX.

1.4.2 Limitation Of Study

The major setback and limitation so far as this work is concern is the unavailability of much needed resources in the form of useful textbooks. There is also the challenge of in-depth information on the subject matter and most importantly very few resource persons. There is the lack of motivation to further expand knowledge of the subject matter and so the work is only limited to the various works of researchers and academicians in the field.

1.5 Organization Of The Thesis

The content of this study is divided into the following chapters:

- Introduction
- Literature Review

- Exposition on Dynamical Systems
- Chaos Theory
- Contribution to open questions, Conclusion and Recommendation

Chapter 2

Literature Review

2.1 Overview

We consider various available literature on field of study. This was achieved through a summary of abstracts on topological dynamics and chaos theory which are of interest in this thesis.

2.2 Topological Dynamics And Dynamical Systems

Mallat defines a dynamical system as a concept in the field of mathematics where a fixed rule is used to describe the time dependence of a point in a geometrical space in (2009). Thompson(2013) describes how Dynamical systems can be studied from a distinct point of view of which one dominant area is topological dynamics. Topological dynamics deals with a space, a topology and a function acts on it. Such functions are usually continuous

For every dynamical system, there is a state space that represent the set of values for which iterations of the system is generated. At any given time the state space is given by a set of real numbers or possibly a vector. Every point or vector used can or should be possible to be represented by a point in an appropriate state space (Nguyen et al, 1989).

(Lin et al, 2011)The pair (X, f) can be used describes a dynamical system. Usually X contains many points possibly of infinite number. Sequences of continuous maps converge uniformly implying the limit map is continuous. Uniform limits exhibit topological transitivity and sensitivity dependence to initial conditons . Sensitive dependence is a global feature of a typical topological system

At a given point in time,it is always possible to predict at least one future state.This makes these rules in dynamical systems deterministic. The rule is deterministic (Ren and Zhang, 2009). The fixed reule that describes the the trajectory of future states from current state.(Ohtsuki et al, 2006)

Time considered in any dynamical system is either discrete or continuous. Dynamical systems is defined as deterministic model for evolving the state of a system forwarded in time. Usually systems are represented by maps which shows vividly variables changes over time. (Wang et al, 2011)

In Sharipov, (2001), he states the fact that Newtonian mechanics is the basis for. Detailed mechanisms of protein folding are not biased for dynamical systems. This the idea of the possibility of predictability from existing rule of evolution. Topological dynamics can be applied to a number of real life systems and more practically and importantly biological sytems((Hofbauar and Sigmund, 1988).

As soon as the system can be solved, given an preliminary point, it's viable to examine all its future point, The entire collection of the path of the iteration is referred to as a trajectory or orbit

Complex techniques in mathematica (Powell, 2007) which only were available to be used for quite a small class of dynamical systems were the option for solving dynamical system before the option of high speed computers were introduced. (Zimmermann et al, 2005).

The path of a dynamical system (trajectory or orbit) is relevant if only , that of individual systems could be obtained and comprehended. Its often difficult due to the nature of complexity of many dynamical systems.

Trajectories may be periodic whiles they move through different states of the system.(Sanz and Miret-Artes, 2008).The difficulties arise on account that the systems studied could handiest be known approximately, the parameters of the system is probably not identified exactly or terms may be missing from the equations. The approximations when used brings about questions of the validity or relevance of numerical solutions.

One significant thing that is needful for application is how trajectories which are functions of a parameter behaves. (Di et al, 2006) Bifurcation , the process where the system exhibits changes in qualitative behavior as a parameter is varied. It obtains bifurcation points. In a typical example, such systems could change from periodic behavior to a more unsteady or random behavior. The orbit of the system , as if random could display erratic nature.

Dynamical systems have different aspects which are all useful in various sciences. The probabilistic aspect of systems served as one the basis and foundation for statistical mechanics. It is also regarded in chaos theory. Poincare did a lot of work through which these dynamical systems themes developed (Araujo et al, 2008).

Yaacov (2008), defined a dynamical system as a continuous self map of a compact metric space. It was introduced that topological dynamics studies the iterations of the sort of periodic map. Its also considered to study equivalently the orbits of the points of the state. Basic properties and concepts in terms of dynamical systems include expansivity, equicontinuity, sensitivity. Most of these forms the foundation and backbone of the various concepts studied under topological dynamical systems

In Ruette (2015) she says Topological transitivity for transitive maps is quite similar to topological mixing. In the case of interval maps, weakly mixing , transitivity and topological mixing are equivalent. Transitivity guarentees sensitivity dependence and for interval maps, the converse is true. Maps with horseshoe have positive topological

entropy . Positive entropy and homoclinic points are equivalent properties. (Wang et al, 2011) An existence of uncountable number of scrambled sets is chaotic as by Li-Yorke's definition. One necessary condition though not sufficient for ergodicity is topological
transitivity.

2.3 Chaos

Small variations in preliminary conditions yield largely diverging results for dynamical system are considered as chaotic,making long-term prediction not possible on many occasions. This behaviour is often called deterministic chaos, or without difficulty chaos.

Edward Lorenz (Danforth, C.M, 2013) simplifies chaos in these short words. "When the present determines the future, but the approximate present does not approximately determine the future."
(Valle Jnr., 2000) Chaos theory was developed from the works of Edward Lorenz around 1960's. Sensitive dependence to initial condition is a core condition and feature of the theory of chaos in systems. Experimentally, minute and insignificant difference or perturbation has high and real great significance on future predictions. Chaos theory as an idea in non-linear mathematics is applicable in both social sciences and natural science.

Chaos is far from randomness but highly deterministic and predictable to a point. Complexities of systems are synonymous with chaotic behavior as well as non-linearity which gives rise to sensitive dependence to initial conditions. Chaos as an property is observed after a period of time which implies a system could possibly display non-chaos as usual in the initial stages of iteration though highly and easily chaotic after a few iteration

(Davis et al,1992) Chaos is defined for a continuous map on the same metric space which is usually not finite. Three distinquished properties are considered as components of chaos. Dense periodic orbits has an element of regularity, Transitivity , sensitive dependence to initial conditions deals with how negligible and insignificant errors in expeiremental values leads to larger and significant divergence. Systems with transitivity and dense orbits can be considered as being sensitive dependent to initial conditions.

(Aulbach et all, 2000) Distinct and several definitions describing chaos theory are designed for specific purpose. Usually, these definitions are based on different backgrounds and levels of complexity in mathematics. Though until now, there is no universally accepted definition to chaos, there is a possibility of one evolving it the near future.

(Bisiwas , 2013) Lyapunov exponent places a measure on the sensitivity dependence. Maps with positive Lyapuov exponents are usually considered chaotic. Typical examples include the tent map, logistic function and doubling map. These maps are usually topologically transitive. The box dimension , Hausdorrf dimension and entropy are also means for determining chaos in systems. One of the simplest maps that is chaotic is the logistic map. Variations of parameters in both the tent map and logistic function allows the maps to behave in several ways including predictability and chaos.

Predictable maps are usually stable while chaos describe on unpredictability.
Thompson(2013) says the chaotic nature of functions on the circle can be deduced through topological mixing. Chaos represents one of the interesting behavior of dynamical system and it shows movement of sets from their
existing position or location.

(Lin et al, 2011) There exist a direct relationship between Li-Yorke chaos and partial weak mixing such that, the latter implies the former but the converse does not hold. (Lim et al, 2008), There is always a convoluted structure produced during the chaotic mixing of fluids. This is produced by the interface that separates the fluids. Viscosity is a means that cuts off chaos in the interface.

Chapter 3

Dynamical System

3.1 Brief History

The interest in the be field of dynamical systems and particularly nonlinear dynamical systems over the last few decades has been very massive and significant so far as pure mathematics is concerned. Scientist in other areas had been able to apply approaches and techniques in this field to a number of relevant nonlinear problems starting from physics, chemistry, biology ,economics among others. In terms of modern dynamical systems and its ideas, there exist a relatively brief historical past. Considered as the main founding father of the field of dynamical systems,the French mathematician Henri Poincaré (1854-1912) revolutionized the be study of nonlinear differential equations by means of introducing the qualitative approaches of geometry and topology instead of analytic methods to discuss the general properties of solutions of these systems. For him as a mathematician, a world and international appreciation and acceptance of of the behavior of all solutions of the system was equally more important than just the solved analytically-precise solutions. Birkhorf continued the exploits of Poncaire in the first part of the twentieth century.Birkhorf came to realize the significance of the study of mappings and placed emphasis on discrete dynamics as a method of figuring out the more difficult continuous dynamics arising from differential equations. As times progressed, the subject of dynamical systems has benefited from a combination of interest and techniques as well as methods and applications from all sort of fields. Remarkable breakthroughs in fields mathematical biology and economics have motivated various scientists to this area of study

Computer images and graphics have proven that the dynamics of simple systems can be at once wonderful, appealing and an exciting adventure to embark on. Contemporary development have made the theory dynamical systems an appealing and predominant branch of mathematics to scientists in many disciplines. The idea have passed through different phases in time until this point. This ranges from the time of Newtonian mechanics to date. Quite a number of people have contributed to the field of dynamics beyond that which is mentioned here.

3.2 General Overview

A dynamical system is a way of describing the passage of time of all points of a given space. Simply it can be thought of as the repetition of events once and again. That is, You know exactly what you will be doing next. We consider anything that evolves over time (changes over time) as a dynamical system. Since life is full of changing events (non-constant events), life could be considered a dynamical system. In dynamical systems, the starting point, the journey along the line as well as the finishing points are all relevant and hence we pay attention to each of them as such.

One of the ways of describing the passage in time of points in a given space S is a dynamical system. The space S varies depending on the area of dynamics. Dynamical systems helps appreciate the relationships between mathematics and various aspect of science.

Example 3.2.1 *A man throws up an orange into air or space. This scene can be described as a system. Here we require to know the height to which it is thrown and the velocity of its movement. Let h be the height of the throw and let v be its velocity, then the vector* (h, v) *describes the system.*

Given an initial position $x \in R^n$, a dynamical system of R indicates where X is located 1unit of time later or before, 2 units of time later or before, etc. We can choose to represent our new position X_1 and X_2 etc for their corresponding time units. The trajectory of X is given by X_t. For a given dynamical system, the function that takes t to Xt results in either a sequence of points or a curve in R^n. This therefore shows the entire movement of X as t varies from zero to

infinity$(0, \infty)$, as t spans the positive real line.

Though in dynamical systems , various functions depends on time, the branches of dynamical systems present this in entirely different ways. These branches are Egordic theory, Topological dynamics , Differentiable dynamics, Hamiltonian dynamics. Basically in egordic theory, we assume that they preserve the measure on R^n. In topological dynamics, we assume the say X_t varies only continuously and for Differentiable systems, we assume the given system will be continuously differentiable. Every dynamical system can be considered or classified into two : discrete time dynamical system or continuous dynamical system.

A dynamical system is best describe in terms of these three words

Phase space

Time

Law of evolution

Phase space : it's a set whose elements (called 'points') present possible states of the systems at any moment of time. The phase space captures the various structures of dynamical system . The various aspect of dynamical systems are obtained based on these structures. They could either be differentiable, topological or considered measure preserving (ergodic).

Time : Time is expressed either as discrete when the values are integers whereas its considered continuous when the set of values are real numbers. Time considered here is either reversible or irreversible depending on its domain.

Law of Evolution : This is the rule that allows us to determine the state of a system at any moment given its current state.

3.3 Types Of Dynamical Systems

3.3.1 Discrete Dynamical System

Given the present current or present state of a system, we expect to be able to know the state of the system given a change in time. A Discrete Dynamical System is defined as as a sequence Xn with

$Xn + 1 = f(Xn)$ for some $f : R \longrightarrow R$

It's called iterative because there is always the possibility of obtaining a recurrence or pattern as we keep changing the values of the time used. The final formula then represents Xn and that describes the state of the system at the nth time.

We have a discrete dynamical system when time is a sequence of separate chucks, each of the next like beads on a string. In such cases one can really distinquish between the position of the bead infront from the bead behind without confusion or ambiguity. In discrete dynamical systems, usually preceeding states can be obtained depending on computations of the current state. It is always important to know where a system will be in the next instant. Also in discrete dynamical systems, there are intervals(big) between two distinct time intervals , hence we say discrete dynamical systems changes in cycles after the expected time periods.

3.3.2 Continuous Dynamical System

Basically they describe systems that changes over time with the understanding of discrete being its exact opposite. They can also be described as systems where time progresses smoothly. It usually involves the analysis of differential equation. In continuous systems, it is usually very difficult if not impossible to describe where the system would in a moment. Continuous systems are therefore mostly represented by differential equations. It is usually represented as $X^0 = f(x)$ which describes the rate at which the system changes with time. Such systems have almost absolute dependency on time hence the derivative X^0. Here our interest is how quickly the system changes with time.

Example 3.3.1 *An orange is thrown up in the air. It will be unfortunate to ask where the mango will be at the next instant, though we have every reason to know how the height and velocity of the mango changes with time.*

We can describe such system by a vector representation of its height or position and velocity or speed. Velocity here is simply the rate of change of position relative to time. As the mango falls back from up there (return to its starting point), it obtains a velocity against gravity.

Mathematically ,

$$X = [h, v]\ \frac{dh}{dv} = v \text{ and } \frac{dv}{dt} = -g$$

The solution of this system indicates the height and velocity of the mango at any time (t) One area or type of system where continuous systems really appears most is in chemical reactions. It is because it deals with reaction of several components and can normally be modeled as differential equation. We note that both discrete and continuous dynamical systems can appear beyond the one dimensional form. As stated earlier, there are various aspects of dynamical system as a result of the nature of their state space.

3.4 Principal Aspects Of Dynamical Systems

3.4.1 Ergodic Theory

Ergodic theory is a branch of mathematics which studies dynamical systems with invariant measures as well as related problems. Ergodic theory is defined as the study of group actions on measure spaces. The principles and foundation of this idea was founded on contributions from Boltzman and French Mathematician Poncaire. Boltzman was into statistical mechanics whereas Poncaire worked on celestial mechanics. Poncaire developed ways and methods to analyze solutions for a given differential equation. Most underlying concepts in ergodic theory has to do with actions that preserve a probability measure. The concept of ergodic theory helps determining if two measure preserving transformation are isomorphic. The phase space under consideration in ergodic theory are usually Lebesgue measurable spaces with a finite measure. Ergodic theory provides an appropriate tool for the studying of asymptotic distribution as well as behavior of orbits for differentiable dynamical systems.

It usually describe the behavior of a function as n becomes infinitely large.

T^n as $n \longrightarrow \infty$.

In Ergodic theory, we consider two basic preseving transformations ; the measure preserving transformation that works on measure spaces and the probability preserving transformation that works on the probability space.

Examples of Ergodic maps include the Bernoulli Shift

3.4.2 Topological Dynamics

Topological dynamics is considered as a branch of dynamical system which studies qualitative as well as asymptotic properties of dynamical systems. It's called topological dynamics because it is studied from the view point of topological spaces or topology. The phase space of a topological dynamics is a metric space. This metric space is usually compact

3.5 Other Concepts In Dynamical Systems

In a dynamical system, the nature of its orbit is always important and worth noting. There is always the tendency to have repetitions in the orbit. This is what is considered as periodicity.

3.5.1 Fixed Point

Consider the theorem below though without proof. Suppose $F : [a, b] \longrightarrow [a, b]$ is continuous. Then $\exists$ a fixed point for F in $[a, b]$.

The continuity of F and the fact that [a,b] is mapped unto itself must be kept as such. This theorem is obtained or generated or extracted from the intermediate value theorem which says: Once a function maps an interval [a,b] to the set of real numbers,$\exists y_0$ which is an image found between the images F(a) and F(b). Then the image y_0 has its domain in the interval under consideration such that

$F(x_0) = y_0$.

In using the fixed point theorem, there is always the guarantee of at least a fixed point for the function within the said interval.

It is usually worth noting that a closed interval for such functions is advisable. Let $F(x) = x^2$ $(a, b) = (0, \frac{1}{2})$ then there are no fixed points since the two points or extremes are not part of the set. Meanwhile $F(0) = 0$ which implies 0 would have been a fixed point if the interval were $[0, \frac{1}{2}]$. Even when there are fixed points, there is a way they behave and we take a look at that.

In analyzing attracting fixed point, we realize the orbits move closer to the attracting fixed point as the iterations are furthered. Irrespective of where you start, as you keep iterating, there is a point where the difference between the iterated or iterative value and the fixed point will be marginal. In the case of the repelling fixed points, the iterations continue. The points or orbits moves away and possibly very far from the said fixed point. The divergence or nature of growth of the orbit could be exponential or considered very fast.

3.5.2 Linear Map

They usually are the easiest and basic dynamical system for modeling especially in the context of population. It is quite easier to deal with and to have a clear long term view or perspective.

Let P_n represent the size of a population at a given time t.

Define P_{n+1} as $\quad P_{n+1} = aP_n \quad$ for $\quad a > 0$

We observe the difference in the behaviour of such models given a 'positive' initial population.

$$P_{n+1} = aP_n$$

$$P_1 = aP_0$$

$$P_2 = aP_1 = a_2P_0$$

$$\dots \quad \dots$$

$$P_n = a^nP_0 \text{ and } \quad n \geq 0$$

The long term behavior of the population is observed as follows:
Let

$$a > 1, \quad P_n \longrightarrow \infty \quad \text{as} \quad n \longrightarrow \infty$$

$$0 < a < 1, \quad P_n \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty$$

$$a = 1, \quad P_n \quad \text{remains unchanged}$$

Chapter 4

Chaos Theory

4.1 The Route To Chaos

4.1.1 Sensitivity Dependence To Initial conditions

Definition 4.1.1 *Let X be a compact metric space and T a continuous map. A dynamical system* (X, T) *has sensitivity dependence on initial conditions if* $\exists \delta > 0$ *such that for* $x \in X$*, and each* $e > 0$*, there is a* $y \in X$ *with* $d(x, y) < e$ *and* $n \in N$ *such that* $d(T^n x, T^n y) > \delta$

The characteristics of sensitivity is usually observed during the iterations.At the start of the iterations,(i.e $f^0 x$ and $f^0 y$) , the points x and y are a distance close. The idea of sensitivity dependence allows the orbits to be far apart as the number of iterations increases.

Let P_1 and P_2 be the two points just a distance apart. If we denote the orbits as $X(P_1)$ and $X(P_2)$ for P_1 and P_2 respectively. Continuing the iteration, the orbits move farther apart from each other, and the distance between the initial points. Usually, sensitivity dependence holds for large time value. The point is, you observe this phenomenon over time but at best over a large time limit. The respective trajectories diverge and this implies that the distance of the orbit increases compared to the preceding points. This idea of sensitivity dependence is otherwise called butterfly effect. This is perhaps due to any of these reasons or even more ; lost patterns and the great effects from marginal or negligible inputs like the flap of the butterfly wings. Generally this is experienced in non-linear science. The butterfly effect is one of the few ideas in mathematics that are referred to in the non-scientific world indirectly. Sensitivity to initial conditions is referred to as butterfly effect after a presentation by Edward

Lorenz, a mathematician and meteorologist from MIT in 1972 which hAd the title : Does the flap of a butterfly wings in Brazil set a Tornado in Texas ?

This looks like the same idea Poncaire was trying to bring across years earlier on. These were some words by Lorenz which says : "i started the computer again and went out for a cup of coffee. When i returned about an hour , after the computer had generated about two months of data, i found out that the new solution did not agree with the original one. I realized if the real atmosphere behaved in the same manner as the model, long range weather predictions would be impossible since most real weather elements were not measured accurately to three decimals places."(Lorenz 1991) Over the years , definitions of chaos or better still the basis of chaos has been built around this particular element of sensitive dependence. Practically and genuinely for the scientist , rounding off numbers shouldn't be a problem but you realize that over quite a long period, it's significance is great. Of course, for most systems (discrete) dynamical system, a final iteration say $f(X_n)$ could serve as the initial point for $f(X_{n+1})$. So for nearby iterations , the significance could be negligible sine the distance between their trajectories is small.

One thing worthy of noting is that if a system is sensitive dependence on initial conditions, our observation is does not necessarily start from the first iterate or better still we are not only interested and expecting a change in trajectory due to initial condition but that the slightest perturbation causes preceding values to differ from the expected.

The smallest error in change in initial condition grows to become as large as the true and actual value of the state. This makes prediction of future behavior impossible but this does not mean the system is not deterministic.

4.1.2 Topological Entropy

Andrei Nickolaevich Kolmogorov is believed to have introduced the idea of topological entropy.He is considered on of the main contributors to the concept of Dynamical systems after French Mathematician Poncaire.

Topological entropy is defined as the exponential growth rate of the nunber of different orbits(periodic orbits as n tends to infinity).

Entropy is one of the most important quantities in dynamical systems so far as numerical values are concern. It basically measures the rate of complexity of the dynamical system as time varies largely and towards infinity.

Definition 4.1.2 *Let $f : X \to X$ be contoinuos map of a compact metric space X. For $\epsilon > 0$ and $n \in Z^+$, we say $E \subset X$ is an (n, ϵ) - separated set if for every $x, y \in E$, there exist i, $0 \leq i \leq n$ such that $d(f^i(x), f^i(y)) > \epsilon$. The entropy of f is given by*

$$h_{top}(f) = \lim_{\epsilon \to 0}\{\limsup_{n \to \infty} \frac{1}{n} \log N(n, \epsilon)\}$$

and $N(n, \epsilon)$ represents the maximum cardinality of all (n, ϵ) - separated sets. We define a set as (n, e) - separated if for any x 6= y in E, $d_n(x, y) > e$

Topological entropy can also be defined and determined in terms of fixed points of f^n and expressed as

$$H(f) = \lim_{n\to\infty} \frac{1}{n} \ln(\text{number of fixed points under the map} f^n)$$

4.1.3 Lyapunov Exponents

In understanding chaos, one tool that is relevant in the is concept of Lyapunov exponents. Positive Lyapunov exponent implies sensitivity dependence to initial conditions. Lyapunov exponents measures the the rate of divergence of orbits away from each other(i.e it gives a means of quantifying the expansion or contraction of nearby trajectories). Generally, two orbits experience different routes and move further away from each other because there is a small change in one which could cause it to behave as such and thats due to sensitivity to initial conditions. Once a system has sensitive dependence to initial conditions, it must have a positive Lyapunov exponents.

Theorem 4.1.1 *(Fotiou A.,2005) Let f* : **R** −→ **R** *be continuous and differentiable map. If f has a positive Lyapunov exponents, then f has also sensitivity to initial conditions.*

Proof

Choose a point $x_0 \in \mathbf{R}$. Consider another point x_0^0 close to x_0. Let x_0 be δ away from $x00$

let

$f^n(x_0 + \delta) = f^n(x_0')$ and $f^n(x_0) = f^n(x_0)$ be the respective iterations for the initial points.

$$\delta X_n = |f^n(x_0 + \delta) - f^n(x_0) = |x_n' - x_n|$$

$$= \delta x_0 e^{n\lambda(x0)} = \delta........................(1)$$

δx_0 is the distnce between the two initial points given as $|x_0' - x_0|$ and $x_0 + \delta$

$$e_n^{\delta(x_0)} = \frac{\delta}{\delta(x_0)}$$

$$n = \frac{1}{\lambda(x_0)} log|\frac{\delta}{\delta x_0}|$$

For some δ for a given x_0 , then

$\exists x_0' \in N_\epsilon(x)$ such that $\forall e > 0$ after a number of iterations, $m > n$

$$|f^m(x_0') - f^m(x_0)| = \delta x_0 e^{m\lambda(x_0)} = \delta x_0 e^{(m-n)\lambda(x_0)} e^{n\lambda(x_0)} = x_0 e^{(m-n)\lambda(x_0)} \delta > \delta$$

This implies sensitivity dependence to initial conditions

4.1.4 Dense Orbits

Definition 4.1.3 *A dynamical system* (X, f) *has a dense orbit if and only if*

$\exists x \in X : \forall y \in X, \forall e > 0, \exists n \in \mathbf{N} : d(f^n(x), y) < e$

x and y represent the distinct initial points for the iteration. $f^n(x)$ represents a specific iteration. The orbits of $x(f^n(x)$ moves arbitrary close to another orbit at a given in time such that the metric between them is significantly small. As the iteration continues$(n \to \infty)$, the possibility of every other point experincing this is high.This implies that respective points as well as orbits of orther points are crowded within a given space such that the distance between points and iterations are less than e. i.e$(d(f^n(x), y) < e)$. At this point , the movement of orbits and points are difficult to distinquish between. Dense orbit for a map is always seen as the equivalent of Topological transitive.

In that the set of orbits moves close to every point

4.1.5 Transitivity

Definition 4.1.4 *A dynamical system* (X, f) *is topologically transitive if and only if for all non-empty subsets U and V of X, there exist* $n \in \mathbf{N}$ *such that* $F^n(U) \cap V$ 6= $\emptyset$*.*

Transitivity usually implies the existence of dense orbit. Topological transitivity guarantees that there always exist a point which results form the intersection of open sets under iterative process of map.

4.1.6 Expansivity

Definition 4.1.5 *Let $f : X \to X$. X is a metric space with a metric d defined on it. The map f is considered to be expansive if there exist a positive number $e > 0$ such that , for distinct $x, y \in X$, then $\exists n > 0$ such that $d(f^n(x), f^n(y)) \geq e$.*

e is called the expansive constant for the map.

There exist an obvious and more direct link between expansivity and sensitivity dependence on initial conditions. Every expansive map exhibits sensitivity dependence to initial conditions. The converse does not hold and the two conditions are never equivalent. Expansivity implies sensitive dependence because expansivity deals with the distance between between two nearby points and how their orbits separate continuously. In expansivity, the separation is observed between two nearby points by at least the constant e . In sensitive dependence the requirement is that, at least there there should be one point whose orbit moves away from the orbit of another close point after the same number of iteration.

Theorem 4.1.2 *Every expanding map, $f : \mathbf{R} \longrightarrow \mathbf{R}$ has sensitive dependence to initial conditions.*

Proof

A map is expanding if $|f'(x)| > 1 \qquad \forall x \in \mathbf{R}$.

At any given point $x \in \mathbf{R}$, the Lyapunov exponent is defined as

$$\lambda(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)| \qquad (4.1)$$

$$|f^0(x_i)| > 1 \qquad \forall x_i \in \mathbf{R}$$

Taking the ln of both sides

$$\ln |f^0(x_i)| > \ln |1| = 0 \;\; n-1$$

$$\sum_{i=0} \ln |f^0(x)| > 0 \tag{4.2}$$

Comparing equations (1) and (2),we divide the L. H. S. and R. H. S. of equation (2) by n. Evaluate the new equation as $n \longrightarrow \infty$ and that gives the Lyapunov exponent and is positive, hence f is sensitive to initial conditions.

4.1.7 Period Three

Definition 4.1.6 *Let (X,f) be a dynamical system and be defined by the map. The map f is said to have a periodic point if for n > 0, $f^n(x) = x$.*

For a given map, since n is a natural number. The map is said to have periodic point of period three when $f^3(x) = x$. Period three is normally associated with chaos of dynamical systems and was first proved by Tien-Yien Li and James A. York in 1975.The theorem below is relevant to the relation of period three and chaos.

Example 4.1.1 *(Fotiou A.,2005))*

$f : [-1, 1] \to [-1, 1], f(x) = 2|x| - 1$ *Let* $x_0 = \frac{-7}{9}$

$$f(x_0) = f(\frac{-7}{9}) = \frac{5}{9}$$

$$x_1 = \frac{5}{9}, f^2(x_0) = f(x_1) = f(\frac{5}{9}) = \frac{1}{9}$$

$$x_2 = \frac{1}{9}, f^3(x_0) = f(x_2) = f(\frac{1}{9}) = \frac{-7}{9}$$

Since

$$f^3(x_0) = (x_0)$$

then the map has period three point.

Example 4.1.2 *(Kulkarni P.R.,Borkar V.C,2015)*

Given

The given map is defined

$$T_2(x) = \begin{cases} 2x & 0 \le x \le \frac{1}{2} \\ 2(1-x) & \frac{1}{2} \le x \le 1 \end{cases}$$

on the interval [0,1]

We want to show that the tent function has period three cycles

Let $x_0 = \frac{2}{7}$

$$T_2(x_0) = T\left(\frac{2}{7}\right) = \frac{4}{7},\ x_1 = \frac{4}{7}$$

$$T_2^2(x_0) = T_2(x_1) = T\left(\frac{4}{7}\right) = \frac{6}{7},\ x_2 = \frac{6}{7}$$

$$T_2^3(x_0) = T_2(x_2) = T\left(\frac{6}{7}\right) = \frac{2}{7},\ x_3 = \frac{2}{7}$$

Since

$$T_2^3(x_0) = (x_0)$$

then the tent function has a period three cycle

Theorem 4.1.3 *Period three theorem*

Let $f : R \to R$ be a continuous function . If f has periodic points of period three then f has periodic point of all other periods.

Sarkovski generalizes this theorem and brings out which period directly implies other periods. Its important to understand his way of ordering of the number system so as to appreciate his contribution.

3, 5, 7, 9, ...(List of all odd numbers except 1)

2.3, 2.5, 2.7, ...(two times the odd naturals)

Continuing the process will yield multiples of two times the odd naturals and this exhaust all the natural numbers.

Theorem 4.1.4 *(Sarkovskii theorem)(Kulkarni P.R.,Borkar V.C,(2015)) If a continuous function has $f : R \rightarrow R$ has a point of period n. Where n preceeds k in the Sarkovskii ordering of natural numbers, then f has a periodic point of period k*

From the Sarkovskii ordering, three comes before all the odd listing. Hence a system with period three will have all other periods.

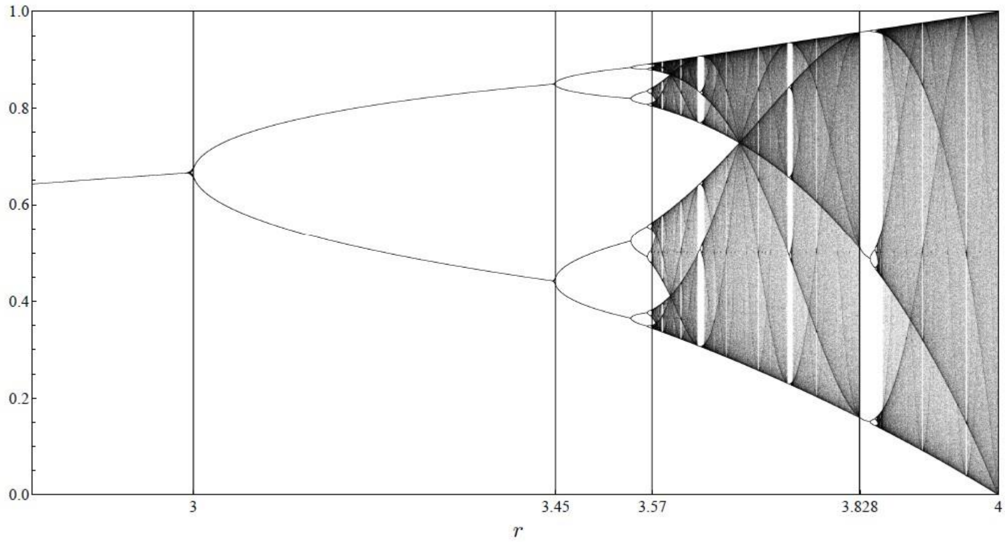

Figure 4.1: Period three window and chaos

We observe the period three window just at the point r = 3.828. From the figure, before that space no patterns could be detected form the diagram. The function is chaotic. Hence the conclusion period three implies chaos.

4.1.8 Bifurcation And Period Doubling

Bifurcation is defined as the changes in the structure of a dynamical system as a result of the changes in the parameter value. This changes is usually sudden and could be topological or qualitative. In such cases , we expect the dynamical system to be a function of of both the dependent variable as well as as the parameter in context. An example is $x^0 f(x, \mu)$

The idea of bifurcation can be grouped into two , which are global bifurcation and local bifurcation.In local bifurcation , the interest is is the changes that happens in the system near the fixed point.It is usually analysed through changes in stability properties, periodic orbits. Global bifurcation occurs when larger invariant sets of the system collide with each other.

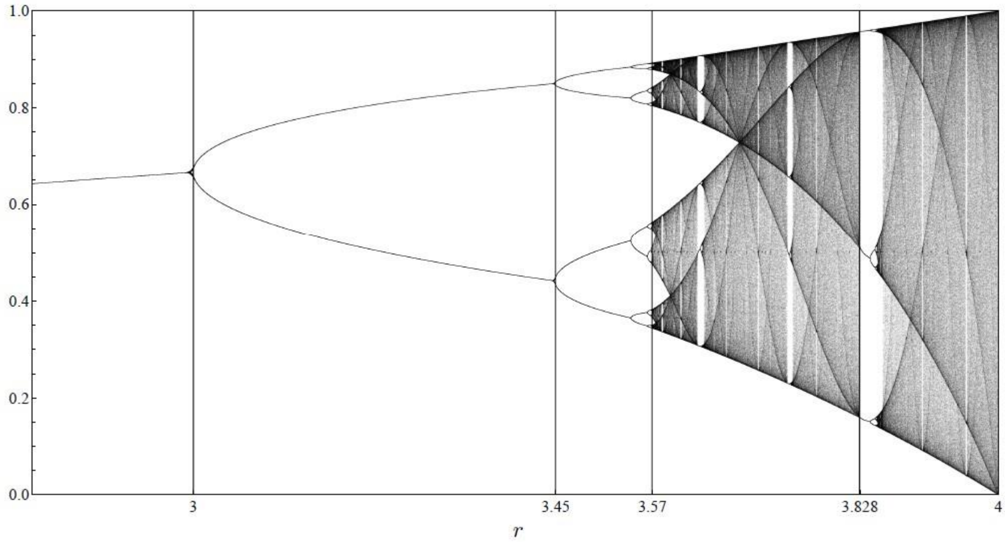

Figure 4.2: Diagram of Period Doubling

From the figure, Period doubling occurs at the point $r = 3$, $r = 3.45$, $r = 3.57$, though some are more visible than others. Period doubling is the splitting of a trajectory into two during iteration. This unusual scene is influenced by the parameter value most of the time. Chaos is observed as the period doubling increases. This is because , the path of trajectory at certain points are mixed up and inseparable making the system chaotic.

4.1.9 Topological Mixing

Definition 4.1.7 *A dynamical systems is topologically mixing if for each pair of open sets* $A, B \subset I$, *there exists* $N_0 > 0$ *such that* $n > N_0$ *implies* $f_n(A) \cap B$ 6= ∅.

A mixing map is obviously topologically transitive but the converse is not true. The mixing property is a strong topological property and can be considered in relation to chaos as a route.

4.2 Types Of Chaos

4.2.1 Li Yorke Chaos

Definition 4.2.1 *(Li,2015) Let* $x, y \in X$*. The pair* $(x, y) \in (X, X)$ *is a Li-Yorke scrambled pair if*

$$\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$$

and

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$$

A map is Li - Yorke chaotic if it has uncountable scrambled set in X.

(x, y) is proximal but not assymptotic and that implies its Li-Yorke sensitive. Since d is a metric imposed on the iterates as n varies through to infinity, we could simply consider the distance between the iterate at some n.

$$d(f^n(x), f^n(y)) = |(f^n(x) - f^n(y)|$$

Ideally and generally, x 6= y(since they are two distinct points in X) are distinct points.The path of the trajectory for the two points irrespective of how close they may to each other at the beginning grows to a positive non-zero value.The closest distance at any point in the iteration is very small and equivalent to zero.

A map with points of discontinuity are usually not Li-Yorke chaotic. Also this type of chaos is seen on subset of the real line(intervals)

4.2.2 Devaney Chaos

Definition 4.2.2 *(J. Banks ,J. Brooks, G Cairns, G. Davis and P. Stacey, 1992) Let X be a metric space. A continuos map f : X −→ X is said to chaostic if*

1. *f is transitive*
2. *the periodic orbit of f are dense in X*
3. *f has sensitivity dependence on initial conditions.*

This definition of chaos is one of the widely known and accepted definitions in chaos theory. The foundation and basis of the definition is in the systems dependence on initial conditions. Later it was discovered that , once a system is transitive and has periodic orbit being dense, then sensitivity is assured. This makes the foundation of the defintion redundant.

The property of transitivity and periodic orbits being dense in the set *X*, are invariant under topological conjugation. It points out the fact that the two are topological properties. A property is considered topological and preserved under topological conjugarion only if the space *X* is defined in terms of topology and as a compact space. Sensitivity to initial conditions is usually defined in terms of metric spaces and is therefore non-invariant under topological conjugation.

4.2.3 Wiggins Chaos

Definition 4.2.3 *(Fotiou A.,2005) Let f : X −→ X be continuos map and X a metric space.*

The map is considered to be choatic if

1. *f is topologically transitive*
2. *f has sensitivity dependence on initial conditions.*

4.2.4 Lyapunov Definition Of Chaos

Definition 4.2.4 *(Fotiou A.,2005) Let* $f : \mathbf{R} \longrightarrow \mathbf{R}$ *be a continuous differentiable map. The map f is said to be chaotic if*

1. *f is topological transitive*
2. *f has positive Lyapunov exponent.*

4.2.5 Knudsen Chaotic System

Definition 4.2.5 *Let* $F : X \longrightarrow X$ *be a continuous map on a metric space* (x, d)*, then the dynamical system* $< X, F >$ *is chaotic according to Knudsen's definition iff*

1. *F has dense orbits*
2. *F is sensitive to initial conditions*

4.2.6 Positive Expansive Chaotic System

Definition 4.2.6 *Let* $F : X \longrightarrow X$ *be a continuous map on a perfect metric space* (x, d).

The dynamical system is positively expansive chaotic (E-chaotic) iff

1. *F is topologically transitive.*
2. *F has dense periodic orbits.*
3. *F is positively expansive.*

Example of E-chaotic maps include the one-sided shift dynamical system on the finite alphabet.

The two-sided shift dynamical system is one of the scenarios when a map is Devaney chaotic but not Expansive chaotic.

4.3 Interelation Of The Various Types Of Chaos

Most characteristics and properties can be inherited in the dynamical systems. Some of these properties are ; the existence of periodic orbits and transitivity. It is inherited via topological conjugacy. This helps in the study of unknown maps such that properties of the known maps can be associated with that of the unknown.Most topological properties are invariant under topological conjugacy in that these properties are preserved. Examples include topological transitivity, dense periodic orbit,Positive Lyapunov exponents. We note that , if via a given mapping say h, g and f are topologically conjugates, then if q is a fixed point of g , then $h(q)$ shall be a fixed point of f

Definition 4.3.1 *Let $p : X \to X$ and $q : Y \to Y$ be two mappings. p is topologically conjugate to q if there exist a homeomorphism $r : Y \to Y$ such that $r \circ p = q \circ r$.*

Now, because of the homeomorphism r, we say that if is topologically conjugate to q, then the converse is true. We describe the homeomorphism r as a topological conjugacy between p and q. In other words, p and q are conjugate via the given mapping r. Below are examples of conjugacy

Example 4.3.1 *(Goodson, G. R. , 2015) Define $f_a : \mathbf{R} \longrightarrow \mathbf{R}$ by $f_a(x) = ax$, for $a \in \mathbf{R}$.*

$$\text{If } h(x) = x^{\frac{1}{3}},\ h(f_8(x)) = h(8x) = (8x)^{\frac{1}{3}} = 2x^{\frac{1}{3}}\quad f_2(h(x)) = f_2(x^{\frac{1}{3}}) = 2x^{\frac{1}{3}}$$

$$h(f_8(x)) = f_2(h(x))$$

f_2 and f_8 are conjugates since h is a homeomorphism.

Since the mappings in all two examples are conjugates topologically, they share same topological properties. Hence if any of the mappings is identified with any topological propety, we can as well associate the other mapping with that same property and vice versa.

Note 4.3.1 *The tent function and the logistic function are topological conjugates via the homeomorphism*

$$h(x) = sin^2(\frac{\pi x}{2})$$

Now we have a look at the direct implications and equivalence in the various types of chaos:

Expansive chaotic systems implies Lyapunov chaos. Expansivity implies positive Lyapunov exponents and hence sensitive dependence to initial conditions. In expansivity, there is a constant moving apart of nearby orbits as the number of iterations increases. For a system to have positive Lyapunov exponent , there should exist at least just a point where nearby orbits move apart. Since the converse is not true , then positive Lyapunov exponent does not guarentee expansivity. Of course , they share a common component of
transitivity.

Devaney chaos implies Wiggins chaos and Knudsen chaos and Lyapunov chaos. Sensitivity to initial conditions is a common component in all three chaos mentioned above whiles its equivalent of positve Lyapunov is used in Lyapunov's definition. Now,Devaney chaos combines both topological transitivity and the existence of dense orbits . Since both Wiggins chaos, Lyapunov and Knudsen chaos depends on either of the two topological conditions , then Devaney chaos implies them all. A system that is Devaney chaotic has to be Wiggens chaotic, Lyapunov chaotic and knudsen chaotic.

Positive expansive implies Wiggins chaos, Knudsen chaos and Lyapunov chaos. Wiggin's definition, Knudsen chaos and Lyapunov's definitions satisfies either the condition of transitivity or dense orbits.

In relation to Wiggins chaos, their distinct conditions are positive expansiveness and sensitive dependence to initial conditions. Expansivity implies sensitive dependence hence Positive expansive chaos implies Wiggin's chaos.

In the case of Knudsen chaos,Beyond the existence of dense orbit, the distinct condition is also sensitivity. Similarly, since expansivity implies sensitive dependence , expansive chaos implies Knudsen chaos. Also for positive Lyapunov exponents , the distinct condition is expansivity and positive Lyapunov exponents. Every expansive map has a positive Lyapunov exponent but the converse is not true.As said earlier, if just two orbits separate apart at a point in the iteration , the Lyapunov will be positive but the map might not necessarily be expansive.

Wiggins chaos and Knudsen chaos imply each another and are quite similar or equivalent. They share a common property of transitivity. They also share an equivalent property of sensitivity and positive Lyapunov exponents. The difference in the two definitions is the space on which it is defined. Wiggins considers a a continuous map on a metric whiles Lyapunov deals with differentiable maps

Knudsen chaos shares an equivalent property of sensitivity dependence to initial condition and positive Lyapunov exponents with Wiggins definition and Lyapunovs definition. In some systems, transitivity is equivalent to dense orbits though not always. In such systems , all three chaos are the same apart from the space on which each is defined.

Devaney chaos and Li York chaos are interrelated via topological entropy.

Deveney chaos implies positive topological entropy and the converse is not true. Positve toplogical entropy implies Li Yorke chaos and here too the converse does not hold. According to the law of transitvity in analysis Deveney chaos implis Li Yorke chaos. On the interval map Deveney chaos is the strongest whereas Li Yorkes chaos is the weakest.

Maps with discontinuity are usually not chaotic.

From above , Devaney chaos implies Wiggins chaos, Lyapunov chaos and Knudsen chaos. Positive Expansive chaos iplies Wiggins chaos, Lyapunov chaos and Knudsen chaos. Possibly then, there should be a link between the two.The two defintions share two common property of topological transitivity and dense periodic orbit. Positive expansivity is a stronger property than sensitive dependence to initial conditions. since positive expansivity implies sensitive dependence, the Positve expansive chaos implies Deveney chaos.

4.4 Chaos Of Some Maps

Here and at this point we try to understand chaos in terms of a few known maps using the various routes like transitivity and positve Lyapunov exponent for sensitive dependence. Perhaps the idea of conjugacy would be relevant and graphical illustration where possible.

4.4.1 The Tent map

Given

The given map is defined $T(x) = \begin{cases} 2x & 0 \le x \le \frac{1}{2} \\ 2(1-x) & \frac{1}{2} \le x \le 1 \end{cases}$ on the interval [0,1]

$$h(x) = \lim_{x \to \infty} \frac{1}{k} \sum_{i=1}^{k} \ln |f^0(x_1)|$$

On the interval is the piece continuous function between 0 and $\frac{1}{2}$, $T^0(x) = 2$

Likewise on the interval between $\frac{1}{2}$ and 1, $T^0(x) = -2$

Now $|T^0(x)| = 2$ in both cases.

Since

$$h(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln |T^0(x_i)|$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln 2$$

$$= \ln 2$$

The tent map has positve Lyapunov exponent.**(Hena Rani Biswas, 2013)**

Consider the tent map $T : [0, 1] \longrightarrow [0, 1]$ given by

$$T(x) = \begin{cases} 2x & 0 \le x \le 0.5 \\ 2 - 2x & 0.5 \le x \le 1 \end{cases}$$

First I will prove that $T(x)$ is transitive. So we choose a positive number d such that $0 < d < \frac{1}{2}$ and the compact interval $I = \left[\frac{1}{2}d, d\right]$. Then $\exists\ k \in \mathbf{N}$ such that $2^{k-1}d < \frac{1}{2} < 2^k d$. The k-th iteration of $T(x)$ gives $T^k(I) = [2^{k-1}d, 2^k d$ and the k+1-th iteration gives $T^{k+1}(I) = T\left(\left[2^{k-1}d, \frac{1}{2}\right] \cup \left[\frac{1}{2}, 2^k d\right]\right) = T\left(\left[2^{k-1}d, \frac{1}{2}\right]\right) \cup T\left(\frac{1}{2}, 2^k d\right] = [2^k d, 1] \cup [2 - 2^{k+1}d, 1)$

Continuing in the same way the k+2-th iteration of T is $T^{k+2}(I) = [0, 2(1 - 2^k d)] \cup T([2 - 2^{k+1}d, 1)) \Rightarrow 1 - 2^k d < \frac{1}{2} \Rightarrow \exists$ $m > 0$ such that $2^m(1 - 2^k d) > \frac{1}{2} \Rightarrow \left[0, \frac{1}{2}\right] \subset [0, 2^m(1 - 2^k d)] \subset T^m([0, 2^m(1 - 2^k d)]) \subset T^{k+m+2}(I)$

So $\exists\ k \in \mathbf{N}$ for every subinterval J of $[0, 1]$ at which $T^k(I) \cap J \neq \emptyset \Rightarrow T$ is transitive and has dense periodic points. We can conclude that the tent map is chaotic at least under Devaney chaos since its also sensitive to initial conditions.

At this point we try to illustrate the idea of chaos in some well known maps(the tent map) using graphical analysis.This will be done using different iterates of the tent function. consider the iterations of the tent function as n gets larger (approaches infinity), the pattern in the diagram is lost gradually and it gets worse and worse.This is observed very clearly up to the seventh iteration. We notice the interval in the diagram reducing as the iteration furthered on. In iteration eight and nine, the behavior of the orbits seems to have changed entirely from the one we could predict as the half of the the previous iteration. At this point the regular periodicity is getting lost and chaos is eminent. Chaos just like its routes are experienced over time.

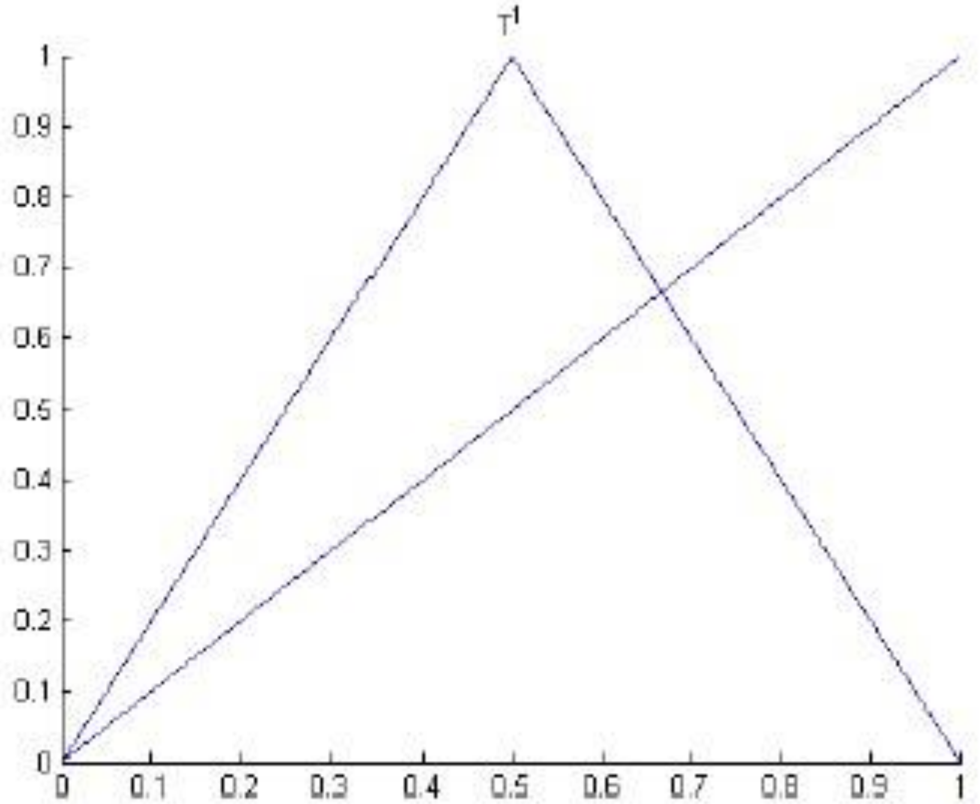

Figure 4.3: The Tent function

From the iterations below there is an observed pattern. The preceding figure for the iteration is obtained by dividing the existing figure into two. This implies the number of fixed points increases as well. The fixed point are the points of intersection of the line $y = x$ with the orbits of the diagram

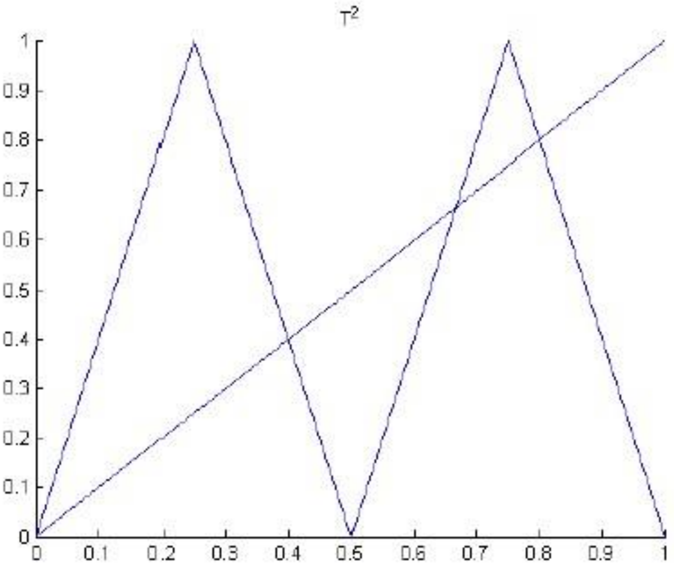

Figure 4.4: Second iteration of the tent function

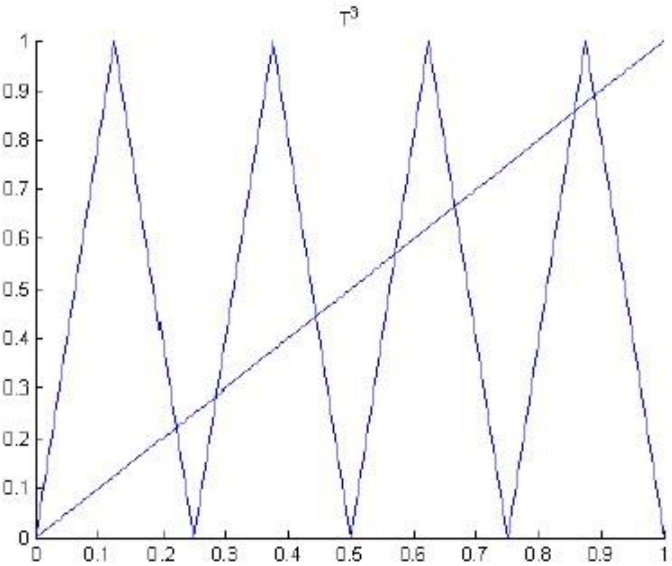

Figure 4.5: Third iteration of the tent function

This behavior continues clearly through from iteration one to seven

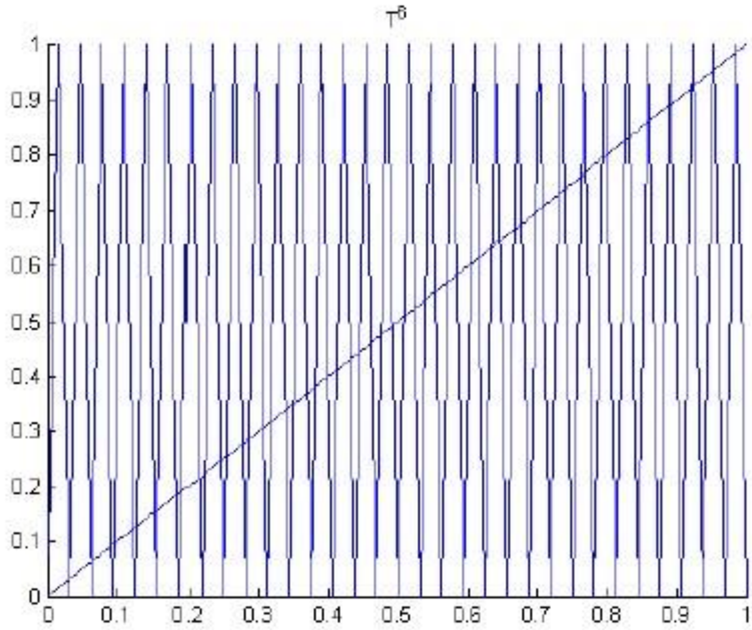

Figure 4.6: Sixth iteration of the tent function

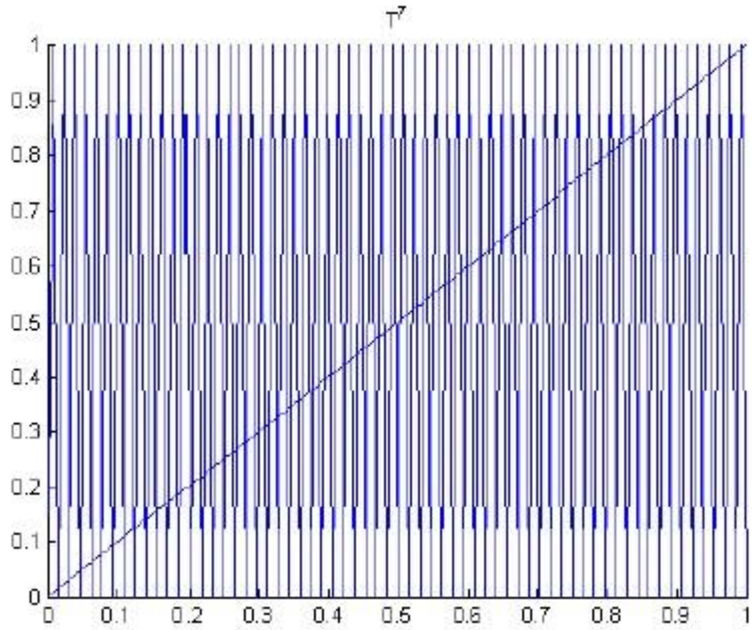

Figure 4.7: Seventh iteration of the tent function

Both iteration eight and nine behave in a same pattern. The movement of their orbits has changed . The regular periodicity is lost. The routes to chaos at this point is becoming obvious and visible.

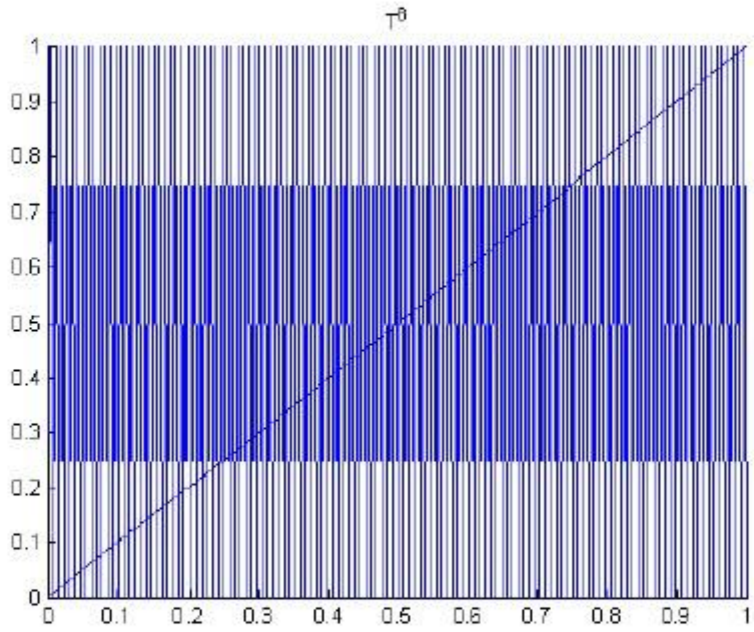

Figure 4.8: Eighth iteration of the tent function

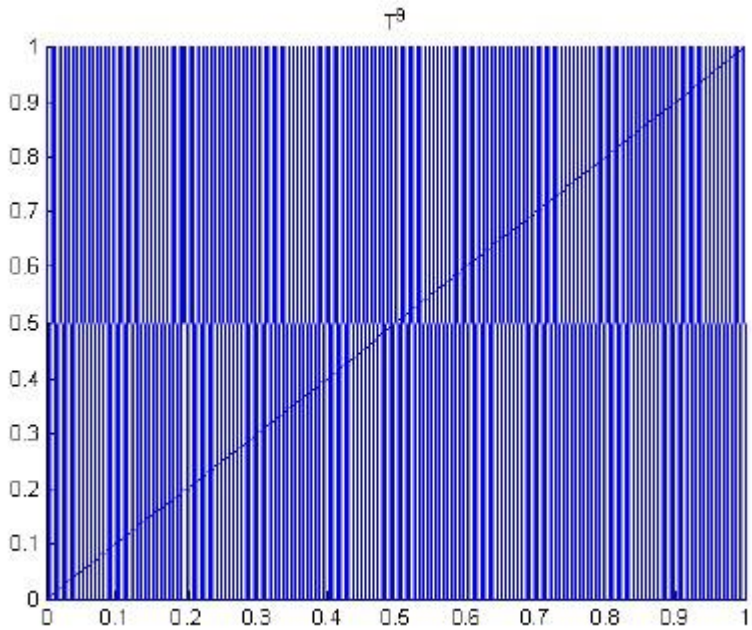

Figure 4.9: Ninth iteration of the tent function

Graphically , the Tent map is observed to be chaotic as the iterations increases. This is typical of almost every chaotic map such that the phenomenon is observed for considerably long iterations depending on the nature of the map. Here we observed lost patterns . The various trajectories or orbits cannot be distinguished. Periodic orbits have become so dense. Hence chaos is in motion and the tent map is therefore considered chaotic.

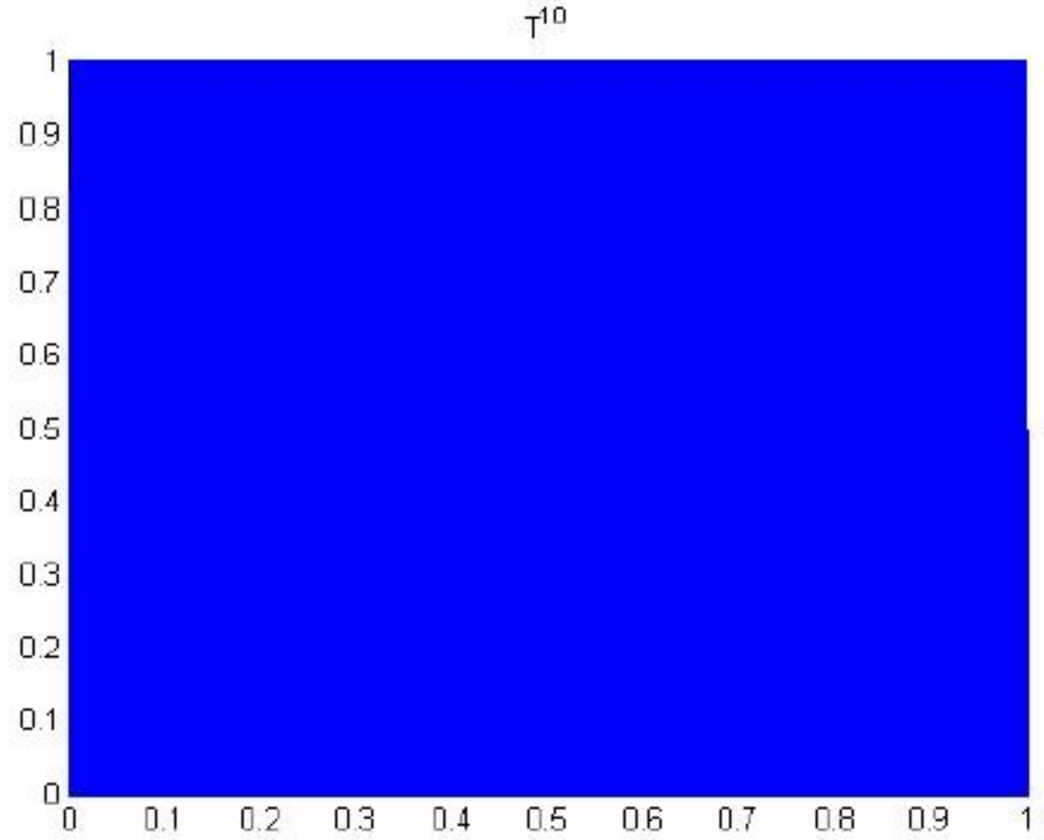

Figure 4.10: Chaos of The Tent Map

Chapter 5

Conclusion and Recommendation

5.1 Contribution To Open Question

Here I intend to contribute to one of the questions in one of the papers I came across in my work.

Are all Li Yorke sensitive systems Li-Yorke chaotic ?(Li, 2015)

Sensitivity is considered a strong route to chaos. It is usually considered as an equivalence of positive Lyapunov exponents such that a system ought to be sensitve dependent on initial conditions once it has a positive Lyapunov exponent. Of course , the converse is always true and holds for all systems. Li-Yorke chaos is a type of chaos defined by James A. Yorke and T. Y. Li. Their definition of chaos was based on the existence of scrambled pairs in a dynamical system.

Definition 5.1.1 *Let X be a compact metric space and T a continuous map. A dynamical system* (X, T) *has sensitivity dependence on initial conditions if* $\exists \delta > 0$ *such that for x* $\in X$*, and each* $e > 0$*, there is a* $y \in X$ *with* $d(x, y) < e$ *and* $n \in N$ *such that* $d(T^n x, T^n y) > \delta$

Definition 5.1.2 *A dynamical system* (X, T) *is called Li-Yorke sensitive if* $\exists$ *some* $\delta > 0$ *such that for any* $x \in X$ *and* $\in > 0$*, there is* $y \in X$*, satisfying* $d(x, y) < \varepsilon$ *such that*

$$\liminf_{n\to\infty} d(T^n x, T^n y) = 0 \quad \textit{and} \quad \limsup_{n\to\infty} d(T^n x, T^n y) > \delta$$

Now Li Yorke sensitivity is a form of the general sensitivity dependence to initial condition which is a route to chaos. Its a specialized or improved from of sensitivity such that the conditions

$$\liminf_{n\to\infty} d(T^n x, T^n y) = 0 \quad \text{and} \quad \limsup_{n\to\infty} d(T^n x, T^n y) > \delta$$

must be satisfied.

Definition 5.1.3 *Let* $x, y \in X$*. The pair* $(x, y) \in (X, X)$ *is a Li-Yorke scrambled pair if*

$$\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$$

and

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$$

A system is considered Li-Yorke chaotic if it has uncountable scramble pair.

Li yorke chaos deals with the existence of uncountable scrambled pairs. For both Li Yorke sensitivity and Li Yorke chaos,

$$\liminf_{n\to\infty} d(T^n x, T^n y) = 0$$

In Li Yorke chaos

$$\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$$

whiles for a Li Yorke sensitive system, ∃ some $\delta > 0$ such that for any $x \in X$ and $\in > 0$, there is $y \in X$, satisfying $d(x, y) < \varepsilon$ such that

$$\limsup_{n\to\infty} d(T^n x, T^n y) > \delta$$

In Li-Yorke sensitivity, the metric $d(x, y) < e$ must be met which implies that the two arbitrary starting points should be a distance less than e close.

Next a positive δ is essential in Li-Yorke sensitivity in the sense that $\exists\delta > 0$ such that $x \in X$, $e > 0$, there exists $y \in X$ that satisfies the metric condition and

$$\limsup d(f^n(x), f^n(y)) > 0 \qquad (5.1)$$

$$\limsup d(T^n(x), T^n(y)) > \delta \qquad (5.2)$$

but $\delta > 0$ and

$$T^n(x) = f^n(x)$$

$$T^n(y) = f^n(y)$$

The combined equation from both (5.1) and (5.2) can be written as

$$\limsup d(T^n(x), T^n(y)) > \delta > 0$$

Equation (5.1) is captured in (5.2) and for a $\delta > 0$, the *right hand side* of the equation below does not exist

$$\limsup d(T^n(x), T^n(y)) > \delta > 0 \; 6= \limsup d(T^n(x), T^n(y)) > 0 > \delta$$

We conclude as follows :

Li-Yorke sensitivity guarantees the existence of scrambled sets.

Li-Yorke sensitivity is a special case of Li-Yorke chaos in that it satisfies all conditions of Li-Yorke chaos and certain extra restrictions.

Hence **All Li-Yorke sensitive system are Li-Yorke.**

I give a personal definition of chaos in the form of an inference.

Every expanding map is chaotic.

Expansivity implies topological mixing which implies transitivity. This is to include transitivity which is considered as one of the strongest and essential condition for chaos.

I strongly think there should always be a way to calculate chaos or numerically determine chaos. Entropy usually is quite difficult to obtain numerically compared to Lyapunov exponent. Since every expansive map has positive Lyapunov , then by obtaining that , chaos is guarenteed.

Lastly i think that gives a more stronger form of chaos in that the existence of sensitive dependence alone in itself would not and cannot guarentee chaos in dynamical systems.

5.2 Summary And Conclusion

Dynamical systems is about the result trends as well changes observed over time concerning a particular real life scenerio and practice. Time is an important factor in the study of dynamical systems. The different behavior of various systems has become relevant to study and understand. The idea of dynamical systems has gone through phases and has even been accorded different names until now. Chaos gradually has become a part of our dairly lives. We still dont have one principled definition for chaos, but its defined based on the set of conditions it satisfies in terms of topology or metric. Different routes leads to the conclusion of systems being chaotic. The routes in terms of metric are usually measurable whiles the topological routes are usually analytical. The topological routes of a known map can be interelated to the properties of an unknown map such that the conclusion for the two maps are same. This is done through topological conjugation. Toplogical conjugation preserves topological properties but the same cannot be said for all metric properties of chaos.

Devaney defintion of chaos is considered as a general and strong definition of chaos. Its is based on the strength of topological transitivity in the discovery of chaos. For a given map or dynamical system, a particular form of chaos may imply the other. It was observed that the tent map exhibits either non-periodicity at higher iterations or a different kind of periodicity. The tent has shown some properties of chaoticity.

The mathematical language expressed by chaotic systems(especially sensitve dependence , transitivity and dense orbits) guarentees that a chaotic system passes the element of regularity, unpridictability abd indecomposability. The chaos of a map cannot be soley based on sensitive dependence or its equivalent relation of positive Lyapunov exponent. Transitivity is about the strongest property amongs all the conditions. Perhaps its the reason most definition has an aspect of transitivity. For most maps, when transitivity fails, its likely if not obvious that the condition of dense orbits might fail. Of course, positve Lyapunov does not depend on transitivity.

5.3 Recommendation

1. It is recommended that extra attention and effort should be given to the study of chaos of other maps especially the lesser known maps like the horseshoe, bernoulli shift and others

2. It is recommended that we explore other numerical mechanisms for determining chaos beyond the known Lyapunov exponents and topological entropy

3. It is recommended that we develop other areas of dynamical in terms of their characteristics beyond chaos, the nature of their orbits and fixed points

Reference

Adams, C., Franzosa, R. (2008), *Introduction to Topology : Pure and Applied* Pearson Prentice Hall, New Jersey.

Araujo, V., Viana, M. (2008). *Hyperbolic Dynamical Systems . Handbook of dynamical systems,* 1(1), 22. Elsevier. Retrieved from http://arxiv.org/abs/0804.3192

Aulbach, B., Kieninger, B.,(2000) *On Three Definitions of Chaos*, NonLinear dynamics and systems theory (1) , Department of Mathematics, University of Augsburg, D-86135 Augsburg, Germany.

Banks, J., Brooks, J., Cairns, G.,Davis, G. and Stacey, p., (1992) *On Devaney's definition of chaos*, Am. Math. Mon. 99 ,p 332–334.

Biswas, H.R., (2013) *One dimensional chaotic dynamical system* Journal of Pure and Applied Mathematics: Advances and Applications Volume

10, Number 1„ Pages 69-101 Department of Mathematics University of Barisal Bangladesh

Cheban, D. N.,(2009)*Asymptotically Almost Periodi cSolutions of Differential Equations*

Devaney, R.L.,(2003)*An Introduction to Chaotic Dynamical Systems*, 2nd Edition Boulder, Co.: Westview Press.

Di Bernardo, M., Budd, C. J., Champneys, A. R., Kowalczyk, P., Nordmark, A. B., Tost, G. O., Piiroinen, P. T. (2006), *Bifurcations in nonsmooth dynamical systems.* SIAM Review, 50(4), 629. Retrieved from http://dx.doi.org/10.1137/050625060

Fatiou A., (2005) *Deterministic chaos* Msc Project , Queen Mary and Westfield College , University of London

Fre, P., Sorin, A. S., Trigiante, M. (2011), *Integrability of Supergravity Black Holes and New Tensor Classifiers of Regular and Nilpotent Orbits Scanning.* Retrieved from http://arxiv.org/abs/11030848.

Goodson, G. R., (2015)*Chaotic dynamics : fractals, tilings ans substitutions* Towson University Mathematics Department.

Hofbauer, J., Sigmund, K. (1988), *The Theory of Evolution and Dynamical Systems,* Cambridge Univ. Press, Cambridge

Kellert, S. H., (1993). In the Wake of Chaos: Unpredictable Order in Dynamical Systems. University of Chicago Press. p. 32. ISBN 0-226-42976-8.

Kulkarni, P. R., Borkar, V. C.,(2015) *Period three cycle and chaos in a dynamical system* Volume :2, Issue :4, pp. 591-594 , www.allsubjectjournal.com eISSN: 2349-4182 p-ISSN: 2349-5979

Li, J., and Ye, X.,(2015)*Recent Development Chaos Theory In Topological Dynamics*

Li, T., Yorke, J. A., (1975) *Period Three Implies Chaos* The American Mathematical Monthly, Vol. 82, No. 10., pp. 985-992. 81

Lim, H., Yu Y., Glimm, J., Li. X-L., Sharp D. H.,(2008) *Chaos , transport and mesh convergence for fluids mixing*

Lin, L.,Zhao, S. , Lang, H. ,(2011)*Uniform convergence and Transitive subsets* , Hindawi Publishing cooperation, Discrete dynamics in nature and society,

Mallat, S. (2009), *Geometrical grouplets. Applied and Computational Harmonic Analysis* , 26(2),p 161-180. Elsevier Inc. Retrieved from http://linkinghub.elsevier.com/ retrieve/pii/S1063520308000444.

Nguyen, A. H., Fraser, S. J. (1989), *Geometrical picture of reaction in enzyme kinetics.* Journal of Medical Physics, (February), 186-193. Retrieved from http://jcp.aip.org/jcpsa6/v91/i1/p186s1

Ohtsuki, H., Hauert, C., Lieberman, E., Nowak, M. A. (2006), *A simple rule for the evolution of cooperation on graphs and social networks.* Nature, 441(7092), 502-505. Nature Publishing Group. Retrieved from http://www.ncbi.nlm.nih.gov/pubmed/16 724065

Ren, J., Zhang, B. (2009), *An Improvement on a Non-deterministic Oneto-n Substitution Scheme in Outsourcing Association Rule Mining Work*, (60502032).

Ruette, S.,(2015) *Chaos for continuous interval maps a survey of relationship between the various kinds of chaos*

Sanz, A. S., Miret-Artes, S. (2008),*A trajectory-based understanding of quantum interference.* Journal of Physics A: Mathematical and Theoretical, 41(43), 27. Retrieved from http://arxiv.org/abs/0806.2105

Sharipov, R. (2001),*A note on Newtonian, Lagrangian and Hamiltonian dynamical systems in Riemannian manifolds.* Dynamical Systems, 14. Retrieved from http://arxiv.org/abs/math/0107212

Thomson, C.A., (2013) *Hierarchy of Chaotic Topological Dynamics* Davidson College,Spring2013.

Valle Jnr., V. (April 200) *Chaos, Complexity and Deterrence* , National War College

Wang, L.,Li, L.,Ding, J.,(2011) *Uniform Convergence, Mixing and Chaos* Studies in Mathematical Sciences Vol. 2, No. 1, pp. 73-79 ISBN 1923-8444 [Print] , ISBN 1923-8452 [Online]

Yaacov, I. B. (2008), *Topometric spaces and perturbations of metric structures. Logic and Analysis,* 1(3-4), 3-4. Retrieved from http://arxiv.org/abs/0802.4458

ZimmermannH, G., Grothmann, R., Sch. A. M., (2005).*Modeling Large Dynamical Systems with Dynamical Consistent Neural Networks.* New Directions in Statistical Signal Processing From Systems to Brain, 203-242.

Appendix A

5.3.1 The Logistic Map

Given the logistic map $f(x) = 4x(1 - x)$ Let $\{x_1, x_2, x_3, x_4, \cdots, x_k\}$ be a periodic orbit of the logistic map f.

The stability of the given map is obtained by the derivative of f^k

The chain rule along a cycle, we have

$$(f_k)0(x_1) = f\,0(x_1) \cdot f\,0(x_2) \cdots f\,0(x_k)$$

$$\text{If } f(x) = 4x(1 - x)\ f(x)$$

$$= 4x - 4x^2 f^0(x) = 4$$

$$- 8x$$

On the interval [0,1]

$$|f^0(x)| \quad \text{is between 0 and 4}$$

$$h(x_1) = {}_x\text{lim}_{\to\infty}\ \_1k\ {}_i\textstyle\sum_{=k1} \ln |f\,0(x_1)|$$

$$= \ln 2 > 0$$

Since $h(x_1)$ is true, the map is sensitive to initial conditions on the interval [0,1] **(Hena Rani Biswas, 2013)**

Also in terms of topological properties, the logistic function has a conjugacy with the tent map via a homeomorphism,

$$h(x) = sin^2\left(\frac{\pi x}{2}\right)$$

This allows us to conclude that, the logistic map and tent map share common topological properties. hence the logistic map is transitive and has dense periodic orbits.

We conclude that the logistic map is chaotic..