MODELLING CUSTOMERS RESPONSE TO SERVICE OFFERS USING GAME THEORY. CASE STUDY: MTN AND VODAFONE IN BOLGATANGA POLYTECHNIC.

By

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DECLARATION

I hereby declare that, this submission is my own work towards the award of a Master of Philosophy(Mphil) Mathematics degree and that, to the best of my knowledge it contains no material previously published by another person nor material which has been accepted for the award of any other degree of the University, except where due acknowledgment has been made in the text.

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ABSTRACT

Game theory is being found to be one of the useful tools for analyzing strategic situations that involve competitors or rivals who struggle against each other for supremacy.

The mobile phone industry is one sector where competitors are always in fierce competition for new customers as well as maintaining their already existing customers. They do this by way of the services and products they offer to the public.

It is against this backdrop that, this thesis explores game theory as tool for assessing the optimal strategies of two telecom companies in Ghana, namely MTN and Vodafone. The services offered to the public by these two companies were considered as their strategies to win more customers. Out of the services provided by these two companies we selected an equal number of eight (8) for the two companies.

Two hundred (200) respondents from Bolgatanga Polytechnic were sampled by simple random means for this research.

Our results provided us with the probabilities with which the two companies will have to play their selected strategies in order to win a larger share of customers. We also determine the optimal value of the game.

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LIST OF ABBREVIATIONS

NCA	National Communications Authority
MVNO	Mobile Virtual Network Operator
CTP	Compulsory Third Party
ISP	Internet Service Provider
ARLID	Alto Rio Lerma Irrigation District
MTBE	Methyl Tert-Butil Ether
OPEC	Organization of Petroleum Exporting Countries
FCMS	Fed Cattle Market Simulator
EDLP	Everyday Low Pricing
HLP	High-Low Pricing
RC	Recycling Company
VC	Virgin-raw-material-using company
ERC	Equity, Reciprocity and Competition

DEDICATION

I dedicate this work to my beloved late mum Ernestina K. Lardi and my dad Mr.Ayebire Edward.

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CHAPTER 1

INTRODUCTION

1.0 Overview

Life is full of conflict and competition. Numerous examples involving adversaries in conflict include military battles, political campaigns, advertising and marketing campaigns by competing business firms Hillier and Lieberman (2001). In all the above examples, individuals or groups concerned struggle against each other in a rational manner for supremacy.

An important mathematical tool that provides the framework to formulate, analyze and understand strategic scenarios is game theory. The relevance of game theory in business and most competitive cases has captured the attention of economists who have devoted much thought on how to play games of strategy, and these ideas, which constitute game theory, influence the thinking of business people, military strategists, and even biologists. They also infiltrate everyday life. Not understanding game theory puts you at a tactical disadvantage when playing against those who do. "Following this, almost all MBA students and undergraduate economics majors will formally encounter game theory in the classroom" Miller (2003).

Game theory has been traditionally divided into two branches: non-cooperative and cooperative. Cooperative game theory deals with situations where there are institutions that make agreements among the players binding. In such a setting the central question becomes one of agreeing on a best joint course of action, where 'best' could have different meanings, such as 'acceptable to all players and coalitions of players' or 'satisfying some desirable properties'.

Non-cooperative game theory, on the other hand, deals with institutional settings where binding agreements are not possible, either it is because communication is impossible or because agreements are illegal or because there is no authority that can enforce compliance Bonanno (2008).

Game theory studies how competitors act, react, and interact in the strategic pursuit of their own self-interest. It is for this reason that game theory has become an essential competitive tool in today's business environment. According to Abd El-Kareem (2010) the object of study in game theory is the game, which is the formal model of an interactive situation in which opponents often base their moves on what they think other people might do. But if your move is based on what your opponents might do, and their moves are based on what they think you are going to do, then your move will in fact be somewhat based on what you think your opponents think that you will do. A primary feature in many of these situations is that, the final outcome depends mainly upon the combination of strategies selected by the opposing groups.

Game theory models attempt to abstract from personal, interpersonal, and institutional details of problems how individuals or groups may behave given a set of given conditions. This modelling allows a researcher or planner to get at the root of complex human interactions. The major assumption underlying most game theory is that people and groups tend to work toward goals that benefit them. That is, they have ends in mind when they take actions (www.enotes.com).

1.1 Background of the Study

The telecommunication sector is now a highly competitive business area in Ghana's economy. Some few years ago, very few people were seen possessing mobile phones. However, today it is common to see a good percentage of the populace using mobile phones. As recently as 1996, the telephone density of Ghana was 0.26% meaning that there were 2.6 telephone lines for every thousand (1,000) people including thirty-five (35) payphones in the entire country out of which thirty-two (32) were located in Accra. This was one of the lowest in Africa. Today there is one phone for every four Ghanaians (www.ghanaweb.com). The number of service providers have also increased significantly. This tremendous increase in the tele-density has been as a result of the establishment of the National Communications Authority (NCA) in 1997 and the subsequent deregulation of the telecom industry which brought about the growth of wireless telephony as a result of significant investment by operators. For the consumer, being in touch simply means being able to purchase a mobile handset and subscribing to a wireless service.

Deregulation also meant opportunities for ambitious entrepreneurs and large telecom companies to establish operations in Ghana and participate in what was to become the biggest boom in Ghana's recent economic history Amoafo-Yeboa, (2007).

The telecom industry has also opened up a lot of job opportunities for various categories of individuals both in the formal and the informal sectors in the country. The contribution of the telecom industry in the growth of the Ghanaian economy cannot be ignored. Currently in Ghana there are about five service providers in this industry namely; MTN, Vodafone, Tigo, Expresso, and Airtel. There are also other providers warming up to storm the Ghanaian market with their service notably Glo.

1.2 Problem Statement

The fierce competition in the telecom industry in Ghana has led to some industry players to lose customers, MTN for one had their customer base reduced by 52,065 and Vodafone by 12,619 at the end of August 2010 (www.thecorporateguardian.com). As part of strategies to gain lager market shares of customers, mobile service providers engage in various forms of service offers that will appeal to users and get them attracted to use their product. The application of game theory to this sector can help the various players to identify their best strategies to employ in the face of growing competition. By identifying their best strategies will help them avoid unnecessary confrontation with Ghana Lotteries which has often landed in court cases in recent times, it will also help them maximize their profits from gaining more customers, help them avoid wastage on offers that do not appeal to customers. Playing optimal strategies will also lead to customer satisfaction.

It is against this background that, this project seeks to examine the strategies of two of the giant players in the industry namely Vodafone and MTN and how consumers react to these strategies using game theory on the backdrop that the concepts of game theory provide a language to formulate, structure, analyze, and understand strategic scenarios.

1.3 Objectives of the Study

The objective of this study is to:

(i) Model customers response to mobile service offers by MTN and Vodafone using game theory

(ii) Determine the optimal strategies and payoffs for the two mobile companies.

1.4 Methodology

In this study, we made use of Linear Programming Problem of game theory to model the response of customers to service offers by MTN and Vodafone. The major source of data for this project was basically primary data. To achieve this, questionnaires were administered to some sampled respondents. A sample size of two hundred (200) was considered for this thesis. The study location was Bolgatanga in the Upper East region of Ghana. Simple random sampling was used to administer questionnaires to respondents with the assumption that 7 out every 10 people had mobile phones. The questionnaires were designed in a way that afforded respondents the opportunity to choose between the two networks which they preferred given a set of service offers from the two networks simultaneously matched. Secondary data was also however utilized in the study to obtain the best results for this thesis. This data was in the form of the various service offers been employed by the two firms under study to win a greater share of customers in the market. These service offers were regarded as strategies in order to use game theory. Primal-dual interior point method based on the Mehrotra's predictor-corrector algorithm was then employed to solve for the optimal strategy and its associated payoff.

We used the software matlab to solve our models to obtain our optimal strategies at equilibrium and the expected payoffs associated with the two players. We relied on textbooks, journals and articles from the library and the internet for information.

1.5 Justification

The massive increase in mobile phone users with the corresponding increase in the number of service providers reveals the usefulness of the telecom industry in the world and for that matter the Ghanaian society.

The mobile phone has become an integral part of our daily lives. It has become an important means of communication by family members, friends and business associates. It is against this backdrop that it is relevant for service providers to tailor their services to offer satisfaction to customers. When this is done mobile phone users will obtain value for money for subscribing to a particular network resulting in the promotion of businesses and relationships.

When telecom operators are able to play their optimal strategies, they will be able to survive the prevailing competition and increase their customer base which will invariably lead to good business. When this happens more job opportunities both in the formal and informal sectors will be created. Also with the introduction of the mobile phone number portability which will give customers the chance to switch service providers without necessarily losing their already existing numbers, service providers are therefore required to address the needs of their customers in order to keep them.

This project will help service providers to identify and play their optimal strategy in order to satisfy their customers. The project will also add to already existing literature on the application of game theory.

1.6 Organization of Thesis

In this chapter we provided an overview of the research undertaken in this thesis. The remainder of the document is organized as follows. Chapter 2 entitled "Literature Review" looks at the historical background of game theory and pertinent literature in the field of game theory.

Chapter 3 entitled Methodology presents some solution concepts on game theory and offers a brief description of the Lemke-Howson algorithms which is used to solve bimatrix games. Chapter 4 is entitled Data Analysis and Results and Chapter 5 is entitled Conclusion and Recommendation makes conclusion based on the study and suggest recommendations.

CHAPTER 2

LITERATURE REVIEW

2.0 Introduction

A classic marketing textbook, by Drucker (1954) written almost 60 years ago, suggested that successful marketing involves seeing the business from the customer's point of view and in today's commercial environment the same could be said with equal force about successful brand management. One possible route into the customer's mind, often recommended in the literature, is the study of market segmentation variables. They provide a method of identifying subgroups of consumers who are likely to respond in a relatively homogeneous way to products or services or brands Brennan (1995).

In trying to understand the factors that could influence individuals preference on a particular item, Moss and Coleman (2001) performed two experiments and the outcome of their findings corroborated with earlier findings and provided evidence that male and female design choices and preferences tend to differ systematically. These findings have been discussed in the context of other psychological gender differences, outlining some of the factors that may underlie these differences and commenting on some of the implications for marketing and brand management. Understanding these factors has the potential to provide companies with a significant competitive advantage.

In competitive situations, it seems important to be able to figure out what one's rival is going to do, that is, to anticipate a rival's moves by assigning probabilities to the different actions a rival might take. Game theory provides this probability perspective Cabral (2000).

2.1 Historical Background of Game Theory

Game theory is a branch of applied mathematics, which deals with multiperson decision making situations. The basic assumption is that, decision makers pursue some well defined objectives and take into account their knowledge or expectations of the other decision makers' behaviour. Many applications of game theory are related to economics, but it has been applied to numerous fields.

Fent et al. (2002) applied game theory as a tool for law enforcement.

Bilbao et al. (2002) also in the same year applied it to voting decisions in European Union.

While game theory has antecedents dating back to the ancient times and the 18th century, it emerged as a scientific discipline early in the 20th century, the result of efforts to apply quantitative analysis to abstract cognitive dilemmas. Because of poorly developed methodologies and a lack of demonstrably consistent application to more widely useful purposes, it was largely ignored until the 1940s.

Cabral (2000) indicates however that, during the early years of the Cold War, significant breakthroughs were made that legitimized it as a science with testable proofs. As a result, certain aspects of game theory could be applied to war games and global military strategy, economic and pricing strategies, social issues, and even labour bargaining.

Modern game theory started somewhere around 1928 with John von Neumann's analysis of two-person zero-sum games, where one player's gain equals the other's loss (as in poker, chess, matching pennies, and possibly war). He proposed solutions with some remarkable properties. However, in collaboration with Oskar Morgenstern, John von Neumann realized that many interesting economic situations lead to games which have more players or are not zero-sum. In their book The Theory of Games and Economic Behaviour by von Neumann and Morgenstern (1944) proposed methods for analyzing games generally and their pioneering contribution is probably the most important milestone in the history of game theory.

2.2 Applications of Game Theory

Game theory is been applied in several areas of real life situations, ranging from business, telecommunication, conflict resolution, economics, agriculture etc . This section reviews some of the work done using game theory.

Hassan et al. (2010) considered a scenario where mobile users may use threat as a strategic tool to force service providers in allocating more resource to fix occasional link quality problems during an active communication session. The effectiveness and feasibility of the proposed threat-based mobile communication scenario was analyzed using Game Theory and Monte Carlo simulation. The outcome of their work showed that, at least from a conceptual point of view, users can benefit from the dynamics of such a system.

Das and Lin (2004) applied game theory to investigate the role and importance of the economic aspects that are vital to the success of wireless services deployment and provider selection by users in a competitive environment. Their work showed how some of the econometric measures can meaningfully capture the user decisions or actions (e.g., churning) that can potentially be utilized by the providers in managing radio resources (e.g., bandwidth) in wireless data networks. In particular, by modelling the interaction between a service provider and its customers (or users) as a non-cooperative game, they proposed a novel cross-layer resource management framework for integrated admission and rate control in code-division multiple access (CDMA) networks. Analytical and simulation results showed that the proposed framework improves the provider's revenue, yet optimizing customer satisfaction and providing differentiated quality of service (QoS) for different classes of users.

Conklin et al; (2004) carried out a research to identify key drivers in customer satisfaction in relation to product quality using tools from Cooperative Game Theory and Risk Analysis. They used Shapley Value and Attributable Risk techniques to identify priorities of key drivers of customer satisfaction, or key dissatisfiers and key enhancers. They demonstrated the usefulness of Shapley Value and Attributable Risk concepts in elaborating optimal marketing strategy.

Conklin et al. (2004) also demonstrated the practical advantages of the Shapley Value as a useful decision tool that can be applied for numerous problems of categorical data modelling arising in various managerial fields. Following the strategy suggested by the Shapley Value for key drivers, managers can choose ~~the best~~ direction toward improving customer acquisition and retention.

Lye and Wing (2002) applied game-theoretic method for analyzing the security of computer networks. They viewed the interactions between an attacker and the administrator as a two-player stochastic game and constructed a model for the game. Using a non-linear program, they computed the Nash equilibrium or best-response strategies for the players (attacker and administrator). They then explained how administrators could use the results to enhance the security of their network.

Lannoo et al. (2009) evaluated a complete business case for the rollout of a municipal wireless access network for offering Mobile Internet of two different business players, a municipality deploying a WiFi network and a mobile virtual network operator (MVNO) offering 3G femtocells in the city of Ghent (Belgium). Different customer classes and service offers were defined. By applying game theory they modelled the influence of competition between these two market players. The optimal rollout strategy as well as the price setting for both players was evaluated. Due to the high initial investments for deploying WiFi access points throughout the whole city, the municipality tend to postpone its rollout one or two years, whereas the 3G femtocell operator could start as soon as the technology allowed a mass deployment. Regarding the price setting strategy, both players balanced each other leading to slightly higher service tariffs, when no extra competitors were considered.

Maillé et al. (2009) studied the price war between two providers in the telecommunication industry in a case where user's decisions were modelled by a Markov chain, with price dependent transition rates. Each provider is assumed to look for maximized revenue, which depends on the strategy of the competitor. Therefore, using the framework of non-cooperative game theory, they showed how the providers take into account the user churn behavior to determine the price they fix for the service, so as to maximize their steady-state revenue.

Olabode (2003) stated that, the deregulation of the telecommunication industry in Jamaica had led to lower rates for mobile phone customers as new mobile phone service providers enter the market. Olabode (2003) modelled the competition among old and new mobile phone service providers as a game of entry deterrence. His work revealed that in equilibrium, the price war between old and new companies will persist until all economic profits have been eroded. Hence, the current trend in mobile phone rate reduction may be expected to continue in the short term.

Lu et al. (2011) employed game theory and co-opetition perspectives, to investigate 2-year part-time college students' dynamic behaviours of mutual trust and knowledge sharing toward learning performance via self-developed questionnaire scales. The results showed that, via trust-building and knowledge sharing processes, the case study group with co-opetition oriented game leads to better learning performance than individual-learning-alone group. Moreover, diversified backgrounds of teammates will also enhance learning performance. These practical game-based implications are valuable to management educators.

Konstantinidis et al. (2007) utilized the concepts of game theory in the insurance industry. They applied game theory to the compulsory third party motor insurance (CTP) where several players operate. According to them insurers in a soft market can find themselves trapped because they are worse off if they try to move to a sound basis. They conclude that it is sometimes dangerous to charge sound premiums because sound premiums are set based on costs only, without reference to key features of markets dynamics, such as what the market will allow you to charge, and what actions your competitors can take against you.

Biczók et al. (2008) examined the impact of customer loyalty on price competition between local Internet Service Providers (ISP) who sells Internet access to end-users using game theory. First, they developed a repeated game, and showed how cooperation between ISPs resulting in higher profits can be enforced through a threat strategy in the presence of customer loyalty. Secondly, they investigated the case of a differentiated customer population by introducing dual reservation values, and showed how it leads to new, pure strategy Nash equilibra for a wide range of demand functions. Thirdly, they developed two novel models for customer loyalty, along with a simulation tool that was capable of demonstrating the impact of the novel models.

Salazar and Rojano (2002) applied Game theory to groundwater management conflict in the Alto Rio Lerma Irrigation District (ARLID) located in the state of Guanajuato in Mexico. The major agricultural constraint in this irrigation district is water. Besides, the current management practices in the irrigation district, like high fertilizers and pesticides application together with the irrigation management, produces environmental problems: nitrates and pesticides percolation that may cause aquifer contamination. On the other hand, farmers need to maximize their net benefit by increasing yields. The problems in the ARLID were classified as economic and environmental and they found the best compromising or satisfying solution that increased in economical benefits and reduced the negative environmental effects as well as using the water in the most efficient way in the region.

Salazar and Rojano (2002) again applied game theory to water problems. Water use in current time brings about the following problem. Urban water demand is growing up while rainfall keeps somehow constant. Agriculture area does not increase and water savings are feasible with better irrigation techniques. Recent experiences in California Valley and Chinese regions give some idea how to deal with these scenarios before the problem exploits by itself. Based on this, they established the optimal solution working on the mathematical form in water problems searching the demands and supplies. Their work revealed that, demand comes from the urban areas and the supply from the agriculture. Demand from surface water and supply from groundwater.

Mongin et al. (2009) provided what appears to be the first game-theoretic modelling of Napoleon's last campaign, which ended dramatically on 18 June 1815 at Waterloo. It was specifically concerned with the decision Napoleon made on 17 June 1815 to detach part of his army against the Prussians he had defeated, though not destroyed, on 16 June at Ligny. Military historians agree that this decision was crucial but disagree about whether it was rational. Hypothesizing a zero-sum game between Napoleon and Blücher, and computing its solution, they showed that it could have been a cautious strategy on the former's part to divide his army, a conclusion which runs counter to the charges of misjudgement commonly heard since Clausewitz.

Krek (2005) demonstrated how game theory can be implemented in a geoinformation trade situation. He set up a business game model where two producers of geoinformation produce and sell their products. The producers were rational in the sense that they tried to maximize their profits, and they can set either a high or a low price for their geoinformation products. He focused on the implementation of different pricing strategies and showed the resulting payoffs in the form of the profits gained by the geoinformation producers. His analysis showed that the producers are better of setting a low price for their geoinformation products. Setting a low price is a strictly dominant strategy. A dominant strategy is the best choice for a player for every possible choice of the other player. According to Krek (2005), game theory can be a powerful tool, which offers insights into how pricing policy affects buyer's behavior with respect to the information he or she has.

Yang (2010) presented a game theoretic model on resource allocation in epidemic control. The model considered the drug allocation problem when faced with the outbreak of an international influenza pandemic. The drugs were for prophylactic use. Since drug stockpiles were scattered in different countries, the outbreak of an epidemic gave rise to a game in which each country must make decisions about how best to allocate its own stockpile in order to protect its population. They developed a two-period multivariate Reed-Frost model to represent the spread of the epidemic within and across countries at its onset. They showed that for small probability of between-country infections, the underlying game is supermodular and that a unique Pareto optimal Nash equilibrium exists. Further, they identified sufficient conditions under which the optimal solution of a central planner constitutes a Pareto improvement over the decentralized equilibrium.

Ghorbani (2008) compared the effect of market sale and contract strategies on yield, using a 90 survey data from tomato farmers in Khorasan-Razavi province, Iran in 2007 by applying the game theory approach. The results showed that the effect of no-contract strategy with tomato processing factories was more than contracted strategy on tomato yield in farm level. With respect to this finding, producing the support-extension services as distributing the high-yield seeds, organizing the production cooperatives from small farms and drawing up the village-collectively contract framework was been recommended to planners and policymakers.

Bailey et al. (2009) seek to find out whether after about thirty years of Munro's 1979 work on the potential applications of game theory to fisheries, had this potential been realized, and in what forms? He provided illustrations on how coalition theory has been brought in over the past decade to allow the modelling of games with greater than two players, and how coalitional externalities, a major issue in international and shared fish stocks management, have been incorporated recently. Finally, he highlighted new areas for game theory to come into its own in fisheries, which will most definitely include the potential for catch rights in international shared-stock fisheries, as well as ecosystem dynamics to be modelled in a game-theoretic context. As game theory can offer insights into the challenges of achieving cooperative fisheries management, it is expected that, the next decade of fisheries economics research will include a bigger leap from academic game theory exercises to impacting policy decisions.

According to Maciel et al. (2009) since the Hubbert model succeeded in forecasting US-48 oil production peak in early 1970s, recent studies have estimated the end of cheap oil era. Environmental restrictions especially the replacement of MTBE (Methyl Tert-Butil Ether) and the need to mitigate the emissions of greenhouse gases have elected ethanol as the main candidate to replace gasoline, one of the most important oil derivatives. The expected worldwide growth of light-duty vehicles' fleet and the still limited capacity of ethanol production suggest that, the total replacement of gasoline is not feasible in the short to medium-term. Based on game theory, the transition process from the former fossil fuel era to an era with wider use of biomass could focus on cooperation and compromise strategies with optimal results to both oil and ethanol industries. They found that 30% ethanol blend to the gasoline could increase the performance of internal combustion engines, decrease the emission of greenhouse gases and alter the rate of depletion for conventional oil reserves, thus extending the oil era.

Brykalov et al. (2004) used game theory to construct a model of investment in gas pipeline projects competing for a regional gas market. The model was designed as a multi-player game with integral payoffs, in which times of entering the market acted as players' strategies. For each player, they identified the location of player's best responses to strategies chosen by other players. On this basis, thy reduced the original game to a game with a finite number of strategies for each player. Finally, they constructed a finite algorithm for finding player's best responses and the Nash equilibrium points in the game. The presented approach can be used to analyze competition of large-scale technological and energy transportation projects in situations where the investment periods precedes the periods of sales and the appearance of every new supplier on the market drastically affects the market price.

Sindik and Vidak (2007) applied game theory to the field of sports. In their work, the level of predictability of the most frequent tactical performance of one player in a team sport game was considered, reflecting outcomes ~~both for the~~ same team's tactical performance (co-players in one player's team), as well as for the opponent team's tactical performance. Four different possible situations during team sport competition could lead to considering utilities of one player's specific decisions. Predictability is in general better than unpredictability, both for the players in the same team and for the opponent's team players.

Weinstein (2003) demonstrated a practical application of game theory in the bankruptcy proceedings of United Airlines and its affiliated companies. Here United Airline's bankruptcy attorneys got the best deals for their client by practically applying the "Prisoner's Dilemma" to United's aircraft lessors.

Haita (2007) employed a stylized static oligopoly model of the world oil market in two versions differentiated by the behaviour of Russia. In the first version Russia was regarded as a price-taker together with the fringe producers and Organization of Petroleum Exporting Countries (OPEC) was a partial monopolist. In the second version, Russia was a strategic player in a quantity leadership game with OPEC, with the fringe producers behaving competitively. The aim of this work was to quantitatively compare the Russian profits in these two behavioural situations. Using game theoretic principles, he concluded that, Russia should be indifferent between being a price-taker and behaving strategically, as the gains from the strategic behaviour are very low, below 1%. In addition, he showed that, price increases in the strategic version of Russia's behaviour as compared with the price-taking version. Furthermore, OPEC was always more better-off than Russia when the latter played strategically.

Lianju and Luyan (2011) investigated the mechanism of bribery behaviour based on non-cooperative static game theory. ~~With the~~ general presumption of "rational player", two bi-matrix game models were established to analyze the strategy choice of the briber and the bribee. After discussing the cost-benefit of the players, they realized that the generation of bribery is due to the fact that the expected revenue of players is more than the costs. According to them, punishing the bribers severely afterwards is a temporary solution and therefore suggest that we should take measures according to the factors that influence the briber's decision, reducing his psychological expected net revenue so that the briber can't obtain extra profit.

Calberg et al. (2009) carried out a research into cattle fed procurement. In their research a game theory model was developed and an empirical model estimated to measure beef packing firm behaviour in cattle procurement. Experimental market data from three semester-long classes using the Fed Cattle Market Simulator (FCMS) were used. Collusive behavior was found for all three data periods though the extent of collusion varied across semester-long data periods. Results may have been influenced by market conditions imposed on the experimental market in two of the three semesters. One was a marketing agreement between the largest packer and two feedlots and the other involved limiting the amount and type of public market information available to participants. Findings underscored the need for applying game theory to real-world transaction-level, fed cattle market data.

Bratvold and Koch (2011) also identified game theory as useful tool in the oil and gas industry. According to them, major projects in the oil and gas sector are mostly joint, requires some negotiations as well as competitive in nature. In all these, game theory can be helpful in helping individuals, companies and countries involved to develop the strategies and tactics to "win" or at least achieve the best attainable outcome.

Tanczos (2005) applied game theory to the Hungarian transport system to identify a way to allocate common cost of road infrastructure to different user groups in order to create an efficient, fair and equal road charging system. To achieve this objective, he tried to maximize user benefits by choosing optimal value of user charges by considering the following constraints: user charges shall not exceed user benefits, revenues from user charges shall exceed occasioned costs of user categories. The result of the generated model showed that, charges were too high if they considered construction, operation and maintenance costs. His result proved the current Hungarian ministerial decree in force which declares that revenues from motorway user charges should cover only operation and maintenance. Within the road transportation, the research focused on motorways, namely motorway users.

Green (2002) tried to investigate whether game theory could aid in forecasting the decision making of parties in a conflict. When put to the test, game theorists' predictions were more accurate than those from unaided judgment but not as accurate as role-play forecasts. Twenty-one game theorists made 99 forecasts of decisions for six conflict situations. The same situations were described to 290 research participants, who made 207 forecasts using unaided judgment, and to 933 participants, who made 158 forecasts in active role-playing. Averaged across the six situations, 37 percent of the game theorists' forecasts, 28 percent of the unaided-judgment forecasts, and 64 percent of the role-play forecasts were correct

Docherty and Yakovlev (2011) looked at game theory along with its possible uses to manage energy consumed by processes ~~within~~ real-time processing and ensure maximum reliability while managing all available power from a source such as an energy harvesting device. As they continued their investigation they speculated that, reliability can be retained with a reduced energy budget and finally concluded that, a game theoretic algorithm can be developed which gives improved performance across systems of unspecified size.

Mu and Ma (2007) carried out an investigation into the pricing strategy of land and real estate based on the application of game theory. The non-cooperative game and cooperative game between developers and their equilibrium solutions were studied. Efficiencies of each price decisions were further compared when government profit as well as housing prices are taken into consideration. The research result indicated that, cooperation is the optimal strategy and regulating the tax rate in a moderate rang is an efficient way for government to increase its profit while decreasing housing prices so as to maintain social stability. Those who have rich economic strength can participate in land and real estate development simultaneously which not only solve the problem of land shortage but also gain better profit margins.

Gyarmati and Trinh (2009) conducted a research on how to price internet access for disloyal users under uncertainty for Hungarian internet service provider (ISP) market using game theory. In their work, they found out that, customer loyalty issue in ISP market is strongly dependent on price difference. They formulated a game theoretic model dealing with price difference loyalty and the conditions of pricing strategies under uncertainty. Also, analysis of their work showed that if the price difference constant of the market is large enough, the Internet Service Providers can cooperate, namely do not compete in the prices, thus they can sell Internet access on the highest possible prices. They concluded that, providers have to change their prices time after time because of the market conditions and if on the other hand each provider knows only her own subscribers, she has to make price decisions based on her beliefs. .

Alder and Smilowitz (2006) sort to develop a framework to analyze global alliances and mergers in the airline industry under competition. According to them, the proposed framework could help airlines identify partners and network structures, and help governments predict changes in social welfare before accepting or rejecting proposed mergers or alliances. In their research they combined profit-maximizing objectives to cost-based network design formulations within a game theoretic framework. The resulting analysis enables merging airlines to choose appropriate international hubs for their integrated network based on their own and their competitors' costs and revenues in the form of best response functions. They concluded that, some mergers may be more successful than others and optimal international gateway choices change according to the number of competitors remaining in the market.

Supalla et al. (2001) observed that the resource management problem for the Middle Platte ecosystem is that there is insufficient water available to meet both in-stream ecological demands and out-of-stream economic needs. This problem of multiple interest groups competing for a limited resource is compounded by sharp disagreement in the scientific community over endangered species needs for in-stream flows. Following this, they developed game theory models for addressing this resource management problem. The results of their study suggested that, the use of game models can improve the prospects for reaching a resource management agreement. The willingness of states to supply environmental water is enhanced if the bargaining process is structured to disallow cheating and provide for political compensation.

Wang and Chang (2010) proposed to develop a systematic heat integration strategy for more than one plant in an industrial park on the basis that greater energy and cost saving could be achieved by expanding the traditional scope of heat integration to the entire industrial park. In order to achieve this the total-site heat integration problem was solved by treating it as a nonzero-sum matrix game, in which the proportions of heat exchanges at different temperature levels were regarded as alternative strategies in a game. According to them, such an approach allows not only cut down of the overall energy consumption rate but also to facilitate every plant to gain maximum achievable benefit under the most appropriate price structure.

Asgari and Afshar (2008) observed that, time as one of the most important factors in a successful construction project, can be traded between subcontractors in sequential projects. In an optimal case for trading time, subcontractors have reasonable incentive to cooperate. In their work they introduced a new problem in the field of subcontractors cooperating and then proposed a game theory model to solve it. Results from the utilization of the proposed model showed that while optimizing total cost, all subcontractors can negotiate to fairly distribute benefits from cooperation.

Chioeanu (2003) based on game theory developed a model in which oligopolistic firms first invest in persuasive advertising in order to create brand loyalty and then compete in a Bertrand fashion. From this model, Chioeamu (2003) was able to determine that in equilibrium one firm will choose a lower advertising level while the remaining ones will choose the same higher advertising. Using the model one could identify how brand advertising maybe used to induce product differentiation and obtain ~~profits~~ above the competitive level.

Jones (2003) used the concepts of game theory to study the U.S. supermarket industry. In his work, two chains of super market groups namely Everyday Low Pricing (EDLP) and High-Low Pricing (HLP) were considered. The game theoretic model constructed showed that the two supermarket chains can reach equilibrium which maximizes profits for none of the chains. His work also suggested that, to guard against lost sales and profits, all chains should simultaneously engage in promotional activities because failure of anyone to undertake a promotional activity will incur an adverse impact.

Okoye (2010) acknowledged that, Politicians and businessmen use various strategies to outwit their opponents. One such strategy is crowd renting, a powerful virtual strategy. Okoye developed a mathematical model using game theory to show that crowd renting would always be the equilibrium strategy in strategic interactions involving political actors and businesses, provided the cost of using such strategies does not exceed a certain critical threshold where using it would no longer be the equilibrium strategy. It is equally shown that crowd renting would improve the electoral/business fortunes of parties/candidates/businesses.

Shehu (2008) also employed game theory principles to provide strategies required for optimal location of two competitive organizational branching systems. His work was able to aid two competing banks to identify location points for the establishments of their new branches.

Aigbokhaevbolo (2011) sort to examine game theory and its usefulness in formulating business strategies in under developed countries, with a specific reference to Nigeria. Results from his work indicated that, game theory model is desirable, if applied, it will no doubt, enhance the solving of business decision problems in businesses of undeveloped countries.

Koroleva and Zenkevich (2010) also in their paper, considered the model of competition by advertising using game theory in a fashion market according to some specifications. The first specification was that customers were divided to those who were loyal to the whole fashion market and to those who were not loyal. The loyal customers to the market were those who buy more than two items a year in each firm from the industry. The second specification was that, it was necessary to advertise the products of the firm only for the potential customers. It was supposed that there were n firms (players) on the market. The model assumed that, the market had its maximum sales potential N. The firms used advertising $a_i(t), i = 1, \ldots, n$, as a strategic instrument with an effectiveness β_i to find new customers and to increase their sales $s_i(t)$. Using game theory, the Nash equilibrium and an expression for the optimal advertising strategies were found for this problem.

Dimitris and Yannis (2009) proved the functionality of the methodology designed for the development of strategies for increasing the market share of recycled products within a game theory context, referring to the case of a paper market, where a recycling company (RC) was in competition with a virgin-raw-material-using company (VC). The strategies of the VC, for increasing its market share, are the strengthening of (and advertisement based on) the high quality (VCl), the high reliability (VC2), the combination of quality and reliability, putting emphasis on the ~~first~~ component ~~(VC3), the~~ combination quality and reliability, putting emphasis on the second component (VC4). The strategies of the RC, for increasing its market share, are proper advertisement based on the low price of produced recycled paper satisfying minimum quality requirements (RCl), the combination of low price with sensitization of the public as regards environmental and materials-saving issues, putting emphasis on the first component (RC2), the same combination, putting emphasis on the second component (RC3).

Hockstra and Miller (1976) were among the first to recognize the interactive nature of decision making in medical consultations, and hence the value of game theory in modelling this decision making process.

Diamond et al. (1986) used game theory to develop prescriptive models of medical decision making. Game theory has the potential, however, to provide a valuable theoretical basis for broader questions about the medical consultation. This approach has received little attention, with the exception of the work by Batifoulier (1997) which explored the relevance of game theory models to the doctor-patient interaction, and drew on this theoretical perspective to address the question of what produces cooperation between the doctor and the patient.

Palombo (1997) used game theory principles as the basis of a discussion on the development of the therapeutic alliance in psychiatry.

Based on game theoretic principles, Gutek (1995) asserted that, continuous relationships between providers and consumers were conceptually distinct from the other modes of service provision and had unique features that helped to promote cooperation and quality of care. Her empirical work provided evidence to support this assertion.

Gutek et al. (2000) through game theory also discovered that, customers who received service within relationships were more likely to trust their providers and recommend their providers to others. They reported more personalized service within relationships and were more likely to direct complaints to their individual providers than to managers.

Gutek et al. (1999) realized through game theory that, service relationships were also found to be linked to higher customer satisfaction and higher frequency of service use. Gutek's work provided an illustration of the use of game theory to develop a theoretical model and to generate and test predictions about service quality. Although this work did not have a specific focus on medical care, it did point to the value of further research using game theory models to identify predictors of quality in health care.

Gutek (1995) inferred that ongoing provider-customer relationships promote mutual cooperation and improved quality of service. Gutek's work highlights the potential for the use of game theory in the organization and provision of health care, but this needs to be developed further. Not all repeated contacts between doctors and patients lead to cooperation; for example, some ongoing relationships are problematic and, in some cases, cooperation is difficult (or impossible) to get going. So-called ''heartsink'' patients are a clear example of this as noted by O' Dowd (1988). There would be value in further research based on predictions drawn from game theory to investigate which features of repeated doctor-patient interactions are more or less likely to lead to mutual cooperation and good quality care.

The of work Tarrant et al. (2004) also provided a basis to indicate that game theory can be applied to medical consultation and used to generate predictions about how the context of a doctor-patient interaction influences cooperation and quality of care. In particular, game theory models indicate that a history of past interactions between a doctor and patient and anticipation of future interactions make cooperation and good quality care more likely. This review has indicated that game theory can be applied to the medical consultation and used to generate predictions about how the context of a doctor-patient interaction influences cooperation and quality of care. In particular, game theory models indicate that a history of past interactions between a doctor and patient and anticipation of future interactions make cooperation and good quality care more likely.

Özkan and Akçaöz (2001) applied game theory in the study of field crops. In this study the highest expected income level under the worst circumstances was determined using a game theory model. For this purpose, gross product values of wheat, barley, cotton, maize, chickpea, groundnut, and sesame were used for the period of 1980-1999 in Antalya province. The results of the game theory model indicated that groundnut and cotton were the most risky crops in the research area as groundnut and cotton provided the highest expected income under the worst conditions, these crops entered the optimum plan. While wheat and barley were less risky crops, they provided the least expected income under the worst conditions. The variation coefficients of the investigated crops were also calculated based on the gross product values of the crops. It was found that groundnut and cotton had the highest variation coefficients while chickpea had the lowest coefficient in terms of gross product value.

Shakeri et al. (2010) utilized the concepts of game theory to come out with the best leadership's style. In their work they were able to choose the most appropriate leadership style in an unknown environment and concluded that, when the situation is not clear in nearly 90% of the time, charismatic leaders act better than the others. After that, pragmatic leaders can have upper performance in nearly 9% of the time. When they compared these results with the external environment and also with the historical data that they obtained, their results emphasized the external fact. This fact is that in the leadership history, when the situation is unclear, usually the charismatic leaders have the best results and act better than the other type of leaders. Showing this fact by a mathematical approach is a big step to join human sciences with the basic sciences like mathematics. In this way, game theory is a new and useful tool to model our nature. If we can extend these kinds of model to other topics of human sciences, we can express our results stronger than when we use just historical data.

According to Cantwell (2003) military decision-making process relies heavily on mission analysis and a thorough understanding of enemy and friendly capabilities. Game theory supports the importance of this process and provides a valuable tool for the commander to organize his analysis and prioritize the relative value, or military worth, of these outcomes. Contrary to what the name implies, game theory does not suggest that things should be left to chance or some roll of the dice to determine the optimal strategy. Game theory relies on quantifiable analysis to determine the optimal course of action against a skilled opponent.

In fact, a critical assumption of game theory is that the opponent will choose the best course of action to minimize the enemy commander's gains. Game theory is not about a game, it is about decision-making. Incident to a military decision is a planning decision to build a plan based on an assessment of enemy capabilities or intentions. The bold and aggressive commander develops an optimistic strategy that plans to capitalize on any enemy mistakes by understanding the enemy's intentions from the onset. This optimistic strategy in turn assumes more risk if the commander misinterprets his opponent's intentions. In contrast, the commander who bases his strategy on enemy capabilities has the benefit of identifying the worst-case scenario and ensuring that the outcome of events will be no worse than anticipated. Any irrational, unwise, or poorly executed actions by an opponent will present an opportunity for that commander to exploit.

The concept of mixed strategies in game theory has further reinforced the importance of thorough mission analysis and the validity of the military decision-making process. Intelligence, deception, and secrecy all have roots that date back to the start of military operations and are integral parts of game theory. The work of Cantwell (2003) had established a basic understanding of two person-zero sum games and developed several practical examples to demonstrate the utility of game theory to military decision-making. The work also demonstrated the concepts of mixed strategies and the importance of capabilities based planning. Until that day, game theory offers a means for the military commander to organize the results of his analysis and make better decisions. As Colonel Haywood stated 60 years ago, "If a commander is not prepared to make a matrix of opposing strategies for the situation, he isn't prepared to make a decision" Oliver (1951).

Karray and Sigue (2007) carried out a work with the main objective of providing a better understanding of competitive behaviours in distribution channels when not only prices but also local advertising decisions are considered. Local advertising is indeed a very important part of any retailer's marketing mix and should be studied in addition to prices when analyzing competitive behaviours in marketing channels. Furthermore, two forms of competition are of interest: horizontal competition (e.g., between firms at the same channel's level) and vertical competition (e.g., between firms at different channel levels). They presented three essays where both competition between channel members and local advertising decisions were considered.

The first essay investigated the effectiveness of co-op advertising programs in a channel with horizontal competition at both the manufacturing and the retailing levels. They concluded that, when competition is introduced at a channel level, the effectiveness of the co-op program is not guaranteed any more for members who operate at that level. Furthermore, for symmetric channel members, they found that, cooperative advertising programs are implemented only under some conditions on brand and store substitution rates.

In the second essay, they studied the situation where a retailer was competing with a manufacturer by introducing its private label. Then, they evaluated the effectiveness of implementing a co-op advertising program to counter the negative effects created by the store brand competition. They found out that the retailer's as well as the channel's payoffs were higher with the store brand option whereas the manufacturer could lose a part of his profits. However, for a specific range of the retailer's advertising efficiency and of the cross-price competition intensity, the manufacturer could profit from the private label introduction. Furthermore, the co-op plan is an efficient counterstrategy for the manufacturer. However, the retailer would accept its implementation only if the national brand competes strongly with the private label.

Finally, in the third essay, they investigated the effect of a private label introduction in a two-manufacturer, one-retailer channel. They researched further into whether advertising for a private label could improve the performance of its introduction and investigated the effects for manufacturers. They went further to solve for three Manufacturer-Stackelberg games and showed that the retailer benefits from advertising the store brand. Another important outcome of their work was that the private label introduction is not always harmful for manufacturers, especially when the private label is weak, and the store brand positioning does not target the national brand.

CHAPTER 3

METHODOLOGY

3.0 Introduction

In this chapter we are interested in defining some of the basic terms in game theory, some basic solutions to two-person zero-sum games and optimal solution based on linear programming formulation. The Lemke-Howson algorithm is briefly highlighted as solution method to two-person zero-sum games. Interior-point method is also been discussed in this chapter.

3.1 Definitions and Assumptions

3.1.1 Game

A game consists of players, the possible actions of the players, and consequences of the actions. The players are decision makers, who choose how they act. The actions of the players result in a consequence or outcome. The players try to ensure the best possible consequence according to their preferences. The preferences of a player can be expressed either with a utility function, which maps every consequence to a real number, or with preference relations, which define the ranking of the consequences. With mild assumptions, a utility function can be constructed if the preference relations of a player are known von Neumann and Morgenstern (1944).

3.1.2 Rationality

The most fundamental assumption in game theory is rationality. Rational players are assumed to maximize their payoff. If the game is not deterministic, the players maximize their expected payoff. The idea of maximizing the expected payoff was justified by the seminal work of von Neumann and Morgenstern (1944).

However, the rationality assumption has been criticized. Experiments have shown that humans do not always act rationally Friedman (1996). In telecommunications, the players usually are devices programmed to operate in a certain way, thus the assumption of rational behaviour is more justified in this area.

The maximization of one's payoff is often referred to as selfishness. This is true in the sense that all the players try to gain the highest possible utility. However, a high utility does not necessarily mean that the player acts selfishly. Any kind of behaviour can be modelled with a suitable utility function. For example, a preference model called equity, reciprocity and competition (ERC) Bolton and Ockenfels (2000) not only pays attention to the benefit of the player, but also the benefit relative to the other players. In many occasions, an ERC model fits experimental data better than simpler models, where the players only try to maximize their own benefit.

It is also assumed that the players are intelligent, which means that they know everything that outsiders know about the game and they can make the same deductions about the situation that outsiders can make.

3.2 Classification of Games

Games can be classified into different categories according to their properties. In this section we look at some common types of games and examples of these games.

3.2.1 Noncooperative and cooperative games

Games can be divided into noncooperative and cooperative games according to their focus. Cooperative games are also called coalition games. In noncooperative games, the actions of the single players are considered. Correspondingly, in coalition games the joint actions of groups are analyzed, i.e. what is the outcome if a group of players cooperate. The interest is in what kind of coalitions are formed. Both the prisoner's dilemma and the battle of the sexes which will be discussed later are noncooperative games. In telecommunications, most game theoretic research has been conducted using noncooperative games, but there are also approaches using coalition games.

3.2.2 Games with perfect and imperfect information

If the players are fully informed about each other's moves, the game has perfect information. Games with simultaneous moves have always imperfect information, thus only extensive games can have perfect information. A game with imperfect information is a good framework in telecommunications, because the users of a network seldom know the exact actions of the other users. However, it is often more convenient to assume perfect information.

3.2.3 Games with complete and incomplete information

In games with complete information the preferences of the players are common knowledge, i.e. all the players know all the utility functions. In a game of incomplete information, in contrast, at least one player is uncertain about another player's preferences.

A sealed-bid auction is a typical game with incomplete information. A player knows his own valuation of the merchandise but does not know the valuations of the other bidders.

3.2.4 Strategic Games

In strategic or static games, the players make their decisions simultaneously at the beginning of the game. While the game may last long and there can be probabilistic events, the players can not react to the events during the game. The prisoner's dilemma and the battle of the sexes are both strategic games. These games are shown in tables 3.1 and 3.2 below. In strategic games, the players first make their decisions and then the outcome of the game is determined. The outcome can be either deterministic or may contain uncertainties. The solution of a strategic game is a Nash equilibrium. Every strategic game with finite number of players each with a finite set of actions has an equilibrium point Nash (1951). This Nash equilibrium is a point from which no single player wants to deviate unilaterally. When a game is played, the rationality assumption will force the game into a Nash equilibrium outcome. If the outcome is not a Nash equilibrium, at least one player would gain a higher payoff by choosing another action. If there are multiple equilibriums, more information on the behavior of the players is needed to determine the outcome of the game.

3.2.4.1 Prisoner's Dilemma

In the prisoner's dilemma, two criminals are arrested and charged with a crime. The police do not have enough evidence to convict the suspects, unless at least one confesses. The criminals are in separate cells, thus they ~~are not~~ able to communicate during the process. If neither confesses, they will be convicted of a minor crime and sentenced for one month. The police offers both the criminals a deal. If one confesses and the other does not, the confessing one will be released and the other will be sentenced for 9 months. If both confess, both will be sentenced for six months. The possible actions and corresponding sentences of the criminals are given in Table 3.1.

Table 3.1: Prisoner's Dilemma

		Criminal II	
		Don't confess	Confess
Criminal I	Don't confess	(-1,-1)	(-9, 0)
	Confess	(0,-9)	(-6,-6)

In the above prisoner's dilemma, outcome (Confess; Confess) is the equilibrium. Outcome (Don't confess; Don't confess) results in higher payoff for both the criminals, but it is not an equilibrium because both the players have an incentive to deviate from it.

3.2.4.2 Battle of the Sexes

Another famous game is the battle of the sexes, in which a couple is going to spend an evening out. She would rather attend an opera and he would prefer a hockey match. However, neither wants to spend the night alone. The preferences are represented with utility values. The possible actions and corresponding utilities of the players are given in Table 3.2.

Table 3.2: Battle of the sexes

		Husband	
		Opera	Match
Wife	Opera	(2, 1)	(0, 0)
	Match	(0, 0)	(1, 2)

In the above battle of the sexes, the pure strategy equilibrium points are (Opera; Opera) and (Match; Match).

3.2.5 Extensive Games

The strategic game model is suitable for representing simple real life events such as auctions. Many more complex situations can be abstracted sufficiently to be modelled as a strategic game. However, the limitations of the strategic games are evident in many cases. A more versatile model is needed, when more complex interactions are occurring between the decision makers. The possibility to react to the actions of the other players is essential in many applications, thus a broader model is needed. In extensive games the players can make decisions during the game and they can react to other players' decisions thus extensive games eliminate the limitation of the simultaneous decisions, thus they make it possible to model a wider range of real life situations. For simplicity the following formulation does not allow simultaneous actions of the players, i.e. the game has perfect information. By perfect information we mean all the players involved in the game are fully informed about each other's moves. An extensive game with imperfect information can be formulated similarly. In order to define the player's behavior in an extensive game, more information is needed. A strategy describes the action of the player in every possible situation of the game. We form an example of a two-stage extensive game.

First, player I chooses between actions L and R. After observing player I's decision, player II decides between actions A and B if player I played L and between C and D if player I played R. Extensive games with two players can be illustrated with matrices similarly to the strategic games. An example of this game is given in Table 3.3. Instead of the actions, the columns and rows are now the strategies of the players. The utilities of the outcomes are also visible. All the relevant information is available in the matrix, but the chronology of events is hard to perceive. A better option is to form a tree illustrating the game as in Figure 3.1.

Table 3.3: An example of an extensive game in matrix form

Player II

Player I		A,C	A,D	B,C	B,D
	L	(4, 3)	(4, 3)	(1, 4)	(1, 4)
	R	(2, 2)	(3, 1)	(2, 2)	(3, 1)

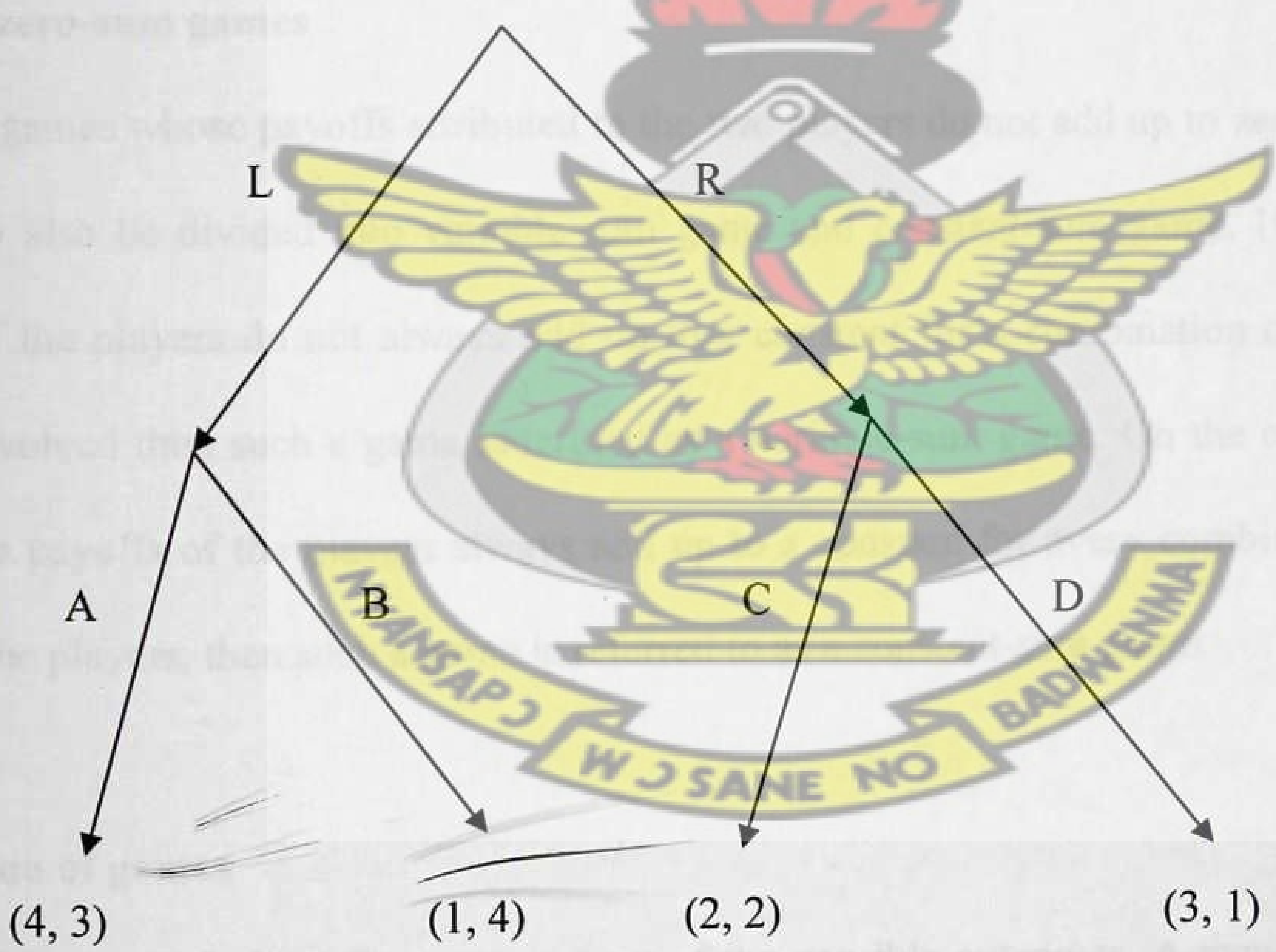

Figure 3.1: An example of an extensive game in tree form

As in the strategic games, the solution of an extensive game is a Nash equilibrium from which no player has an incentive to deviate unilaterally. The solution of the above game can be deduced easily. If player I chooses L it is optimal for player II to choose B. Respectively, if player I chooses R player II prefers C, hence the optimal strategy of player II is B;C. Since player I is intelligent, he can deduce that choosing L leads to utility 1 and choosing R to utility 2, hence his optimal strategy is to choose R.

3.2.6 Zero-sum games

Games can be divided according to their payoff structures. A game is called zero-sum game, if the sum of the utilities is zero in every outcome. Whatever is gained by one player is lost by the other player. Gambling is a typical zero-sum game. Zero-sum games are also called strictly competitive games.

3.2.7 Nonzero-sum games

These are games whose payoffs attributed to the two players do not add up to zero. Nonzero-sum game may also be divided into variable-sum game and constant-sum game. If the sum of the payoffs of the players do not always add up to a constant for a combination of actions by the players involved then such a game is termed as a variable-sum game. On the other hand, if the sum of the payoffs of the players always add up to a constant for every combination of actions taken by the players, then such a game is referred to as a constant-sum game.

3.3 Solution of games

In game theory, a solution of a game is a set of the possible outcomes. A game describes what actions the players can take and what the consequences of the actions are. The solution of a game is a description of outcomes that may emerge in the game if the players act rationally and intelligently. Generally, a solution is an outcome from which no player wants to deviate unilaterally.

3.3.1 Pareto Efficiency

An outcome of a game is Pareto efficient if there is no other outcome that makes every player at least as well off and at least one player strictly better off. That is, there is no other outcome that would make all players better off. Pareto optimal outcomes cannot be improved without hurting at least one player. In the prisoner's dilemma, all the outcomes except (Confess; Confess) are Pareto efficient. In the battle of the sexes, the outcomes in which both attend the same event are Pareto efficient. In implementation theory, the aim is typically to design a game that will end in a Pareto efficient outcome.

3.3.2 Nash Equilibrium

Nash equilibrium is one of the most common solution used in solving game theory problems. It is the point where no player can unilaterally change his strategy and get a better payoff. That is, no player can gain by a change of strategy as long as all the other players hold on to their strategies. In the battle of the sexes if the Husband is playing Opera then the best thing for the Wife is also to play Opera. Thus Opera is best-response for Wife against Opera. Similarly, Opera is best-response for Husband against Opera. Thus, at (Opera, Opera), neither players wants to take a different action. This is Nash Equilibrium.

3.4 Pure and Mixed Strategies

In game theory players can either use a pure or mixed strategy. A pure strategy defines a specific move or action that a player will follow in every possible attainable situation in a game. Such moves are or may not be random or drawn from a distribution. Mixed strategy on the other hand, consist of possible moves and a probability distribution which corresponds to how frequently each move is to played. A player would only use a mixed strategy when he is indifferent between several pure strategies and keeping the opponent guessing is desirable. That is, when the opponent can benefit from knowing the next move. When mixed strategies are used, the players maximize their expected payoff.

3.4.1 Optimal Solution of Two-Person Zero-Sum Games

A two person Zero-sum game is one of the form $(\{1,2\}, X, Y, u_1, -u_1)$. In a zero-sum when a player tries to maximize his payoff, he is also simultaneously minimizing the payoff of the other player. For this reason these games are often referred to as strictly competitive games. Player 1 is usually called the row player and player 2 is called the column player.

Let $X = \{1,2, \dots, m\}$ and $Y = \{1,2, \dots, n\}$.

Example

Player 1	Player 2			
	y_1	y_2	y_3	y_4
x_1	8, -8	2, -2	9,-9	5,-5
x_2	6,-6	5,-5	7,-7	18,-18
x_3	7,-7	3,-3	-4,4	10,-10

Since $u_1(x_i, y_j) = -u_1(x_i, y_j)\ \forall x_i \in X, \forall y_j \in Y$, such a payoff matrix can also be specified by a simpler matrix A where $a_{ij} = u_1(i, j)$. The above game can then be presented as

Player 1	Player 2			
	y_1	y_2	y_3	y_4
x_1	8	2	9	5
x_2	6	5	7	18
x_3	7	3	-4	10

An immediate generalization of a zero-sum game is a constant sum game: $(\{1,2\}, X, Y, u_1, u_2)$ such that $u_1(x_i, y_j) + u_2(x_i, y_j) = C, \forall x_i \in X, \forall y_j \in Y$ with C a given constant.

3.4.2 A Zero-sum Game with a Pure Strategy Nash Equilibrium

Definition: Saddle Point of a Matrix

Given a matrix $A = [a_{ij}]$, the element a_{ij} is called a saddle point of A if

$$a_{ij} \leq a_{il} \; \forall l = 1, \dots, n$$

$$a_{ij} \geq a_{kj} \; \forall k = 1, \dots, m$$

That is, the element a_{ij} is simultaneously a minimum in its row and a maximum in its column.

Proposition: For a matrix game with payoff matrix A, a_{ij} is a saddle point if and only if the outcome (i, j) is a pure strategy Nash equilibrium.

Proof: Let a_{ij} be a saddle point, then

a_{ij} is a row minimum and a_{ij} is a column maximum

That is the column player is playing a best response with respect to strategy i of the row player and the row player is playing a best response with respect to strategy j of the column player.

Then (i, j) is a Nash equilibrium.

The theorem below gives a necessary and sufficient condition for the existence of a pure strategy Nash equilibrium or saddle point.

Theorem: In a matrix $A = [a_{i,j}]$, let

$$u_R = \max_i \min_j a_{ij}$$

$$u_C = \min_j \max_i a_{ij}$$

Then the matrix A has a saddle point if and only if $u_R = u_C$

Example

We consider the following payoff matrix, which represents the row player's gain.

Player B

$$
\begin{array}{lccccl}
 & & & & & \text{Row Minimum} \\
 & \left[\begin{matrix} 8 & 2 & 9 & 5 \\ 6 & 5 & 7 & 18 \\ 7 & 3 & -4 & 10 \end{matrix}\right. & & & & \begin{matrix} 2 \\ 5 \leftarrow \text{Maximin} \\ -4 \end{matrix}
\end{array}
$$

$$
\begin{array}{lcccccc}
 & \multicolumn{4}{c}{\text{Player B}} & & \\
 & & & & & \text{Row Minimum} & \\
 & 8 & 2 & 9 & 5 & 2 & \\
\text{Player A} & 6 & 5 & 7 & 18 & 5 & \leftarrow \text{Maximin} \\
 & 7 & 3 & -4 & 10 & -4 & \\
\text{Column Maximum} & 8 & 5 & 9 & 18 & & \\
 & & \uparrow & & & & \\
 & & \text{Minimax} & & & &
\end{array}
$$

$$u_R = \max_i \ \min_j a_{ij}$$

$$= \min_j a_{ij} = \{2, 5, -4\}$$

$$= \max_i \{2, 5, -4\}$$

$$u_R = 5$$

$$u_C = \min_j \ \max_i a_{ij}$$

$$= \max_i a_{ij} = \{8,5,9,18\}$$

$$= \min_j \{8,5,9,18\}$$

$$u_C = 5$$

Therefore $u_R = u_C$ with $a_{22} = 5$ is the saddle point. This implies that the value of the game is 5.

3.5 MIXED STRATEGIES

If no saddle point is found in a game there is no single safest strategy for each player. In this case a mixture of strategies is used. A mixed strategy for player A consists of a set of probabilities x_i $(for\ i = 1\ to\ m)$, such that $\sum_1^m x_i = 1$. Each x_i represents the probability of using a pure strategy A_i. The objective for player A is to obtain an expected value V (the value of the game) as large as possible. He can only be sure of the expected value V if his strategy will guarantee that, regardless of what strategy his opponent adopts; he will obtain an expectation of V or more. A mixed strategy for X is a vector

$$X = \begin{bmatrix} x_i \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_m \end{bmatrix}$$

where x_i, the probability of selecting the *ith* strategy, satisfy $x_i \geq 0, i = 1 \ldots n$ and

$$\sum_{i=1}^{m} x_i = 1$$

A mixed strategy for Y is a vector y, which is similarly defined. Let A be the payoff matrix and let X' be the transpose of X. Then the payoff to X from strategy X is easily shown to be $X'AY$.

The failure of the minimax-maximin (pure) strategies to give an optimal solution to the game has led to the idea of using mixed strategies. Each player, instead of selecting a pure strategy only, may play all his strategies according to a predetermined set of probabilities.

Let $x_1, x_2, \ldots, x_m$ and $y_1, y_2, \ldots, y_n$ be the row and column probabilities by which A and B, respectively, select their pure strategies. Then

$$\sum_{i=1}^{m} x_i = \sum_{j=1}^{n} y_j = 1,$$

$x_i, y_j \geq 0$ for all i and j. Thus, if a_{ij} represents the (i,j) entry of the game matrix, then it appears as in the following matrix

Player B

Player A

$$\begin{array}{c|c} & \begin{matrix} y_1 & y_2 & \cdots & y_n \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_m \end{matrix} & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

The solution of the mixed strategy problem is based also on the minimax-criterion. The only difference is that A selects x_i that maximize the smallest expected payoff in a column, whereas B selects y_j that minimize the largest expected payoff in a row. Mathematically, the minimax criterion for a mixed strategy case is given as follows.

Player A selects $x_i (x_i \geq 0, \sum_{i=1}^{m} x_i = 1)$, that will yield

$$\max_{x_i} \left\{ \min \left(\sum_{i=1}^{m} a_{i1} x_i , \sum_{i=1}^{m} a_{i2} x_i , \ldots, \sum_{i=1}^{m} a_{in} x_i \right) \right\}$$

Such that

$$\sum_{i=1}^{m} a_{ij} x_i \geq V$$

and

Player B selects $y_j (y_j \geq 0, \sum_{j=1}^{n} y_j = 1)$ that will yield

$$\min_{y_j} \left\{ \left(\max \left(\sum_{j=1}^{n} a_{1j} y_j , \sum_{j=1}^{n} a_{2j} y_j , \ldots, \sum_{j=1}^{n} a_{mj} y_j \right) \right) \right\}$$

Such that

$$\sum_{j=1}^{n} a_{ij} y_j \leq V$$

These values are referred to as the maximin and minimax expected payoffs, respectively. As in the pure strategies case, we have the relationship

Minimax expected payoff $\geq$ maximin expected payoff

When x_i and y_j correspond to the optimal solution. The equality holds and the resulting values become equal to the (optimal) expected value of the game.

3.6 Formulation of Two-Person Zero-Sum Games Using Linear Programming

Game theory bears a strong relationship to linear programming since every finite two-person zero-sum game can be expressed as a linear programming problem. A two-person zero-sum game can be converted into an equivalent linear programming problem. In a two-person zero-sum game the objective of one player is to maximize his expected gain while the other player tries to minimize his expected loss. In other words, the aim of the players in game theory is either to maximize or minimize gains. The objective of the game is a linear function of the decision variables. In linear programming the players wish to optimize their gain subject to given constraints and the variables must be always non-negative. When both players select the optimal strategies in a two-person zero-sum game, one player's highest expected gain is equal to the other player's lowest expected loss. Therefore the value of the maximization problem is exactly the same as that of the minimization problem. This is the same as the primal-dual relationship in linear programming. For example if player A's optimum mixed strategies satisfy

$$\max_{x_i}\left\{\min\left(\sum_{i=1}^{m} a_{i1}x_i, \sum_{i=1}^{m} a_{i2}x_i, \dots, \sum_{i=1}^{m} a_{in}x_i\right)\right\}$$

Subject to the constraints $x_i \geq 0, i = 1, \dots, m$ and $\sum_1^m x_i = 1$, then we can then formulate a linear programming for this as:

Let

$$V = \min\left(\sum_{i=1}^{m} a_{i1}x_i, \sum_{i=1}^{m} a_{i2}x_i, \dots, \sum_{i=1}^{m} a_{in}x_i\right)$$

The problem then becomes

$$maximize\ z = v$$

Subject to

$$\sum_{i=1}^{m} a_{i1}x_i \geq V, j = 1,2, \dots, n$$

$$\sum_{i=1}^{m} x_i = 1$$

$$x_i \geq 0, i = 1, \dots m$$

V represents the value of the game in this case.

For example, if player B were to adopt B_1, then $A's$ strategy must be such that

$$a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + \dots + a_{m1}x_m \geq V$$

Similarly if player B uses B_2, then to guarantee V, player A must have

$$a_{12}x_1 + a_{22}x_2 + a_{32}x_3 + \dots\dots\dots + a_{m2}x_m \geq V$$

A similar condition holds for any strategy player B may play. Hence the linear programming problem for player A is

$$Maximize\ V$$

$Subject\ to$

$$a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + \cdots + a_{m1}x_m \geq V$$

$$a_{12}x_1 + a_{22}x_2 + a_{32}x_3 + \cdots + a_{m2}x_m \geq V$$

.

.

.

$$a_{1n}x_1 + a_{2n}x_2 + a_{3n}x_3 + \cdots + a_{mn}x_m \geq V$$

$$x_1 + x_2 + x_3 + \cdots + x_m = 1$$

$$all\ x_i \geq 0$$

The last equation guarantees that the probabilities add up to one. The solution to this problem gives the equilibrium mixed strategy $(x_1, x_2, \dots\dots x_m)$ for player A and the value of the game V. If assuming $V > 0$, the constraints of the linear program becomes

$$a_{11}\frac{x_1}{V} + a_{21}\frac{x_2}{V} + a_{31}\frac{x_3}{V} + \cdots + a_{m1}\frac{x_m}{V} \geq 1$$

$$a_{12}\frac{x_1}{V} + a_{22}\frac{x_2}{V} + a_{32}\frac{x_3}{V} + \cdots + a_{m2}\frac{x_m}{V} \geq 1$$

.

.

.

$$a_{1n}\frac{x_1}{V} + a_{2n}\frac{x_2}{V} + a_{3n}\frac{x_3}{V} + \ldots + a_{mn}\frac{x_m}{V} \geq 1$$

$$\frac{x_1}{V} + \frac{x_2}{V} + \frac{x_3}{V} + \ldots + \frac{x_m}{V} = \frac{1}{V}$$

$$all\ x_i \geq 0$$

Let

$X_i = \frac{x_i}{V}, i = 1,2, \ldots m$

.

Since

$$Max\ V = min\frac{1}{V} = min\{X_1 + X_2 + X_3 \ldots + X_m\}$$

the problem becomes

$$Minimize\ Z = \{X_1 + X_2 + X_3 \ldots + X_m\}$$

Subject to

$$a_{11}X_1 + a_{21}X_2 + a_{31}X_3 + \cdots + a_{m1}X_m \geq 1$$

$$a_{12}X_1 + a_{22}X_2 + a_{32}X_3 + \cdots + a_{m1}X_m \geq 1$$

.

.

.

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$$a_{1n}X_1 + a_{2n}X_2 + a_{31}X_3 + \cdots + a_{mn}X_m \geq 1$$

all

$$X_i \geq 0, for\ i = 1,2,3, \dots, m$$

Players B's problem is given by

$$\min_{y_j}\left\{(\max\left(\sum_{j=1}^{n} a_{1j}y_j, \sum_{j=1}^{n} a_{2j}y_j, \dots, \sum_{j=1}^{n} a_{mj}y_j\right)\right\}$$

Subject to the constraints

$$y_1 + y_2 + \cdots + y_n = 1$$

$$y_j \geq 0, for\ j = 1, \dots, n$$

This can also be expressed as a linear program as follows

$$Maximise\ W = Y_1 + Y_2 + \cdots + Y_n$$

subject to

$$a_{11}Y_1 + a_{12}Y_2 + a_{13}Y_3 + \cdots + a_{1n}Y_n \leq 1$$

$$a_{21}Y + a_{22}Y_2 + a_{23}Y_3 + \cdots + a_{2n}Y_n \leq 1$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

$$a_{m1}Y + a_{m2}Y + a_{m3}Y + \cdots + a_{mn}Y_n \leq 1$$

all

$$Y_j \geq 0, for\ j = 1,2,3, \ldots, n$$

Where

$$W = \frac{1}{V}, Y_j = \frac{y_j}{V}, j = 1,2, \ldots, n$$

The dual of the linear programming problem for player A is the primal problem from player $B's$ point of view, thus the optimal solution of one problem automatically yields the optimal solution to the other. Player $B's$ problem can be solved by the regular simplex method, and player $A's$ problem is solved by the dual simplex method.

3.7 The Lemke-Howson Algorithm

This section illustrates the solution of game problems by the Lemke-Howson algorithm.

Information for this section is taken from Lemke and Howson (1964) and von Stengel (2002)

The algorithm can be used to find the Nash equilibrium of 2-player, finite, strategic games.

This algorithm originally apperared in the paper of Lemke and Howson (1964). The Lemke-Howson algorithm resembles the simplex algorithm especially as the algorithm consists of iterated pivoting.

Let player 1 have m actions labeled $M = \{1, \dots, m\}$ and player 2 have n actions labeled $N = \{m+1, \dots, m+n\}$. The payoff for the two-player game is represented with $m \times n$ matrices. The matrix A represents the payoff for player 1, and the matrix B represents the payoff for player 2. We think of player 1 picking rows and player 2 picking columns; so a mixed strategy for player 1 is an m-element row vector that is stochastic (the entries are nonnegative and sum to 1) and similarly a mixed strategy for player 2 is a n-element stochastic column vector. With these notations, the payoff to player 1 (resp. 2) under mixed action profile (x^T, y) is $x^T Ay$ $(resp.\ x^T By)$. We use X to denote the set of all mixed strategies of player 1 and Y is defined similarly.

Assumption: We assume that all entries of A and B are nonnegative, and that A has no all-zero columns, and B has no all-zero rows.

The basic idea of the Lemke-Howson algorithm is guessing the support of the equilibrium, we then maintain a single guess as to what the supports should be, and in each iteration we change the guess only a little bit. The easiest description of the algorithm, and the easiest proof of Nash's theorem for two player games, relies on two polytopes which we define as follows. Let B_j denote the column of B corresponding to action j and let A^i denote the row of A corresponding to action i.

$$P_1 = \{x \in \mathbb{R}^M \mid (\overline{i} \in M: x_i \geq 0) \& (j \in N: x^T B_j \leq 1)\}$$

$$P_2 = \{y \in \mathbb{R}^N \mid (i \in M: y_i \geq 0) \& (i \in M: A^i y \leq 1)\}$$

We don't restrict x and y to be stochastic here, only nonnegative. For a nonzero nonnegative x, we can normalize it to a stochastic vector $\mathrm{nrml}(x)$ as follows

$$\mathrm{nrml}(x) = \left(\sum_i x_i\right)^{-1} x.$$

The inequalities that define P_1 have the following meaning:

- If $x \in P_1$ meets $x_i \geq 0$ with equality then i is not in the support of x
- If $x \in P_1$ meets $x^T B_j \leq 1$ with equality then j is a best response to $\mathrm{nrml}(x)$

Let us say that $x \in P_1$ has label k, where $k \in M \cup N = \{1, \dots, m+n\}$, if either $k \in M$ and $x_k = 0$, or $k \in N$ and $x^T B_k = 1$. Similarly $y \in P_2$ has label k if either $k \in N$ and $y_k = 0$, or $k \in M$ and $A^k y = 1$. As a consequence of the Support Characterization, we have the following.

Theorem: Suppose that $x \in P_1$ and $y \in P_2$, and neither x nor y is the all-zero vector, then x and y together have all labels from 1 to k if and only if (nrml (x), nrml (y)) is a Nash equilibrium. All Nash equilibrium arise in this way.

The Lemke-Howson algorithm is stated as follows:

The Lemke-Howson Algorithm

1. Let x (resp. y) be the all-zero vector **0** of length m (resp. n)
2. Let k_0 be any label x
3. Let $k = k_0$
4. **loop**

5. In P_1, remove the label k from x; let x' be the new vertex and k' the label added

6. In $x = x'$

7. If $k' = k_0$, stop looping

8. In P_2, remove the label k' from y; let y' be the new vertex and k'' the label added

9. Let $y = y'$

10. If $k'' = k_0$, stop looping

11. Let $k = k''$

12. **End loop**

13. Output (nrml(x),nrml(y))

Tableau Method and Example

To apply the tableau method to find Nash equilibrium using the Lemke-Howson algorithm, we use the following four steps.

(a) Preprocessing

(b) Initialization of tableaux

(c) Repeated pivoting

(d) Recover Nash equilibrium from final tableaux

In the tableau method, we introduce *slack variables*, and use the terminology basic and *non − basic variables*. For our purposes the *basic variables* and *set of labels* have opposite meanings since labels imply a tight inequality and basic variables are not tight. Hence, "enter the basis" means the same as "label is removed" and "leaves the basis" means that "label added."

Step 1. Preprocessing

Recall that iterated elimination of strictly dominated strategies preserves all Nash equilibria. Elimination reduces the size of the game, and therefore will reduce the amount of work involved with the pivoting later on. Hence, one should apply this elimination before beginning.

Next, to ensure that the game satisfies the conditions of Theorem 1, add a suitably large constant to the entries of each matrix.

Step 2. Initialization of Tableaux

For the purposes of solving the game we need two tableaux, one for each player. Let r_i be the slack in the constraint $A^i y \leq 1$ and let s_j be the slack in the constraint $x^T B_j \leq 1$. We then obtain the system

$$Ay + r = 1$$

$$B^T x + s = 1$$

$$x, y, r, s \text{ are nonnegative}$$

In the initial tableaux, the basis is $\{r_i \mid i \in M\} \cup s_j \mid j \in N\}$ and so we write the equations so as to solve for them.

Example

P_1 \ P_2	4	5	6
1	1,2	3,1	0,0
2	0,1	0,3	2,1
3	2,0	1,0	1,3

Table 3.4: Payoff table for Player I (A)

	y_4	y_5	y_6
x_1	1	3	0
x_2	0	0	2
x_3	2	1	1

Table 3.5: Payoff table for Player II (B)

	y_4	y_5	y_6
x_1	2	1	0
x_2	1	3	1
x_3	0	0	3

The entries are positive, no strict domination occurs, and furthermore there are no pure Nash equilibria.

The initial tableaux are $r = 1 - Ay$,

$$r_1 = 1 - y_4 - 3y_5 \qquad [A1]$$

$$r_2 = 1 \qquad\qquad - 2y_6 \qquad [A2]$$

$$r_1 = 1 - 2y_4 - y_5 \qquad - y_6 \qquad [A3]$$

and $s = 1 - B^T x$,

$$s_4 = 1 - 2x_1 - x_2 \qquad [B1]$$

$$s_5 = 1 - x_1 - 3x_2 \qquad [B2]$$

$$s_6 = 1 \qquad - x_2 - 3x_3 \qquad [B3]$$

Step 3.Pivoting

We arbitrarily choose some x or y variable to bring in to the basis, corresponding to the arbitrary choice k_0 of the label that we remove. Let's bring x_1 in. By considering the min-ratio rule (i.e., looking at the coefficients of x_1 in the $[B]$ tableau) it is s_4 that must leave the basis. Therefore we solve $[B1]$ for x_1, obtaining a new equation $[B']$, and we substitute the new equation into $[B2]$ and $[B3]$ obtaining

$$x_1 = 1/2 - 1/2s_4 - 1/2x_2 \qquad [B'1]$$

$$s_5 = 1/2 + 1/2s_4 - 5/2x_2 \qquad [B'2]$$

$$s_6 = 1 \qquad - x_2 - 3x_3 \qquad [B'3]$$

The main feature the Lemke-Howson algorithm is that the variable which just left the basis determines the variable to enter the basis next. There are $m + n$ complementary pairs of variables:$\{r_i x_i\}$ for $i \in M$ and $\{s_j y_j\}$ for $i \in N$. Each pair corresponds (in an inverse sense) to the labels we mentioned earlier, e.g., x_i is basic $iff\ x$ does not have label i and s_j is basic $iff\ x$ does not have label j.

The $m + n$ complementary conditions

$r_i x_i = 0,\ i \in M \qquad s_j y_j\ ,\ i \in N.$

tells us when to stop. Initially, all complementarity conditions are satisfied. We keep performing pivots until the complementarity conditions are again satisfied. Equivalently, we pivot until, between the two tableaux, in each complementary pair of variables, exactly one is basic and exactly one is non-basic.

In this case, since s_4 just left the basis, y_4 must be brought in. Examining the [A] tableau we see that r_3 is the winner of the min-ratio, and therefore leaves the basis. We obtain the following:

$$r_1 = 1/2 + 1/2r_3 - 5/2y_5 + 1/2y_6$$

$$r_2 = 1 \qquad - 2y_6$$

$$y_4 = 1/2 - 1/2r_3 - 1/2y_5 - 1/2y_6$$

Since r_3 left, now x_3 enters the other tableau, and by the min-ratio rule s_6 leaves.

$$x_1 = 1/2 - 1/2s_4 - 1/2x_2$$

$$s_5 = 1/2 + 1/2s_4 - 5/2x_2$$

$$x_3 = 1/3 \qquad - 1/3x_2 - 1/3s_6$$

Since s_6 left, now y_6 enters, and by the min-ratio rule r_2 leaves.

$$r_1 = 3/4 + 1/2r_3 - 5/2y_5 - 1/4r_2$$

$$y_6 = 1/2 \qquad - 1/2r_2$$

$$y_4 = 1/4 - 1/2r_3 - 1/2y_5 + 1/4r_2$$

Since r_2 left, now x_2 enters, and by the min-ratio rule s_5 leaves.

$$x_1 = 2/5 - 3/5s_4 + 1/5s_5$$

$$x_2 = 1/5 + 1/5s_4 \quad - 2/5s_5$$

$$x_3 = 4/15 - 1/15s_4 + 2/15s_5 - 1/3s_6$$

Since s_5 left, now y_5 enters, and by the min-ratio rule r_1 leaves.

$$y_5 = 3/10 + 1/5r_3 - 2/5r_1 - 1/10r_2$$

$$y_6 = 1/2 \qquad - 1/2r_2$$

$$y_4 = 1/10 - 3/5r_3 + 1/5r_1 + 3/10r_2$$

Step 4. Output

Since x_1 was the initial variable to enter the basis, and r_1 just left, the complementarity conditions are now satisfied. (More generally, if x_i was the initial variable to enter, we stop when x_i or its complement leaves). In a tableau, we obtain values for the basic variables by setting the non-basic variables to zero. Hence the variables values are

$r = (0,0,0), s = (0,0,0), x = (2/5,1/5,4/15), y = (1/10,3/10,1/2).$

Therefore, the Nash equilibrium we just found is

$(\text{nrml}(x), \text{nrml}(y)) = ((6/13,3/13,4/13), (1/9,3/9,5/9)).$

The above problem is also solved on Matlab by developing Matlab codes which is shown in appendix II. The matlab codes developed were based on the algorithm below

Step 1: Enter player 1 and 2 matrix as A and B

Step 2: Create a tableau X for A as $r = 1 - Ay$ and B as $s = 1 - B^T x$.

Step 3: Choose a pivot by either choosing player 1 (A) or player 2 (B)

Step 4: From step 3 select a basis thus if A then pick y otherwise pick x in the second column and on the first row. Divide that entire row by the coefficient of the basis. Find min(in the 2nd column) and choose that row that recorded the minimum. This row then becomes the next ~~column to be~~ selected in the other matrix.

Step 5: Repeat step 4 if a tie is found as the minimum value in the column then check if that number corresponding to the row has already been selected.

Step 6: If yes choose the unselected value and go to step 4 until all rows in both players (A and B) has been used up.

Step 7: Create a column vector from r and s by choosing all the first column in A and B respectively.

Step 8: Normalize this vectors.

Matlab Solution to the above problem

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 2 \\ 2 & 1 & 1 \end{bmatrix} \quad and \quad B = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

1[st] iteration

$$s = \begin{bmatrix} 1/2 & -1/2 & -1/2 & 0 \\ 1/2 & 1/2 & -5/2 & 0 \\ 1 & 0 & -1 & -3 \end{bmatrix}$$

2[nd] iteration

$$r = \begin{bmatrix} 1/2 & -1/2 & -5/2 & 1/2 \\ 1 & 0 & 0 & -2 \\ 1/2 & -1/2 & -1/2 & -1/2 \end{bmatrix}$$

3[rd] iteration

$$s = \begin{bmatrix} 1/2 & -1/2 & -1/2 & 0 \\ 1/2 & 1/2 & -5/2 & 0 \\ 1/3 & 0 & -1/3 & -1/3 \end{bmatrix}$$

4[th] iteration

$$r = \begin{bmatrix} 3/4 & 1/2 & -5/2 & 1/4 \\ 1/2 & 0 & 0 & -1/2 \\ 1/4 & -1/2 & -1/2 & 1/4 \end{bmatrix}$$

5^{th} iteration

$$S = \begin{bmatrix} 2/5 & -3/5 & 1/5 & 0 \\ 1/5 & 1/5 & -2/5 & 0 \\ 4/15 & -1/15 & 2/15 & -1/3 \end{bmatrix}$$

6^{th} iteration

$$r = \begin{bmatrix} 3/10 & 1/5 & -2/5 & -1/10 \\ 1/2 & 0 & 0 & -1/2 \\ 1/10 & -3/5 & 1/5 & 3/10 \end{bmatrix}$$

nrml_x =(6/13, 3/13, 4/13) and nrml_y = (1/9, 1/3, 5/9)

3.8 The Interior-Point Method

Iinterior-point methods were widely used in the past in the form of barrier methods. In linear programming the simplex method dominated, mainly due to inefficiencies of the barrier methods. Interior-point methods became quite popular again after 1984, when Narendra Karmarkar announced a fast polynomial-time interior-point method for nonlinear programming (Karmarkar, 1984). One advantage of interior-point methods is that the number of iterations for a problem with large functional constraints is relatively small compared to that of the simplex algorithm. "For example, a problem with 10,000 functional constraints probably will require well under 100 iterations. Even considering the very substantial computer time per iteration needed for a problem of this size, such a small number of iterations makes the problem quite tractable. By contrast, the simplex method might need 20,000 iterations and so might not finish within a reasonable amount of computer time. Therefore, interior-point algorithms often are faster than the simplex method for such huge problems." (Hillier and Lieberman, 2000) There are several classes of the interior-point methods today. These methods can broadly be classified into three; Affine-Scaling methods, Potential reduction methods and Path-following (or Central trajectory) methods. However, by early 1990s one subclass of the Path-following called the primal-dual methods had distinguished themselves as being the most efficient in practice and implementation. Most computer software today that employ the interior-point method to solve linear programming problems exploit the techniques of this subclass of path-following interior-point algorithms.

3.9 Primal-Dual Interior-Point Methods

We discuss in this section the main ideas behind primal-dual interior-point methods, laying emphasis on the practically efficient class, the path-following methods.

Consider the Linear Programming problem in the standard form as:

$$\min_x c^T x, subject\ to\ Ax = b, x \geq 0 \qquad \text{(PP)}$$

Where c and x are vectors in $\mathbb{R}^n$, b is a vector in $\mathbb{R}^m$ and A is an $m \times n$ matrix. The dual to problem (PP) is given by

$$\max_y b^T, subject\ to\ A^T y \leq c \qquad (1)$$

or in standard form

$$\max_{(y,s)} b^T y, subject\ to\ A^T y + s = c, s \geq 0 \qquad \text{(DP)}$$

Where y is a vector in $\mathbb{R}^m$ and s is a vector in $\mathbb{R}^n$.

We define some terminologies here:

Definition: The set of primal feasible points is the set of feasible solutions to the primal problem (PP) defined by $P = \{x : Ax = b, x \geq 0\}$

Definition: The set of dual feasible points is the set of feasible solutions to the dual problem (DP) defined by $D = \{(y, s): A^T y + s = c, s \geq 0\}$

Definition: The set P^0 of primal interior points is the set of primal feasible points excluding those on the boundary of the feasible region, defined as $P^0 = \{x \in P: x > 0\}$

Definition: The set D^0 of dual interior points is the set of dual feasible points excluding those on the boundary of the feasible region. This set is defined by $D^0 = \{(y, s) \in D: s > 0\}$

Definition: The set of feasible primal-dual points is the Cartesian product of the set of primal feasible points and the set of dual feasible points. This set is defined by $F = P \times D$

Definition: The set of primal-dual interior points is the set of feasible primal-dual points that are not on the boundary of the feasible region. This set is a subset of the set of feasible primal-dual points defined by $F^0 = \{(y, s) \in F: (y, s) > 0\}$

We state here some lemmas concerning the primal-dual feasible interior points.

Lemma 3.8.1 $(Weak\ Duality)$: $For\ every\ \omega = (x, y, s) \in F\ we\ have\ c^T x \geq b^T y. That\ is\ the$

$primal\ and\ dual\ feasible\ interior\ points\ bound\ each\ other.$

Proof Since $x \in P\ and\ (x, y) \in D\ then\ we\ have:$

$c^T x - b^T y = x^T - x^T A^T y = x^T(c - A^T y) = x^T s \geq 0$

$\Rightarrow c^T x - b^T y \geq 0\ and\ thus\ c^T x \geq b^T y$

The difference $c^T x - b^T y$ is called the duality gap. When the duality gap is zero then the primal-dual solution is optimal.

Lemma 3.8.2 ($Strong\ Duality$): $A\ point\ x \in P\ is\ an\ optimal\ solution\ if\ and\ only\ if\ there$

$exist\ (y, s) \in D such\ that\ c^T x = b^T y$

The problem (P) has a solution if $\mathcal{P} \neq \emptyset\ and\ \mathcal{D} \neq \emptyset$ and both (P) and (D) have an optimal solution $\omega^* = (x^*, y^*, s^*)$. If either $\mathcal{P}$ or $\mathcal{D}$ is empty then either the other is empty too or it is unbounded. The difference

$$c^T x - b^T y = x^T s \qquad (3)$$

is called the complementarity gap which measures the distance of the current primal-dual point from the optimal solution. Thus a zero complementarity gap implies optimality. Hence, the duality gap and the complementarity gap achieve equal values at optimal feasible solutions.

To proceed with the development of the primal-dual interior-point methods we make the assumption that $\neq 0\ and\ \mathcal{D} \neq 0$. This assumption is known as the interior-point assumption.

The primal-dual solutions to (PP) and (DP) can be shown to satisfy the following set of equations known as the Karush-Kuhn-Tucker conditions.

$$\boldsymbol{A^T y + s = c} \qquad (2a)$$

$$\boldsymbol{Ax = b} \qquad (2b)$$

$$\boldsymbol{XSe = 0} \qquad (2c)$$

$$\boldsymbol{(x, s) \geq 0} \qquad (2d)$$

Where $\boldsymbol{X}$ is used to denote the diagonal matrix with diagonal $\boldsymbol{x}$

$$X = \begin{pmatrix} [x]_1 & & \\ & [x]_2 & \\ & & [x]_n \end{pmatrix}$$

And analogous notation is used for the other quantities. Vector $\boldsymbol{e}$ is used to denote a vector of all ones, whose dimension usually varies depending on the context. Condition (2a) is known as dual feasibility, condition (2b) is known as primal feasibility, and (2c) corresponds to complementarity.

A point $\boldsymbol{x}^*$ is a solution of the problem PP if and only if there exists vectors $\boldsymbol{s}^*$ and $\boldsymbol{y}^*$ such that conditions (2) hold for $(x, y, s) = (x^*, y^*, s^*)$. On the other hand, a point (y^*, s^*) is a solution of the problem DP if and only if there exists vector x^* such that conditions (2) hold for

$(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{s}) = (\boldsymbol{x}^*, \boldsymbol{y}^*, \boldsymbol{s}^*)$. Vector (x^*, y^*, s^*) is called primal-dual solution.

Primal-dual interior-point methods find primal-dual solutions (x^*, y^*, s^*) by applying variants of Newtons's method to conditions (2a)-(2c), and modifying search directions and step lengths, so that (2d) is strictly satisfied at each iteration. It is more convenient to write the optimality conditions in terms of a mapping $\mathcal{F}$ from

$$\mathcal{F}(x, \lambda, s) = \begin{bmatrix} A^T\lambda + s - c \\ Ax - b \\ XSe \end{bmatrix} = 0 \qquad (4)$$

$$(x, s) \geq 0$$

Where $\boldsymbol{X} = diag(x_1, x_2, \dots, x_n), \boldsymbol{S} = diag(s_1, s_2, \dots, s_n) and\ \boldsymbol{e} = (1,1, \dots, 1)^T$

Newton's method forms a linear model for $\mathcal{F}$ around the current iterate $(\boldsymbol{x}^{(k)}, \boldsymbol{y}^{(k)}, \boldsymbol{s}^{(k)})$, and obtains a search direction $(\Delta\boldsymbol{x}, \Delta\boldsymbol{y}, \Delta\boldsymbol{s})$ by solving

$$J(x,\lambda,s) = \begin{bmatrix} \Delta x \\ \Delta\lambda \\ \Delta s \end{bmatrix} = -F(x,\lambda,s)\,, \tag{5}$$

where J is the Jacobian of $\mathcal{F}$. Newton's method is applicable for systems of equalities not inequalities. If we assume that we have strictly feasible iterates, the previous equation becomes

$$\begin{bmatrix} 0 & A^T & I \\ A & 0 & 0 \\ S & 0 & X \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta\lambda \\ \Delta s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -XSe \end{bmatrix} \tag{6}$$

The next feasible iterate is given by $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{s}) + \boldsymbol{\alpha}(\boldsymbol{\Delta x}, \boldsymbol{\Delta\lambda}, \boldsymbol{\Delta s})$

for a line search parameter $\alpha \in (0,1]$. $\boldsymbol{\alpha}$ is chosen such that the new iterate is kept within the interior of the feasible region. If the next iterate is strictly feasible, then we can apply Newton's method iteratively.

Central path

The central path is an important aspect of interior-point methods that will help us to establish a general ~~algo~~rithm for primal-dual methods.

A central path is an arc of strictly feasible points that is parameterized by a positive scalar ρ, and each point $(x(\rho), \lambda(\rho), s(\rho))$ that belongs to the path, solves

$$A^T y + s = c \tag{7a}$$

$$Ax = b \tag{7b}$$

$$[x]_i[s]_i = \rho, i = 1, \dots, n \tag{7c}$$

$$(x, s) > 0 \tag{7d}$$

Equations (7) differ from equations (2) only in the complementarity condition (7c), where the condition now is that products $[x]_i[s]_i$ have the same value for all $i = 1, \dots, n$. But as ρ goes to zero, then equation (7) approximate equation (2). Therefore points on the central path converge to a primal-dual solution of the linear problem, as $\rho \to 0^-$.

The Barrier Problem

In the primal problem (PP), the nonnegativity constraints $x \geq 0$ can be replaced, by adding a barrier term in the objective function, that looks like

$$B(x) = \sum_{j=1}^{n} log x_j$$

This barrier term is called the logarithmic barrier term, it is finite as long as x_j is positive, and approaches negative infinity as x_j approaches zero. The primal problem can now be rewritten as

$$\max_x c^T x - \rho B(x) \tag{PLBP}$$

$subject\ to$

$$Ax = b$$

For $x > 0$. The lagrangian for this problem is

$$\mathcal{L}(x,y) = c^T x - B(x) + y^T(b - Ax)$$

The KKT conditions for problem (PLBP) are

$$c - \rho X^{-1}e - A^T y$$

$$b - Ax = 0$$

for $x > 0$. If we introduce an extra vector defined as $s = \rho X^{-1}e$, we can rewrite the KKT conditions as

$$A^T y + s = c$$

$$Ax = b$$

$$s = \rho X^{-1}e$$

or, if we multiply the last equation to the left by X

$$A^T y + s = c \tag{8a}$$

$$Ax = b \tag{8b}$$

$$XSe = \rho e \tag{8c}$$

for $(x, s) > 0$. This system of equations is the same as equations (7).

Primal-Dual Interior-Point Algorithm

The Newton equations (6) have been derived for the first order optimality conditions of problem (PP), under the assumption that the iterates (x, s) are strictly feasible. In a similar fashion, the Newton equations for the logarithmic barrier reformation of problem (PP) are

$$\begin{bmatrix} 0 & A^T & I \\ A & 0 & 0 \\ S & 0 & X \end{bmatrix} \begin{bmatrix} \Delta x \\ \nabla\lambda \\ \Delta s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -XSe + \tau\gamma e \end{bmatrix} \qquad (9)$$

In these equations we have used $\rho = \tau\gamma$, where $\tau \in (0,1]$ is a centering parameter and

$$\gamma = \frac{x^T s}{n} \qquad \text{(DM)}$$

is a duality measure, i.e. it measures the average value of the complementarity products $[x]_i[s]_i$.

Algorithm 1 General primal-dual interior-point method

1. Determine $(x^{(0)}, y^{(0)}, s^{(0)})$ strictly feasible

2. Set $k := 0$

3. **repeat**

4. Set $\tau^{(k)} \in [0,1]$ and $\gamma^{(k)} = \frac{x^T s}{n}$

5. Solve system (9) to obtain $(\Delta x^{(k)}, \Delta y^{(k)}, \Delta s^{(k)})$

6. Set

$$(x^{(k+1)}, y^{(K+1)}, s^{(k+1)}) = (x^{(k)}, y^{(k)}, s^{(k)}) + \alpha^{(k)}(\Delta x^{(k)}, \Delta y^{(k)}, \Delta s^{(k)})$$

choosing $\alpha^{(k)}$ so that $(x^{(k+1)}, y^{(k+1)}, s^{(k+1)}) > 0$.

7. Set $k := k + 1$

8. **until** Convergence

The algorithm is implemented on MATLAB as the lipsol function. In order to use the interior-point method we change the options in the linprog function by turning on the LargeScale option. That is we set options=optimset('LargeScale' 'on');. Turning on the 'LargeScale' option makes MATLAB use the interior-point algorithm whilst turning it off makes MATLAB use the simplex algorithm.

Example

We consider a (3×3) two player zero sum game as shown below

Player B

$$\text{Player A}\quad \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 8 & 4 & 2 \\ 2 & 8 & 4 \\ 1 & 2 & 8 \end{bmatrix} \end{array}$$

Player B's game is to minimize his expected loss by formulating the linear programming problem as follows

$$\begin{bmatrix} 8 & 4 & 2 \\ 2 & 8 & 4 \\ 1 & 2 & 8 \end{bmatrix} \begin{matrix} y_1 \\ y_2 \\ y_3 \end{matrix}$$

$$8y_1 + 4y_2 + 2y_3 \le V$$

$$2y_1 + 8y_2 + 4y_3 \le V$$

$$y_1 + 2y_2 + 8y_3 \le V$$

Player B's optimum solution is obtained by solving the above linear programming

We let $y_4 = V$

Minimize y_4

subject to

$$8y_1 + 4y_2 + 2y_3 \le y_4$$

$$2y_1 + 8y_2 + 4y_3 \le y_4$$

$$y_1 + 2y_2 + 8y_3 \le y_4$$

$$y_1 + y_2 + y_3 = 1$$

and

$$y_1 \ge 0, y_2 \ge 0, y_3 \ge 0$$

The above system of equations is rearranged as follows

$$8y_1 + 4y_2 + 2y_3 - y_4 \leq 0$$

$$2y_1 + 8y_2 + 4y_3 - y_4 \leq 0$$

$$y_1 + 2y_2 + 8y_3 - y_4 \leq 0$$

$$y_1 + y_2 + y_3 = 1$$

$$y_1 \geq 0, y_2 \geq 0, y_3 \geq 0$$

The above problem is then solved on matlab using linprog function as shown in appendix III.

The following results are then generated by matlab

Y = { 0.3111, 0.2444, 0.4444, 4.3556}

rats(Y) = {14/45, 11/45, 4/9, 196/45}

That is $\{y_1 = 14/45, y_2 = 11/45, y_3 = 4/9\}$ and the value of the game is $y_4 = V = 196/45$

On the other hand, Player A's optimal strategies are obtained by the dual solution to the above problem. We do this by first taking the transpose of the original payoff matrix and this gives us the results below

$$\begin{array}{cl} \begin{bmatrix} 8 & 2 & 1 \\ 4 & 8 & 2 \\ 2 & 4 & 8 \end{bmatrix} & \begin{array}{l} x_1 \\ x_2 \\ x_3 \end{array} \end{array}$$

$$8x_1 + 2x_2 + x_3 \geq V$$

$$4x_1 + 8x_2 + 2x_3 \geq V$$

$$2x_1 + 4x_2 + 8x_3 \geq V$$

Player A's optimum solution is obtained by solving the above linear programming

We let $x_4 = V$

Maximize x_4

subject to

$$8x_1 + 2x_2 + x_3 \geq x_4$$

$$4x_1 + 8x_2 + 2x_3 \geq x_4$$

$$2x_1 + 4x_2 + 8x_3 \geq x_4$$

$$x_1 + x_2 + x_3 = 1$$

and

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

Then above system of equations is rearranged as follows

$$8x_1 + 2x_2 + x_3 - x_4 \geq 0$$

$$4x_1 + 8x_2 + 2x_3 - x_4 \geq 0$$

$$2x_1 + 4x_2 + 8x_3 - x_4 \geq 0$$

$$x_1 + x_2 + x_3 = 1$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

The above problem is then solved on matlab using linprog function as shown in appendix III.

The following results are then generated by matlab

X = {0.4444, 0.2444, 0.3111, 4.3556}

rats(X) = {4/9, 11/45, 14/45, 196/45}

That is $\{ x_1 = 4/9, x_2 = 11/45, x_3 = 14/45\}$ and the value of the game is $x_4 = V = 196/45$

The results indicate that, Player A makes an expected gain of $V = 196/45$ and Player B makes an expected loss of $V = 196/45$.

CHAPTER 4

DATA ANALYSIS AND RESULTS

4.0 Introduction

This chapter presents the data used for the study and the results of analysis been carried out. The data took the form of customers preference for one of the service providers based on the service offers available to them. The number of customers going in for a particular service offer from a service provider was captured as the payoff for that service provider.

4.1 Data Presentation and Analysis

We first present the secondary data that was used for this work. This data was obtained from the brochures of the two companies and took the form of the available services offered by the two telecom companies to mobile phone users in the country. However, out of the various services they each offer to the public, we selected eight of these services for both companies. Our choice of these services was based on the services that were frequently advertised in the electronic media like the radio, television and the internet. Apart from the advertisement on the electronic media, we also held some preliminary discussions with some individuals concerning their awareness of the services by these companies which also guided us in our selection of these services. These services were considered as the strategies that the two companies considered useful to enable them capture the huge market of mobile phone users in the country.

Our primary data on the other hand, was obtained by designing a questionnaire which was administered to two hundred (200) respondents from Bolgatanga Polytechnic to indicate their preferred network based on the services made available to them by the two telecom companies.

The hundred (200) respondents were sampled by simple random means. Before the administration of the questionnaires, respondents were briefed on the nature of the questionnaires and how they were expected to answer the questionnaires. Respondents who had no knowledge about some of the services were offered some explanation on these services.

We engaged in all these activities to ensure that we generated the best data as much as possible for this work.

The tables 4.1 and 4.2 below show the services selected for this thesis for MTN and Vodafone respectively.

The payoffs for the two players were captured and presented in a tabular form as shown in table 4.3 below.

Table 4.1: Strategies (Services) employed by MTN

Services	MTN Zone	MTN Free Night Call	MTN Family and Friends	MTN Per Minute Billing	MTN "pay 4 me"	MTN "Me 2 U"	MTN Call me Back	MTN Voice SMS
Labels	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8

Table 4.2: Strategies (Services) employed by Vodafone

Services	Vodafone 4040	Vodafone Free Night Call	Vodafone Family and Friends	Vodafone Per Minute Billing	Vodafone 5050	Vodafone "Me 2 U"	Vodafone Call me Back	Vodafone 6060
Labels	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

Table 4.3: Payoff table for MTN and Vodafone

Vodafone / MTN	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
x_1	(77,123)	(136,64)	(131,69)	(79,121)	(76,124)	(121,79)	(127,73)	(78,122)
x_2	(72,128)	(79,121)	(94,106)	(69,131)	(75,125)	(115,85)	(123,77)	(82,102)
x_3	(69,131)	(103,97)	(90,110)	(62,138)	(74,126)	(122,78)	(118,82)	(70,130)
x_4	(95,105)	(123,77)	(117,83)	(73,127)	(92,108)	(124,76)	(143,57)	(71,129)
x_5	(69,131)	(97,103)	(123,77)	(58,142)	(62,138)	(110,90)	(106,94)	(66,134)
x_6	(63,137)	(96,104)	(112,88)	(63,137)	(69,131)	(107,93)	(102,98)	(67,133)
x_7	(58,142)	(93,107)	(96,104)	(66,134)	(64,136)	(103,97)	(100,100)	(68,132)
x_8	(70,130)	(98,102)	(99,101)	(73,127)	(65,135)	(103,97)	(102,98)	(72,128)

The payoff table for MTN is shown in the table below

Table 4.4: Payoff table for MTN

MTN	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
x_1	77	136	131	79	76	121	127	78
x_2	72	79	94	69	75	115	123	82
x_3	69	103	90	62	74	122	118	70
x_4	95	123	117	73	92	124	143	71
x_5	69	97	123	58	62	110	106	66
x_6	63	96	112	63	69	107	102	67
x_7	58	93	96	66	64	103	100	68
x_8	70	98	99	73	65	103	102	72

The payoff table for Vodafone is shown in the table below

Table 4.5: Payoff table for Vodafone

Vodafone	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
y_1	123	128	131	105	131	137	142	130
y_2	64	121	97	77	103	104	107	102
y_3	69	106	110	83	77	88	104	101
y_4	121	131	138	127	142	137	134	127
y_5	124	125	126	108	138	131	136	135
y_6	79	85	68	66	90	93	97	97
y_7	73	77	82	57	94	98	100	98
y_8	122	118	130	129	134	133	132	128

4.2 Model Formulation

For purposes of solving the above game, we reformulate the above constant-sum game as a zero-sum game by considering the payoffs of the individual players separately. First we consider the payoff matrix table of MTN and then formulate our linear programming problem.

The following system of inequalities is generated from the MTN payoff table with MTN as the row player.

$$Maximize\ x_9$$

$$Subject\ to$$

$$77x_1 + 72x_2 + 69x_3 + 95x_4 + 69x_5 + 63x_6 + 58x_7 + 70x_8 - x_9 \geq 0$$

$$136x_1 + 79x_2 + 103x_3 + 123x_4 + 97x_5 + 96x_6 + 93x_7 + 98x_8 - x_9 \geq 0$$

$$131x_1 + 94x_2 + 90x_3 + 117x_4 + 123x_5 + 112x_6 + 96x_7 + 99x_8 - x_9 \geq 0$$

$$79x_1 + 69x_2 + 62x_3 + 73x_4 + 58x_5 + 63x_6 + 66x_7 + 73x_8 - x_9 \geq 0$$

$$76x_1 + 75x_2 + 74x_3 + 92x_4 + 62x_5 + 69x_6 + 64x_7 + 65x_8 - x_9 \geq 0$$

$$121x_1 + 115x_2 + 122x_3 + 124x_4 + 110x_5 + 107x_6 + 103x_7 + 103x_8 - x_9 \geq 0$$

$$127x_1 + 123x_2 + 118x_3 + 143x_4 + 106x_5 + 102x_6 + 100x_7 + 102x_8 - x_9 \geq 0$$

$$78x_1 + 82x_2 + 70x_3 + 71x_4 + 66x_5 + 67x_6 + 68x_7 + 72x_8 - x_9 \geq 0$$

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 \geq 0$$

and

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0, x_6 \geq 0, x_7 \geq 0, x_8 \geq 0$$

In the second case, we consider the payoff table of Vodafone with Vodafone as the row player.

In the case of Vodafone, we have the following maximization problem

$$Maximize\ y_9$$

subject to

$$123y_1 + 64y_2 + 69y_3 + 121y_4 + 124y_5 + 79y_6 + 73y_7 + 122y_8 - y_9 \leq 0$$

$$64y_1 + 121y_2 + 106y_3 + 131y_4 + 125y_5 + 85y_6 + 77y_7 + 118y_8 - y_9 \leq 0$$

$$131y_1 + 97y_2 + 110y_3 + 138y_4 + 126y_5 + 68y_6 + 82y_7 + 130y_8 - y_9 \leq 0$$

$$105y_1 + 77y_2 + 83y_3 + 127y_4 + 108y_5 + 66y_6 + 57y_7 + 129y_8 - y_9 \leq 0$$

$$131y_1 + 103y_2 + 77y_3 + 142y_4 + 138y_5 + 90y_6 + 94y_7 + 134y_8 - y_9 \leq 0$$

$$137y_1 + 104y_2 + 88y_3 + 137y_4 + 131y_5 + 93y_6 + 98y_7 + 133y_8 - y_9 \leq 0$$

$$142y_1 + 107y_2 + 104y_3 + 134y_4 + 136y_5 + 97y_6 + 100y_7 + 132y_8 - y_9 \leq 0$$

$$130y_1 + 102y_2 + 101y_3 + 127y_4 + 135y_5 + 97y_6 + 98y_7 + 128y_8 - y_9 \leq 0$$

$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 \geq 0$$

and

$$y_1 \geq 0, y_2 \geq 0, y_3 \geq 0, y_4 \geq 0, y_5 \geq 0, y_6 \geq 0, y_7 \geq 0, y_8 \geq 0$$

4.3 Computational Procedure

The two systems of inequalities are then solved on the matlab platform by invoking the linprog function. The computer used to perform this analysis was Toshiba AMD with 2.00 GHz CPU, 2.00 GB RAM and 32-bit Operating System.

The input data was from table 4.4 and table 4.5.

4.4 Results of Data Analysis

The following results were generated as the optimal solution for MTN using data from Table 4.4.

$$\{x_1 = \frac{67}{82}, x_2 = \frac{25}{317}, x_3 = 0, x_4 = \frac{5}{48}, x_5 = 0, x_6 = 0, x_7 = 0, x_8 = 0 \textit{ and } x_9 = \frac{9388}{121}\}$$

$x_9 = \frac{9388}{121}$, is the optimal objective function value for MTN.

The results for optimal solution for Vodafone using data from Table 4.5 are given below:

$$\{y_1 = 0, y_2 = 0, y_3 = 0, y_4 = \frac{16}{89}, y_5 = \frac{51}{172}, y_6 = 0, y_7 = 0, y_8 = \frac{166}{317} \textit{ and } y_9 = \frac{14812}{121}\}$$

$y_9 = \frac{14812}{121}$, is the optimal objective function value for Vodafone.

4.5 Discussions of Results

The results indicates that, the optimal mixed strategies for MTN is $(x_1 = \frac{67}{82}, x_2 = \frac{25}{317}, x_3 = 0, x_4 = \frac{5}{48}, x_5 = 0, x_6 = 0, x_7 = 0, x_8 = 0)$ and the value of the game $V = x_9 = \frac{9388}{121}$. This means that, MTN will have to adopt his pure strategies with the following probabilities: MTN Zone = 0.8171, MTN Free Night Call = 0.0789, MTN Family and Friends

= 0, MTN Per Minute Billing = 0.1042, MTN "pay 4 me" = 0, MTN "Me 2 U" = 0, MTN call me back = 0, and MTN Voice SMS = 0. This is to ensure an expected gain of 78 customers out of the 200 mobile phone users used in the study from Bolgatanga Polytechnic.

The results can be interpreted as; MTN must pursue the strategy MTN Zone aggressively as that stands out tall amongst the other strategies, followed by his strategy MTN Per Minute Billing and lastly his strategy MTN Free Night Call. Those services with probabilities zeros are dominated strategies and do not contribute to the value of the game.

On the other hand, the optimal mixed strategies for Vodafone in response to MTN is the dual of MTN's primal problem which is $(y_1 = 0, y_2 = 0, y_3 = 0, y_4 = \frac{16}{89}, y_5 = \frac{51}{172}, y_6 = 0, y_7 = 0, y_8 = \frac{166}{317})$ and the value of the game is $V = y_9 = \frac{9388}{121}$. That is Vodafone must adopt his pure strategies with the following probabilities: Vodafone 4040 = 0, Vodafone Free Night call = 0, Vodafone Family and Friends = 0, Vodafone Per Minute Billing = 0.1798, Vodafone 5050 = 0.2965, Vodafone "Me 2 U" = 0, Vodafone call me back = 0, and Vodafone 6060 = 0.5237. This is to ensure an expected loss of 78 customers.

From the results, Vodafone is required to adopt more of his strategy Vodafone 6060, followed by his strategy Vodafone 5050 and lastly his strategy Vodafone Per Minute Billing.

However, the results obtained from table 4.5 indicates that the optimal mixed strategy for Vodafone is $(y_1 = 0, y_2 = 0, y_3 = 0, y_4 = \frac{16}{89}, y_5 = \frac{51}{172}, y_6 = 0, y_7 = 0, y_8 = \frac{166}{317})$ and the value of the game in this case is $V = y_9 = \frac{14812}{121}$. This means that, Vodafone will have to adopt his pure strategies with the following probabilities: Vodafone 4040 = 0, Vodafone Free Night call = 0, Vodafone Family and Friends = 0, Vodafone Per Minute Billing = 0.1798, Vodafone

5050 = 0.2965, Vodafone "Me 2 U" = 0, Vodafone call me back = 0, and Vodafone 6060 = 0.5237. This is to ensure an expected gain of 122 customers.

Under this scenario, the optimal strategies for MTN in response to Vodafone is the dual of the primal of Vodafone's problem which is $\{x_1 = \frac{67}{82}, x_2 = \frac{3}{38}, x_3 = 0, x_4 = \frac{5}{48}, x_5 = 0, x_6 = 0, x_7 = 0, x_8\}$ and the value of the game is $V = x_9 = \frac{14812}{121}$. This means that MTN will have to adopt his pure strategies with the following probabilities: MTN Zone = 0.8171, MTN Free Night Call = 0.0789, MTN Family and Friends = 0, MTN Per Minute Billing = 0.1042, MTN "pay 4 me" = 0, MTN "Me 2 U" = 0, MTN call me back = 0, and MTN Voice SMS = 0. This is to ensure an expected loss of 122 customers out of the 200 mobile phone users used in the study from Bolgatanga Polytechnic.

We summarized the results of both the primal and the dual of the two telecom companies in a tabular form as in Tables 4.6 and 4.7. The results on these tables reveal the resultant payoffs that the two companies obtain for any row or column of strategies they adopt. The results on Table 4.6 show that, the maximin and minimax value is 78. On the other hand, results on Table 4.7 also show that, the maximin and minimax value is 122.

It is interesting to observe that, the optimal strategies obtained from the dual of Vodafone is equal to the optimal strategies of the primal of MTN. Similarly, the optimal strategies from the dual of MTN is equal to the optimal strategies of the primal of Vodafone. Also, the value of the game from the primal and dual of the two cases add up 200 which is the total number of respondents used in the study.

Table 4.6: Nash Equilibrium table for MTN

MTN	0	0	0	$\frac{16}{89}$	$\frac{51}{172}$	0	0	$\frac{166}{317}$	$\sum_{j=1}^{n} a_{ij}y_j$
67/82	77	136	131	79	76	121	127	78	**77.58**
25/317	72	79	94	69	75	115	123	82	**77.58**
0	69	103	90	62	74	122	118	70	**69.74**
5/48	95	123	117	73	92	124	143	71	**77.58**
0	69	97	123	58	62	110	106	66	**63.37**
0	63	96	112	63	69	107	102	67	**66.87**
0	58	93	96	66	64	103	100	68	**66.45**
0	70	98	99	73	65	103	102	72	**70.10**
$\sum_{i=1}^{m} a_{ij}x_i$	**78.45**	**130.16**	**126.64**	**77.59**	**77.59**	**120.85**	**128.36**	**77.59**	

Table 4.7: Nash Equilibrium table for Vodafone

Vodafone	$\frac{67}{82}$	$\frac{3}{38}$	0	$\frac{5}{48}$	0	0	0	0	$\sum_{j=1}^{n} a_{ij}x_j$
0	123	128	131	105	131	137	142	130	**121.54**
0	64	121	97	77	103	104	107	102	**69.87**
0	69	106	110	83	77	88	104	101	**73.39**
16/89	121	131	138	127	142	137	134	127	**122.44**
51/172	124	125	126	108	138	131	136	135	**122.44**
0	79	85	68	66	90	93	97	97	**79.18**
0	73	77	82	57	94	98	100	98	**71.66**
166/317	122	118	130	129	134	133	132	128	**122.44**
$\sum_{i=1}^{m} a_{ij}y_i$	**122.41**	**122.41**	**130.25**	**122.41**	**136.62**	**133.12**	**133.54**	**129.89**	

CHAPTER 5

CONCLUSION AND RECOMMENDATIONS

5.0 Conclusion

In this project we formulated two models to find optimal strategies at equilibrium for two telecom companies operating in Ghana in section 4.2. We considered eight different service offers by these companies to customers as a way of obtaining a larger share of customers in our model.

Outcome of our work indicates that, at equilibrium when MTN adopts his available strategies, Vodafone will respond by playing his strategies with probabilities as $y_1 = 0, y_2 = 0, y_3 = 0, y_4 = {}^{16}/_{89}, y_5 = {}^{51}/_{172}, y_6 = 0, y_7 = 0$ *and* $y_8 = {}^{166}/_{317}$. On the other hand, at equilibrium when Vodafone adopts his available strategies, MTN will respond by playing his strategies with probabilities as $x_1 = {}^{67}/_{82}, x_2 = {}^{25}/_{317}, x_3 = 0, x_4 = {}^{5}/_{48}, x_5 = 0, x_6 = 0, x_7 = 0$ *and* $x_8 = 0$. From our model formulated we realized that at equilibrium, MTN will gain an expected payoff of 78 and Vodafone will lose an expected payoff of 78. In other words, MTN will gain 78 customers and lose 122 customers whereas Vodafone will lose 78 customers and gain 122 customers.

We also realized that, when the payoff matrix of Vodafone is considered in which case Vodafone becomes the row player and MTN the column player, the expected value of the game is 122 customers. That is, MTN's expected loss will be 122 customers and Vodafone's expected gain will be 122 customers.

The value of the game in the first scenario where Vodafone is a column player and the second scenario where he is row player adds up to 200 which corresponds to the total number of respondents used for the study.

5.2 Recommendations

From the outcome of this research, the following recommendations are made:

1. Telecom companies should adopt game theory in analyzing their business strategies in relation to their rivals in the sector. There is the need for the integration of game theory with the existing management techniques in organizations to enhance decision making in strategic management. When managers are able to do this they will be in a better position to choose the best options toward improving customer acquisition and retention.
2. More research should be done in game theory with emphasis on solution methods to linear programming problems that arise from the game formulation.
3. We suggest that further research that will provide a platform where all the operators in the industry could be modelled using game theory or some other decision making tool to enable them identify their optimal strategies and payoffs be carried out.

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APPENDICES

APPENDIX I : Research Questionnaire

KWAME NKRUMAH UNIVERSITY OF SCIENCE AND TECHNOLOGY

COLLEGE OF PHYISCAL SCIENCE, DEPARTMENT OF MATHEMATICS

P lease the questions below are designed to help me complete my Mphil research on the topic: Modelling Customers Response to Service offers by two Telecom Operators. A case of MTN and Vodafone in Bolgatanga in the Upper East Region of Ghana.

Respondents Personal Details

1. Age

a. 18-25 [ ] b. 26-30 [ ] c. 31-35 [ ] d. 36 and above [ ]

2. Programme of study

a. Accounting [ ] b. Marketing [ ] c. Statistics [ ] d. Others [ ]

3. Religious background

a. Christianity [ ] b. Islam [ ] c. Traditional [ ] d. Others [ ]

4. Marital status

a. Married [ ] b. Single [ ] c. Divorce [ ] d. Others [ ]

Information on Service offers by MTN and Vodafone

The following are services offered by MTN and Vodafone, please indicate which service you will prefer by ticking the appropriate box.

5. MTN Zone [ ]
   Vodafone 4040 [ ]

6. MTN Zone [ ]
   Vodafone Free Night Call [ ]

7. MTN Zone [ ]
   Vodafone Family and Friends [ ]

8. MTN Zone [ ]
   Vodafone Per minute Billing [ ]

9. MTN Zone [ ]

Vodafone 5050 [ ]

10. MTN Zone [ ]

Vodafone ""Me 2 U"" [ ]

11. MTN Zone [ ]

Vodafone call me back [ ]

12. MTN Zone [ ]

Vodafone 6060 [ ]

13. MTN Free Night call [ ]

Vodafone 4040 [ ]

14. MTN Free Night call [ ]

Vodafone Free Night Call [ ]

15. MTN Free Night call [ ]

Vodafone Family and Family [ ]

16. MTN Free Night call [ ]

Vodafone Per minute Billing [ ]

17. MTN Free Night call [ ]

Vodafone 5050 [ ]

18. MTN Free Night call [ ]

Vodafone ""Me 2 U"" [ ]

19. MTN Free Night call [ ]

Vodafone call me back [ ]

20. MTN Free Night call [ ]

Vodafone 6060 [ ]

21. MTN Family and Friends [ ]

Vodafone 4040 [ ]

22. MTN Family and Friends [ ]

Vodafone Free Night Call [ ]

23. MTN Family and Friends [ ]

Vodafone Family and Family [ ]

24. MTN Family and Friends [ ]

Vodafone Per minute Billing [ ]

25. MTN Family and Friends [ ]

Vodafone 5050 [ ]

26. MTN Family and Friends [ ]

Vodafone "Me 2 U" [ ]

27. MTN Family and Friends [ ]

Vodafone call me back [ ]

28. MTN Family and Friends [ ]

Vodafone 6060 [ ]

29. MTN Per minute Billing [ ]

Vodafone 4040 [ ]

30 . MTN Per minute Billing [ ]

Vodafone Free Night Call [ ]

31. MTN Per minute Billing [ ]

Vodafone Family and Family [ ]

32. MTN Per minute Billing [ ]

Vodafone Per minute Billing [ ]

33. MTN per minute Billing [ ]

34. MTN Per minute Billing [ ]

Vodafone 5050 [ ] Vodafone "Me 2 U" [ ]

35. MTN Per minute Billing [ ]

Vodafone Call me back [ ]

36. MTN Per minute Billing [ ]

Vodafone 6060 [ ]

37. MTN "pay 4 me" [ ]

Vodafone 4040 [ ]

38. MTN "pay 4 me" [ ]

Vodafone Free Night Call [ ]

39. MTN "pay 4 me" [ ]

Vodafone Family and Family [ ]

40. MTN "pay 4 me" [ ]

Vodafone Per minute Billing [ ]

41. MTN "pay 4 me" [ ]

Vodafone 5050 [ ]

42. MTN Pay for me [ ]

Vodafone "Me 2 U" [ ]

43. MTN "pay 4 me" [ ]

Vodafone Call me back [ ]

44. MTN "pay 4 me" [ ]

Vodafone 6060 [ ]

45. MTN me 2U [ ]

Vodafone 4040 [ ]

46. MTN "Me 2 U" [ ]

Vodafone Free Night Call [ ]

47. MTN me 2U [ ]

Vodafone Family and Family [ ]

48. MTN "Me 2 U" [ ]

Vodafone Per minute Billing [ ]

49. MTN me 2U [ ]

Vodafone 5050 [ ]

50. MTN "Me 2 U" [ ]

Vodafone "Me 2 U" [ ]

51. MTN me 2U [ ]

Vodafone Call me back [ ]

52. MTN "Me 2 U" [ ]

Vodafone 6060 [ ]

53. MTN call me back [ ]

Vodafone 4040 [ ]

54. MTN call me back [ ]

Vodafone Free Night Call [ ]

55. MTN call me back [ ]

Vodafone Family and Friends [ ]

56. MTN call me back [ ]

Vodafone Per minute Billing [ ]

57. MTN call me back [ ]

Vodafone 5050 [ ]

58. MTN call me back [ ]

Vodafone "Me 2 U" [ ]

59. MTN call me back [ ]

Vodafone call me back [ ]

60. MTN call me back [ ]

Vodafone 6060 [ ]

61. MTN Voice SMS [ ]

Vodafone 4040 [ ]

62. MTN Voice SMS [ ]

Vodafone Free Night Call [ ]

63. MTN Voice SMS [ ]

Vodafone Family and Friends [ ]

64. MTN Voice SMS [ ]

Vodafone Per minute Billing [ ]

65. MTN Voice SMS [ ]

Vodafone 5050 [ ]

66. MTN Voice SMS [ ]

Vodafone "Me 2 U" [ ]

67. MTN Voice SMS [ ]

Vodafone call me Back [ ]

68. MTN Voice SMS [ ]

Vodafone 6060 [ ]

APPENDIX II: Matlab codes for Lemke-Howson's Algorithm

```
clc, clear, format rat

% A=input('Enter player1 as a matrix...   ');
% B=input('Enter player2 as a matix...   ');

% A=[1 3 0;0 0 2 ;2 1 1 ]  % this is the i-th row
% B=[2 1 0;1 3 1 ;0 0 3 ]   %this is the j-th column

   A=[74 110 120 60 65 121 108 70;70 80 106 69 77 117 110 78;72 103 90 79 74
122 118 63;95 123 117 80 92 124 143 65;75 97 123 67 62 110 106 64;70 95 115
68 72 110 102 62;58 93 96 72 64 103 100 68;76 98 99 78 85 103 102 89] ; %
this is the i-th row
    B=[126 90 80 140 135 79 92 130;130 120 94 131 123 83 90 122;128 97 110
121 126 78 82 137;105 76 83 120 108 77 57 135;125 103 77 133 138 87 94
136;130 105 85 132 128 90 98 138;142 107 104 128 136 103 100 132;124 102 101
122 115 97 98 111];   %this is the j-th column

r=[ones(length(A),1),-1*A];
s=-1*B;

s=[ones(length(s),1),s'];

c=1;
k=1;
q=2;

storeB=[];

while c <= 2*length(A)

    if mod(k,2)~=0

    curr_vec=1./abs(s(:,q));
    get_min=min(curr_vec);
    get_indx=find(curr_vec==get_min);

        if length(get_indx) > 1
            for ix=1:length(get_indx)
                if isempty(storeB)
                    get_indx=get_indx(ix);
                    storeB(get_indx)=get_indx;
                    break
                else
                    for ip=1:length(storeB)
                        if get_indx(ix)~=storeB(ip)
                            get_indx=get_indx(ix);
                            storeB(get_indx)=get_indx;
                            break
                        end
                    end
                end
            end
```

```
else
    if isempty(storeB)
        storeB(get_indx)=get_indx;
%         break
    else
        for ip2=1:length(storeB)
            if storeB(ip2)~=0
                if get_indx > storeB(ip2)
                    storeB(ip2+1)=get_indx;
                    break
                elseif get_indx==storeB(1)
                    storeB(2)=ip2;
                else
                    storeB(ip2-1)=get_indx;
                    break
                end
            end
        end
    end
end


for i=get_indx:length(B)

    if i==get_indx
        get_vec=s(i,:)/abs(s(i,q));
        get_vec(q)=get_vec(q)/abs(s(i,q));
        s(i,:)=get_vec;
        flagB=i;
    else
        if s(i,q)~=0
            temp_varB=s(i,q)*s(flagB,:);
            s(i,q)=0;
            s(i,:)=s(i,:) + temp_varB;
        end
    end

end


for i=get_indx:-1:1

    if i~=get_indx

        if s(i,q)~=0
            temp_varB=s(i,q)*s(flagB,:);
            s(i,q)=0;
            s(i,:)=s(i,:) + temp_varB;
        end

    end

end

s=s
```

```
    p=get_indx+1;

else

    curr_vecA=1./abs(r(:,p));
    get_minA=min(curr_vecA);
    get_indxA=find(curr_vecA==get_minA);

    for i=get_indxA:length(A)

        if i==get_indxA
            get_vecA=r(get_indxA,:)/abs(r(i,p));
            get_vecA(p)=get_vecA(p)/abs(r(i,p));
            r(i,:)=get_vecA;
            flagA=i;
        else

            if r(i,p)~=0
                temp_varA=r(i,p)*r(flagA,:);
                r(i,p)=0;
                r(i,:)=r(i,:) + temp_varA;
            end

        end

    end
    for i=get_indxA:-1:1

        if i~=get_indxA

            if r(i,p)~=0
                temp_varA=r(i,p)*r(flagA,:);
                r(i,p)=0;
                r(i,:)=r(i,:) + temp_varA;
            end

        end

    end

    r=r

    j=j+1;
    q=get_indxA+1;

end

c=c+1;
k=k+1;
```

```
end

x=s(:,1
y=r(:,1

% sumX=  (x)
% sumY=  (y)

% format short
%
% outX=  und(1/sumX)
% outY=  und(1/sumY)
%
% outX=  tY

% % for  t short
% %
% % nrm  x=(1/sum(x))*(x)
% % rm  =(1/sum(y))*(y)

format  t

nrml_xx  1/sum(x))*(x)
nrml_yy  1/sum(y))*(y)

outtee=  *nrml_xx

sumAll  m(nrml_xx)
format  ort
```

APPENDIX III: Primal –Dual Interior-Point Method for the (3×3) Zero-sum game

```
f=[0;0;0;1];

A=[8 4 2 -1;2 8 4 -1;1 2 8 -1];

b=[0;0;0];

Aeq=[1 1 1 0];beq=[1];

lb=zeros(4,1);lb(4)=-inf;

ub=[];

Y=linprog(f,A,b,Aeq,beq,lb,[])


f=[0;0;0;1];

A=[8 2 1 -1;4 8 2 -1;2 4 8 -1];

b=[0;0;0];

Aeq=[1 1 1 0];beq=[1];

lb=zeros(4,1);lb(4)=-inf;

ub=[];

X=linprog(-f,-A,b,Aeq,beq,lb,[])
```

APPENDIX IV: Primal-Dual Interior-Point Method for MTN

```
f=[0;0;0;0;0;0;0;0;1];
A=[77 136 131 79 76 121 127 78 -1;72 79 94 69 75 115 123 82 -1;69 103 90 62
74 122 118 70 -1;95 123 117 73 92 124 143 71 -1;69 97 123 58 62 110 106 66 -
1;63 96 112 63 69 107 102 67 -1;58 93 96 66 64 103 100 68 -1;70 98 99 73 65
103 102 72 -1];
b=[0;0;0;0;0;0;0;0];
Aeq=[1 1 1 1 1 1 1 1 0];beq=[1];
lb=zeros(9,1);lb(9)=-inf;
ub=[];
z=linprog(f,A,b,Aeq,beq,lb,[])
```

```
f=[0;0;0;0;0;0;0;0;1];
A=[77 72 69 95 69 63 58 70 -1;136 79 103 123 97 96 93 98 -1;131 94 90 117 123
112 96 99 -1;79 69 62 73 58 63 66 73 -1;76 75 74 92 62 69 64 65 -1;121 115
122 124 110 107 103 103 -1;127 123 118 143 106 102 100 102 -1;78 82 70 71 66
67 68 72 -1];
b=[0;0;0;0;0;0;0;0];
Aeq=[1 1 1 1 1 1 1 1 0];beq=[1];
lb=zeros(9,1);lb(9)=-inf;
ub=[];
j=linprog(-f,-A,b,Aeq,beq,lb,[])
```

APPENDIX V: Primal-Dual Interior-Point Method for Vodafone

```
f=[0;0;0;0;0;0;0;0;1];
A=[123 64 69 121 124 79 73 122 -1;128 121 106 131 125 85 77 118 -1;131 97 110
138 126 68 82 130 -1;105 77 83 127 108 66 57 129 -1;131 103 77 142 138 90 94
134 -1;137 104 88 137 131 93 98 133 -1;142 107 104 134 136 97 100 132 -1;130
102 101 127 135 97 98 128 -1];
b=[0;0;0;0;0;0;0;0];
Aeq=[1 1 1 1 1 1 1 1 0];beq=[1];
lb=zeros(9,1);lb(9)=-inf;
ub=[];
m=linprog(-f,-A,b,Aeq,beq,lb,[])
```



```
f=[0;0;0;0;0;0;0;0;1];
A=[123 128 131 105 131 137 142 130 -1;64 121 97 77 103 104 107 102 -1;69 106
110 83 77 88 104 101 -1;121 131 138 127 142 137 134 127 -1;124 125 126 108
138 131 136 135 -1;79 85 68 66 90 93 97 97 -1;73 77 82 57 94 98 100 98 -1;122
118 130 129 134 133 132 128 -1];
b=[0;0;0;0;0;0;0;0];
Aeq=[1 1 1 1 1 1 1 1 0];beq=[1];
lb=zeros(9,1);lb(9)=-inf;
ub=[];
s=linprog(f,A,b,Aeq,beq,lb,[])
```