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OPTIMAL FUND ALLOCATION OF LOAN

(CASE STUDY: UNIBANK (GHANA) LTD)

By

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CHAPTER 1

INTRODUCTION

1.0 BACKGROUND OF STUDY

The first banks were the religious temples of the ancient world, and were established in the third millennium B.C. Banks probably brought about the invention of money. Deposits initially consisted of grain and later other goods including cattle, agricultural implements, and eventually precious metals such as gold, in the form of easy-to-carry compressed plates. Temples and palaces were the safest places to store gold as they were constantly attended and well built. As sacred places, temples presented an extra deterrent to would-be thieves. There are extant records of loans from the 18th century BC in Babylon that were made by temple priests and monks to merchants. (Wikimedia Foundation, Inc, 2010)

Banking in the modern sense can be traced to medieval and early Renaissance Italy, to the rich cities in the north like Florence, Venice and Genoa. The Bardi and Peruzzi families dominated banking in 14th century Florence, establishing branches in many other parts of Europe. Perhaps the most famous Italian bank was the Medici bank, set up by Giovanni Medici in 1397. The earliest known state deposit bank, Bancodi San Giorgio (Bank of St. George), was founded in 1407 at Genoa, Italy. (Wikimedia Foundation, Inc, 2008)

The Central Bank of Ghana traces its roots to the Bank of the Gold Coast (BCG), where it was nurtured. On the 4th March 1957, just two days before the declaration of political independence, the Bank of Ghana was formally established by the Bank of Ghana Ordinance (No. 34) of 1957, passed by the British Parliament. Frantic preparations then began to put in place an organisational structure for the new central bank. By the middle of July 1957, all was set for the official commissioning of the new Head Office of the Bank on the High Street. (Wikimedia Foundation, Inc, 2010)

In his opening address at the end of July 1957, the then Leader of Government Business (Prime Minister) stated with pleasure that the occasion marked the beginning of independent monetary administration in the newly independent Ghana – a cherished dream had at long last become a reality. The Leader of Government Business had put the aspiration of the country in establishing the central bank as follows: "In the modern world a central bank plays a very important and decisive role in the life of a country. It is essential to our own independence that we have a government-owned bank and that the central bank follows a policy designed to secure our economic independence and to further the general development of our country." (Wikimedia Foundation, Inc, 2010)

The principal objects of the new central bank, as enshrined in the 1957 Ordinance, were "to issue and redeem bank notes and coins, to keep and use reserves and to influence the credit situation with a view to maintaining monetary stability in Ghana and the external value of the Ghana cedis and to act as banker and financial adviser to the Government. The opening ceremony paved the way for the Bank to commence formal banking operations on 1st August 1957, when the Banking Department opened for business. The Issue Department did not commence operations until July 1958. (Wikimedia Foundation, Inc, 2008)

The (Bank of Ghana) has since 1957 undergone various legislative changes. The Bank of Ghana Ordinance (No.34) of 1957 was repealed by the Bank of Ghana Act (1963), Act 182. This Act was subsequently amended by the Bank of Ghana (Amendment Act) 1965, (Act 282).The Bank of Ghana Law, 1992 PNDCL 291 repealed Acts 182 and 282. ([Wikimedia Foundation](), Inc, 2008)

The current law under which the Bank operates is the Bank of Ghana Act, 2002 (Act 612). This enable suitable locally incorporated bodies to file applications for licences to operate as banking institutions. Subsequently, a number of corporate entities were licensed to operate as banks, in the country (Ghana). (Wikimedia Foundation, Inc, 2010)

1.1 TYPES OF BANKS

There are various types of banks which operate in the country to meet the financial requirements of different categories of people engaged in agriculture, business, profession, etc. On the basis of functions, the banking institutions in Ghana may be divided into the following types; (ghanaWeb.com, 2010)

1.1.1 CENTRAL BANK

Central Bank, Reserve Bank or Monetary Authority is a public institution that usually issues the currency, regulates the money supply and controls the interest rates in a country. Central Banks often also oversee the commercial banking system within its country's borders. A central bank is distinguished from a normal commercial bank because it has a monopoly on creating the currency of that nation, which is usually that nation's legal tender. The primary function of a central bank is to provide the nation's money supply, but more active duties include controlling interest rates, and acting as a lender of last resort to the banking sector during times of financial crisis. It may also have supervisory powers, to ensure that banks and other financial institutions do not behave recklessly or fraudulently. Example is Bank of Ghana (Wikimedia Foundation, Inc, 2009)

1.1.2 COMMERCIAL BANKS

A commercial bank is owned by a group of individuals or an individual or the state, for-profit entity that is licensed by the Central Bank (Bank of Ghana) to conduct overall banking activities. These activities include deposit taking, lending, bank-owned securities and various banking services for its clientele. Commercial banks, if properly licensed, may also conduct trust, insurance and portfolio management business. All commercial banks are dependent upon the acquisition of deposits in order to make loans and provide a source of liquidity. The basic types of deposits offered are checking, savings and time deposits. (Carl Wolf, 2010). Examples are; Ghana Commercial Bank, Standard Chartered Bank (Gh) Ltd., Barclays Bank (Gh) Ltd., SG-SSB Limited, Metropolitan Allied Commercial Bank, the Trust Bank, Zenith Bank, Intercontinental Bank, Standard Trust Bank, Fidelity Bank, Guaranty Trust Bank (Ghana) Limited. (GhanaWeb.com, 2010)

1.1.3 DEVELOPMENT BANKS

The Development Bank fosters, empowers and finances up coming and already exiting ventures. The Bank provides finance for private sector start-ups and expansions, equity deals, bridging finance, enterprise development finance, trade finance, small and medium enterprises, public private partnerships, public sector infrastructure, local authorities, and bulk finance to responsible micro-finance providers. Examples are; National Investment Bank, Agricultural Development Bank, International Commercial Bank, the Trust Bank, Prudential Bank, Amalgamated Bank, Ghana Commercial Bank, ARB Apex Bank. (GhanaWeb, 2010)

1.1.4 MERCHANT BANKS

Merchant bank is a financial institution primarily engaged in offering financial services and give advice to corporations and to individuals. The term can also be used to describe the private equity activities of banking. The Merchant bank is that a merchant bank invests its own capital in client companies. Merchant banks provide fee-based corporate advisory services, including those in mergers and acquisitions. Examples are; Merchant Bank of Ghana Ltd., Ecobank Ghana Ltd., Continental Acceptances Ltd. and First Atlantic Merchant Bank, CAL Bank, HFC Bank. (GhanaWeb, 2010)

1.2 MONEY

Money is any object that is generally accepted as payment for goods and services and repayment of debts in a given country or socio-economic context. Money originated as commodity money, but nearly all contemporary money systems are based on fiat money. Fiat money is without intrinsic use value as a physical commodity, and derives its value by being declared by a government to be legal tender; that is, it must be accepted as a form of payment within the boundaries of the country for all debts, public and private. (Wikimedia Foundation, Inc, 2010)

1.3 LOAN

A loan is a type of debt. Like all debt instruments, a loan entails the redistribution of financial assets over time, between the lender and the borrower. In a loan, the borrower initially receives or borrows an amount of money, called the principal, from the lender, and is obligated to pay back or repay an equal amount of money to the lender at a later time. Typically, the money is paid back in regular instalments, or partial repayments; in an annuity, each instalment is the same amount. (Wikimedia Foundation, Inc, 2010)

The loan is generally provided at a cost, referred to as interest on the debt, which provides an incentive for the lender to engage in the loan. In a legal loan, each of these obligations and restrictions is enforced by contract, which can also place the borrower under additional restrictions known as loan covenants. Although this article focuses on monetary loans, in practice any material object might be lent. Acting as a provider of loans is one of the principal tasks for financial institutions. For other institutions, issuing of debt contracts such as bonds is a typical source of funding. (Wikimedia Foundation, Inc, 2010)

1.4 ADVANTAGES OF LOAN

Loan is a form of debt, often with interest. There are several reasons why people apply for loans. Usually they borrow money to purchase a house, buy a car, or start a business. Often, applying for a loan is necessary because most do not have available financial resources they need to make a purchase. Other forms of loans, like the student loans have helped a lot of students get through school. Those who use student loan debt consolidation clearly have multiple student loans. They do this to manage their obligations better. (hubpages.inc, 2010)

1.5 DISADVANTAGES OF LOAN

Since loan is borrowed, the lender expects to receive payment with the interest specified. In addition, borrowers should make the payments at the specified date for a certain period. This is where most people have problems. Most problems start when people cannot make the monthly payments required due to different circumstance. Some finds it difficult to pay their loan because of the many other debts they have. Some encounter additional problems such as medical emergencies and job loss. (hubpages.inc, 2010)

1.1.6 COLLATERAL

In lending agreements, collateral is a borrower's pledge of specific property to a lender, to secure repayment of a loan. The collateral serves as protection for a lender against a borrower's default - that is, any borrower failing to pay the principal and interest under the terms of a loan obligation. If a borrower does default on a loan (due to insolvency or other event), that borrower forfeits (gives up) the property pledged as collateral and the lender then becomes the owner of the collateral. In a typical mortgage loan transaction, for instance, the real estate being acquired with the help of the loan serves as collateral. Should the buyer fail to pay the loan under the mortgage loan agreement, the ownership of the real estate is transferred to the bank. The bank uses a legal process called foreclosure to obtain real estate from a borrower who defaults on a mortgage loan obligation. (Wikimedia Foundation, Inc, 2010)

1.7 INTEREST RATE

An interest rate is the rate at which interest is paid by a borrower for the use of money that they borrow from a lender. For example, a small company borrows capital from a bank to buy new assets for their business, and in return the lender receives interest at a predetermined interest rate for deferring the use of funds and instead lending it to the borrower. Interest's rates are fundamental to a capitalist society. Interest rates are normally expressed as a percentage rate over the period of one year. Interest rates targets are also a vital tool of monetary policy and are taken into account when dealing with variables like investment, inflation, and unemployment. (Wikimedia Foundation. Inc, 2010)

1.8 INFLATION

In economics, inflation is a rise in the general level of prices of goods and services in an economy over a period of time. When the general price level rises, each unit of currency buys fewer goods and services; consequently, inflation is also erosion in the purchasing power of money – a loss of real value in the internal medium of exchange and unit of account in the economy. A chief measure of price inflation is the inflation rate, the annualized percentage change in a general price index (normally the Consumer Price Index) over time. (Wikimedia Foundation. Inc, 2010)

1.9 PROFILE OF UNIBANK

Unibank (Ghana) Limited was incorporated as a private company in December 1997 to operate as a bank. It is a wholly owned Ghanaian and authorized to undertake a broad range of banking business. The Bank opened its door to customers in January 2001.UniBank has carved a niche for itself in the Small and Medium Enterprises (SME) sector. The bank (Unibank) objective is to see the growth of small and medium sized enterprises into giants that can propel the economy to great heights. The bank has shown remarkable strength in the face of stiff competition and endeared itself to the hearts of customers.

1.10 PROBLEM STATEMENT

Unibank has the welfare of it customers at interest, hence provide a flexible loan payment term across all the types of loans at their door steps. As time went on, it was identified by Management Board of the bank, that a section of loan types always end up in bad debt, both the principal and the interest which can never being retrieved. Management of the bank decided to give loan to its customers on just a few number of loan types which can be retrieved. Furthermore, some customers bank in a particular bank because of it favourable and reliable loan policy in their favour, so when this favour does not exist anymore he or she (customer) finds its way to other banks. This reduces the number of account holders as against the amount of money the bank (Unibank) generate as profit which affect the development of the bank.

1.11 OBJECTIVES

1. To model disbursement of loan funds of Unibank to their customers as a Linear Programming Problem.
2. To optimize the loan disbursement model using Karmarkar interior point method.

1.12 METHODOLOGY

The main challenge of the loan portfolio is to minimize, bad debt and maximize returns on the loans disbursed. This challenge could be solved by properly modeling funds allocated for loan to its customers. For this thesis to be successful and achieve its goals a secondary data on loan formulation portfolio was collected from Unibank and interior point method categorically, the potential reduction algorithm (karmarkar's algorithm) was used to analyse the data. Resource materials would be from required books, programming language used was Matlab and more importantly the internet.

1.13 JUSTIFICATION

The relevance of this research was to come out with a long lasting programme or model for the disbursement of loan as against it returns to enable the bank (Unibank) continue with it developmental projects and proper services to its customers. Achieving this goal, could lead a long way in creating a favourable opportunity for the populace to go for loan irrespective of the purpose of it. Thus, Customers' burden with domestic problems can amicably resolve it with an ease and whoever has the intention to invest or create a job which would go a long way to reduce unemployment in the country. This research would improve the maximization of profit on loans a as well as proper allocation of funds for the various types of loans the bank (Unibank) operate on, which would increase the number of account holders for the bank (Unibank).

1.14 STRUCTURE OF THE THESIS

The thesis consist five (5) chapters where Chapter 1 sheds light on the introduction, problem statement, objectives of the thesis, justification, methodology and structure of the thesis. Chapter 2 reviews work done by other people on the topic (literature review). Chapter 3 contains the method used to carry out this research. Chapter 4 talks about the analysis and results. Finally chapter 5 contain conclusion and recommendation.

CHAPTER TWO

LITERATURE REVIEW

This chapter of the thesis talks about people's works of various fields of research using linear programming programs will be considered.

Quintan et al. (2000), stated that since Karmarkar's first successful interior-point algorithm for linear programming in 1984, the interest and consequently the numbers of publications in the area have increased tremendously. They reviewed and classified major publications on interior-point methods theory, on the practical implementation of the most successful interior-point algorithms and on their applications to power systems optimization problems.

Xihui et al. (1996), dealt with the application of a feasible interior point method to optimal power flow problems. Besides discussing the problem formulation, the project offers a detailed description of the feasible interior point algorithm; it also addresses some important implementation issues such as the determination of the barrier parameter, the choice of the initial point, and the efficient solution of the optimality-condition equations by sparse matrix techniques. Some suggestions are proposed that significantly improve the performance of the algorithm. Computational results on large-scale power systems have shown that the algorithm is fast and robust, suitable to real-time applications.

Panos and Mauricio (1996), explained how interior point methods, originally invented in the context of linear programming, have found a much broader range of applications, including global optimization problems that arise in engineering, computer science, operations research, and other disciplines. The project also overviews the conceptual basis and applications of interior point methods for some classes of global optimization problems. During the last decade, the field of mathematical programming has evolved rapidly. New approaches have been developed and increasingly difficult problems are being solved with efficient implementations of new algorithms.

Todd (1990), showed variant of Karmarkar's projective algorithm for linear programming can be viewed as following the approach of Dantzig-Wolfe decomposition. At each iteration, the current primal feasible solution generates prices which are used to form a simple sub problem. The solution to the sub problem is then incorporated into the current feasible solution. With a suitable choice of step size a constant reduction in potential function is achieved at each iteration.

Ferris and Philpott (1988), described how a new polynomial-time algorithm for linear programming was announced by Narendra Karmarkar of Bell Laboratories in 1984. This algorithm is claimed by Bell Labs significantly to outperform the simplex method. Many numerical experiments have been carried out by other workers in the field which show a much smaller iteration count than the simplex method but larger computational times. Some have shown that, by using advanced numerical linear algebra and heuristics to exploit the problem structure, it is possible occasionally to beat the simplex method even in terms of computation time. A brief description of the main features of Karmarkar's algorithm is presented, along with the results of some numerical experiments. Another closely related interior-point method which involves the rescaling of the variables is also discussed, and some details of the sparse matrix manipulations involved in an implementation of the algorithm are mentioned.

Monteiro (1991), analyzed the convergence and boundary behaviour of the continuous trajectories of the vector field induced by the projective scaling algorithm as applied to (possibly degenerate) linear programming problems in Karmarkar's standard form. They showed that a projective scaling trajectory tends to an optimal solution which in general depends on the starting point. When the optimal solution is unique, they prove that all projective scaling trajectories approach the optimal solution through the same asymptotic direction. The analysis was based on the affine scaling trajectories for the homogeneous standard form that arises from Karmarkar's standard form by removing the unique non homogeneous constraint.

Ponnambalam et al. (1989), said that optimization of multi-reservoir systems operations is typically a very large scale optimization problem. The following are the three types of optimization problems solved using linear programming (LP):

(i) deterministic optimization for multiple periods involving fine stage intervals, for example, from an hour to a week.

(ii) implicit stochastic optimization using multiple years of inflow data

(iii) explicit stochastic optimization using probability distributions of inflow data.

Until recently, the revised simplex method has been the most efficient solution method available for solving large scale LP problems. They showed that an implementation of the Karmarkar's interior-point LP algorithm with a newly developed stopping criterion solves optimization problems of large multi-reservoir operations more efficiently than the simplex method. For example, using a Micro VAX II minicomputer, a 40 year, monthly stage, two-reservoir system optimization problem is solved 7.8 times faster than the advanced simplex code in MINOS 5.0. The advantage of this method is expected to be greater as the size of the problem grows from two reservoirs to multiples of reservoirs. This presents the details of the implementation and testing and in addition, some other features of the Karmarkar's algorithm which makes it a valuable optimization tool are illuminated.

Zhenghua et al. (1998), presented a new interior point method—combined homotopy interior point method (CHIP method) for convex nonlinear programming. Without strict convexity of the logarithmic barrier function and boundedness and non emptiness of the solution set, they prove that for any $\varepsilon > 0$, an ε-solution of the problem can be obtained by the CHIP method. To our knowledge, strict convexity of the logarithmic barrier function and non emptiness and boundedness of the solution set are the essential assumptions of the well-known center path-following method. Therefore, the CHIP method essentially reduces the assumptions of the center path-following method and can be applied to more general problems.

Wei-Tai and Ue-Pyng (2001), presented a modified interior point algorithm for solving linear optimization over the efficient set problems. Using computational experiments, they showed that the modified algorithm provides an effective and accurate approach for solving the linear optimization over the efficient set problem.

Kojima (1999), said Karmarkar proposed a new interior-point method for linear programs in 1984, the interior-point method has made dramatic progress in these fifteen years. Competing with the traditional simplex method, the interior-point method is now known as the most powerful computational method for solving huge scale linear programs. In the field of continuous optimization, the interior-point method has been successfully extended to convex quadratic programs, semi-definite programs, and more general convex programs, while, in the field of discrete optimization, the interior-point method has been playing an important role in terms of the semi-definite programming relaxation of 0-1 integer and nonconvex quadratic programs.

Jabr et al. (2002), explained how the solution of the optimal power flow dispatching (OPFD) problem by a primal-dual interior point method is considered. Several primal-dual methods for optimal power flow (OPF) have been suggested, all of which are essentially direct extensions of primal-dual methods for linear programming. The aim of the work is to enhance convergence through two modifications, a filter technique to guide the choice of the step length and an altered search direction in order to avoid convergence to a non minimizing stationary point. A reduction in computational time is also gained through solving a positive definite matrix for the search direction. Numerical tests on standard IEEE systems and on a realistic network are very encouraging and show that the new algorithm converges where other algorithms fail.

Rider et al. (2004), used an interior point method (IPM) to solve the optimal power flow (OPF) problem. The IPM uses a combination of the predictor corrector, multiple predictor corrector and multiple centrality correction methods (all belong to the family of higher order interior point methods). The proposed IPM uses the best properties of each method to obtain a more robust IPM with faster convergence characteristics. The active power loss minimisation, minimum load shedding and maximum load ability problems are formulated as an OPF problem and solved with the proposed methodology. The IEEE 30, 57, 118, and 300 bus systems, and two realistic power systems, a 464 bus corresponding to the interconnected Peruvian system, and a 2256 bus corresponding to part (South-Southeast) of the interconnected Brazilian system were tested successfully. Results have indicated that good convergence performance is obtained and the computational time is small.

Mitchell et al. (2006), discussed how beneficiary interior point methods for large-scale linear programming, were useful for problems arising in telecommunications. They gave the basic framework of a primal-dual interior point method, and consider the numerical issues involved in calculating the search direction in each iteration, including the use of factorization methods and/or preconditioned conjugate gradient methods. They also looked at interior point column generation methods which can be used for very large scale linear programs or for problems where the data is generated only as needed.

de Miguel et al. (2004), stated how an interior-point method can solve mathematical programs with equilibrium constraints (MPECs). At each iteration of the algorithm, a single primal dual step is computed from each sub problem of a sequence. Each sub problem is defined as a relaxation of the MPEC with a nonempty strictly feasible region. In contrast to previous approaches, the proposed relaxation scheme preserves the nonempty strict feasibility of each sub problem even in the limit. Local and super linear convergence of the algorithm is proved even with a less restrictive strict complementarily condition than the standard one. Moreover, mechanisms for inducing global convergence in practice are proposed.

Jaan-Willem and Dieter (2010), said that a new interior-point algorithm for the computation of shakedown loads has recently been developed. The analytical formulation is based on the statical shakedown theorem which leads to a nonlinear convex optimization problem. The algorithm's efficiency results from the close adaption of the solution procedure to the specific problem of shakedown analysis. This project focuses on algorithmic aspects of the proposed method.

Smelyanskiy et al. (2007), described how parallelization of interior-point method (IPM) aimed at achieving high scalability on large-scale chip-multiprocessors (CMPs). IPM is an important computational technique used to solve optimization problems in many areas of science, engineering and finance. IPM spends most of its computation time in a few sparse linear algebra kernels. While each of these kernels contains a large amount of parallelism, sparse irregular datasets seen in many optimization problems make parallelism difficult to exploit. As a result, most researchers have shown only a relatively low scalability of 4X-12X on medium to large scale parallel machines. This project proposes and evaluates several algorithmic and hardware features to improve IPM parallel performance on large-scale CMPs. Through detailed simulations, they demonstrated how exploring multiple levels of parallelism with hardware support for low overhead task queues and parallel reduction enables IPM to achieve up to 48X parallel speedup on a 64-core CMP.

Fonseca et al. (2010), presented a primal-dual interior-point algorithm to solve a class of multi-objective network flow problems. More precisely, our algorithm is an extension of the single-objective primal infeasible dual feasible inexact interior point method for multi-objective linear network flow problems. Our algorithm is contrasted with standard interior point methods and experimental results on bi-objective instances are reported. The multi-objective instances are converted into single objective problems with the aid of an achievement function, which is particularly adequate for interactive decision-making methods.

Lukšan et al. (2005), proposed a primal interior-point method for large sparse minimax optimization. After a short introduction, the complete algorithm is introduced and important implementation details were given. They proved that this algorithm is globally convergent under standard mild assumptions. Thus the large sparse non convex minimax optimization problems can be solved successfully. The results of extensive computational experiments given in this project confirm efficiency and robustness of the proposed method.

Forsgren et al. (2002), said that interior point methods are an omnipresent, conspicuous feature of the constrained optimization landscape today. Primarily in the form of barrier methods, interior-point techniques were popular during the 1960s for solving nonlinearly constrained problems. However, their use for linear programming was not even contemplated because of the total dominance of the simplex method. Vague but continuing anxiety about barrier methods eventually led to their abandonment in favour of newly emerging, apparently more efficient alternatives such as augmented Lagrangian and sequential quadratic programming methods. By the early 1980s, barrier methods were almost without exception regarded as a closed chapter in the history of optimization. This picture changed dramatically with Karmarkar's widely publicized announcement in 1984 of a fast polynomial-time interior method for linear programming; in 1985, a formal connection was established between his method and classical barrier methods. Since then, interior point methods have advanced so far, so fast, that their influence has transformed both the theory and practice of constrained optimization. This research provides a condensed, selective look at classical material and recent research about interior point methods for nonlinearly constrained optimization.

Xinwei and Jie (2003), explained how mathematical program with equilibrium constraints (MPEC) has extensive applications in practical areas such as traffic control, engineering design, and economic modeling. Some generalized stationary points of MPEC were studied to better describe the limiting points produced by interior point methods for MPEC.A primal-dual interior point method is then proposed, which solves a sequence of relaxed barrier problems derived from MPEC. Global convergence results are deduced without assuming strict complementarity or linear independence constraint qualification. Under very general assumptions, the algorithm can always find some point with strong or weak stationarity. In particular, it is shown that every limiting point of the generated sequence is a piece-wise stationary point of MPEC if the penalty parameter of the merit function is bounded. Otherwise, a certain point with weak stationarity can be obtained.

Srijuntongsiri and Vavasis (2004), showed a way to exploit sparsity in the problem data in a primal-dual potential reduction method for solving a class of semidefinite programs. When the problem data is sparse, the dual variable is also sparse, but the primal one is not. To avoid working with the dense primal variable, they apply Fukuda theory of partial matrix completion and work with partial matrices instead. The other place in the algorithm where sparsity should be exploited is in the computation of the search direction, where the gradient and the Hessian-matrix product of the primal and dual barrier functions must be computed in every iteration. By using an idea from automatic differentiation in backward mode, both the gradient and the Hessian-matrix product can be computed in time proportional to the time needed to compute the barrier functions of sparse variables itself. Moreover, the high space complexity that is normally associated with the use of automatic differentiation in backward mode can be avoided in this case. In addition, they suggested a technique to efficiently compute the determinant of the positive definite matrix completion that is required to compute primal search directions. The method of obtaining one of the primal search directions that minimizes the number of the evaluations of the determinant of the positive definite completion is also proposed. They then implement the algorithm and test it on the problem of finding the maximum cut of a graph.

Silva et al. (2008), analyzed the rate of local convergence of the Newton primal-dual interior-point method when the iterates are kept strictly feasible with respect to the inequality constraints. It is shown under the classical conditions that the rate is Q-quadratic when the functions associated to the binding inequality constraints are concave. In general, the Q-quadratic rate is achieved provided the step in the primal variables does not become asymptotically orthogonal to any of the gradients of the binding inequality constraints. Some preliminary numerical experience showed that the feasible method can be implemented in a relatively efficient way, requiring a reduced number of function and derivative evaluations. Moreover, the feasible method is competitive with the classical infeasible primal-dual interior-point method in terms of number of iterations and robustness.

El-Bakry et al. (1995), studied the formulation of the primal-dual interior- point method for linear programming. They showed that, it cannot be viewed as the damped Newton method applied to the Karush-Kuhn-Tucker conditions for the logarithmic barrier function problem. Next they extend the formulation to general nonlinear programming, and then validate this extension by demonstrating that this algorithm can be implemented so that it is locally and Q-quadratically convergent under only the standard Newton's method assumptions. They also establish a global convergence theory for this algorithm and include promising numerical experimentation.

Portugal et al. (2000), introduced the truncated primal-infeasible dual-feasible interior point algorithm for linear programming and described an implementation of this algorithm for solving the minimum-cost network flow problem. In each iteration, the linear system that determines the search direction is computed inexactly, and the norm of the resulting residual vector is used in the stopping criteria of the iterative solver employed for the solution of the system. In the implementation, a preconditioned conjugate gradient method was used as the iterative solver. The details of the implementation are described and the code PDNET is tested on a large set of standard minimum-cost network flow test problems. Computational results indicate that the implementation is competitive with state-of-the-art network flow codes

Zhi-jun et al. (2008), explained how a victorial implementation of dynamic optimal power flow (DOPF) was established, by arranging the control variables and state variables according to the variable types and time intervals. A step-controlled primal-dual interior point framework with upper and lower inequality constrains was used to solve this DOPF model. The gradient and Hessian matrices of each time interval had relative non-zeros position with the admittance matrix, which was constant during iterations. Hence a sparse data structure and memory allocation strategy was utilized to accelerate the construction of Karush-Kuhn-Tucker (KKT) system. The effect of ramping rates and generation contract constrains on solving KKT system was analyzed. Through computation statistics, it is confirmed that approximate minimum degree (AMD) reordering algorithm is most efficient with only ramping rate constrains, and column approximate minimum degree (COLAMD) reordering algorithm is most efficient with both ramping rate and generation contract constrains. Numerical simulations on test systems ranging in size from 14 to 1 040 buses over 12~96 time intervals validate the correctness and efficiency of the proposed method. Vectorization technique with step-controlled primal-dual interior point method improves the calculation speed and convergence performance of DOPF.

Wright (2005), explained how interior methods are a pervasive feature of the optimization landscape today. Although interior-point techniques, primarily in the form of barrier methods, were widely used during the 1960s for problems with nonlinear constraints, their use for the fundamental problem of linear programming was unthinkable because of the total dominance of the Simplex Method. During the 1970s, barrier methods were superseded, nearly to the point of oblivion, by newly emerging and seemingly more efficient alternatives such as augmented Lagrangian and sequential quadratic programming methods. By the early 1980s, barrier methods were almost universally regarded as a closed chapter in the history of optimization. This picture changed dramatically in 1984, when Narendra Karmarkar announced a fast polynomial-time interior method for linear programming; in 1985, a formal connection was established between his method and classical barrier methods. Since then, interior methods have continued to transform both the theory and practice of constrained optimization. We present a condensed, unavoidably incomplete look at classical material and recent research about interior methods.

Dekrajangpetch and Sheble (2000), stated that interior-point programming (IPP) has been applied to many power system problems because of its efficiency for big problems. This project illustrates application of interior-point linear programming (IPLP) to auction methods. This extended IPLP algorithm can find the exact optimal solution (i.e., exact optimal vertex) and can recover the optimal basis. Sensitivity analysis can be performed after the optimal basis is found. The sensitivity analysis performed in this paper is increase in the bid price and increase in the flow limit of the transmission line. This extended algorithm is expanded from the affine-scaling primal algorithm. The concept used in this extended algorithm to find the optimal vertex and optimal basis is simple

Rothberg and Hendrickson (1998), state that main cost of solving a linear programming problem using an interior point method is usually the cost of solving a series of sparse, symmetric linear systems of equations, $A\Theta A^T x = b$. These systems are typically solved using a sparse direct method. The first step in such a method is a reordering of the rows and columns of the matrix to reduce fill in the factor and/or reduce the required work. This project evaluates several methods for performing fill-reducing ordering on a variety of large-scale linear programming problems. They found out that a new method, based on the nested dissection heuristic, provides significantly better orderings than the most commonly used ordering method, minimum degree.

Bayer and Lagarias (1989), described a geometric structure underlying Karmarkar's projective scaling algorithm for solving linear programming problems. A basic feature of the projective scaling algorithm is a vector field depending on the objective function which is defined on the interior of the polytope of feasible solutions of the linear program. The geometric structure studied is the set of trajectories obtained by integrating this vector field, which we call P-trajectories. They also study a related vector field, the affine scaling vector field, and its associated trajectories, called A-trajectories. The affine scaling vector field is associated to another linear programming algorithm, the affine scaling algorithm. Affine and projective scaling vector fields are each defined for linear programs of a special form, called strict standard form and canonical form, respectively. This derives basic properties of P-trajectories and A-trajectories. It reviews the projective and affine scaling algorithms, defines the projective and affine scaling vector fields, and gives differential equations for P-trajectories and A-trajectories. It shows that projective transformations map P-trajectories into P-trajectories. It presents Karmarkar's interpretation of A-trajectories as steepest descent paths of the objective function $\langle C, X \rangle$ with respect to the Riemannian geometry $ds^2 = \sum_{i=1}^{n} dx_i dx_i / x_i^2$ restricted to the relative interior of the polytope of feasible solutions. P-trajectories of a canonical form linear program are radial projections of A-trajectories of an associated standard form linear program. As a consequence there is a polynomial time linear programming algorithm using the affine scaling vector field of this associated linear program: This algorithm is essentially Karmarkar's algorithm. These trajectories are studied in subsequent papers by two nonlinear changes of variables called Legendre transform coordinates and projective Legendre transform coordinates, respectively. It will be shown that P-trajectories have an algebraic and a geometric interpretation. They are algebraic curves, and they are geodesics (actually distinguished chords) of a geometry isometric to Hubert geometry on a polytope combinatorially dual to the polytope of feasible solutions. The A-trajectories of strict standard form linear programs have similar interpretations: They are algebraic curves, and are geodesics of a geometry isometric to Euclidean geometry.

Oliveira and Lyra (2004), narrated how interior point methods specialized to the L_∞ fitting problem were surveyed, improved, and compared with the traditional simplex approach. A primal affine-scaling interior point method was presented, completing the affine-scaling interior point family approach to the L_∞ fitting problem. Computational complexity and data storage are reduced for interior point approaches when dealing with polynomial fitting problems. Numerical experiments indicate that interior point approaches rarely perform better than the Simplex Method for the tested problems.

Joo-Siong and Chuan (2005), described the importance of numerical stability for interior-point methods applied to Linear Programming LP and Semidefinite Programming SDP. They analyzed the difficulties inherent in current methods and presented robust algorithms. They started with the error bound analysis of the search directions for the normal equation approach for LP. Their error analysis explains the surprising fact that the ill-conditioning was not a significant problem for the normal equation system. They also explained why most of the popular LP solvers have a default stop tolerance of only 10-8 when the machine precision on a 32-bit computer was approximately 10-16. They then proposed a simple alternative approach for the normal equation based interior-point method. This approach has better numerical stability than the normal equation based method. Although, their approach was not competitive in terms of CPU time for the NETLIB problem set, they do obtain higher accuracy. In addition, they obtain significantly smaller CPU times compared to the normal equation based direct solver, when we solve well-conditioned, huge, and sparse problems by using our iterative based linear solver. Additional techniques discussed are: crossover; purification step; and no backtracking. Finally, they presented an algorithm to construct SDP problem instances with prescribed strict complementarity gaps and then introduce two measures of strict complementarity gaps. We empirically show that:

i. these measures can be evaluated accurately.
ii. the size of the strict complementarity gaps correlate well with the number of iteration for the SDPT3 solver, as well as with the local asymptotic convergence rate.
iii. large strict complementarity gaps, coupled with the failure of Slater's condition, correlate well with loss of accuracy in the solutions. In addition, the numerical tests show that there is no correlation between the strict complementarity gaps and the geometrical measure used in, or with Renegar's condition number.

Lesaja (2009), explained how in recent years the introduction and development of Interior-Point Methods has had a profound impact on optimization theory as well as practice, influencing the field of Operations Research and related areas. Development of these methods has quickly led to the design of new and efficient optimization codes particularly for Linear Programming.

Consequently, there has been an increasing need to introduce theory and methods of this new area in optimization into the appropriate undergraduate and first year graduate courses such as introductory Operations Research and/or Linear Programming courses, Industrial Engineering courses and Math Modeling courses. The objective of this paper is to discuss the ways of simplifying the introduction of Interior-Point Methods for students who have various backgrounds or who are not necessarily mathematics majors.

Friedlander et al. (2004), proposed an interior-point method for solving mathematical programs with equilibrium constraints (MPECs). At each iteration of the algorithm, a single primal-dual step is computed from each subproblem of a sequence. Each subproblem is defined as a relaxation of the MPEC with a nonempty strictly feasible region. In contrast to previous approaches, the proposed relaxation scheme preserves the nonempty strict feasibility of each subproblem even in the limit. Local and superlinear convergence of the algorithm is proved even with a less restrictive strict complementarity condition than the standard one. Moreover, mechanisms for inducing global convergence in practice are proposed. Numerical results on the MacMPEC test problem set demonstrate the fast-local convergence properties of the algorithm.

CHAPTER 3

METHODOLOGY

3.0 INTRODUCTION

The problem of solving a system of linear inequalities dates back at least as far as Fourier, after whom the method of Fourier-Motzkin elimination is named. Linear programming arose as a mathematical model developed during World War II to plan expenditures and returns in order to reduce costs to the army and increase losses to the enemy. It was kept secret until 1947. After the war, many industries found its use in their daily planning. (Wikimedia Foundation, Inc, 2010)

The founders of the linear programming are Leonid Kantorovich, a Russian mathematician who developed linear programming problems in 1939, George B. Dantzig, who published the simplex method in 1947, and John von Neumann, who developed the theory of the duality in the same year. The linear programming problem was first shown to be solvable in polynomial time by

Leonid Khachiyan in 1979, but a larger theoretical and practical breakthrough in the field came in 1984 when Narendra Karmarkar introduced a new interior point method for solving linear programming problems. (Wikimedia Foundation, Inc, 2010)

Linear programming (LP) is a mathematical method for determining a way to achieve the best outcome (such as maximum profit or minimum cost) in a given mathematical model for some list of requirements represented as linear relationships. Linear programming is a specific case of mathematical programming.

More formally, linear programming is a technique for the optimization of a linear objective function, subject to linear equality or linear inequality constraints. Linear programs are problems that can be expressed in canonical form:

$$\text{Maximize} \quad C^T X$$

$$\text{Subject to} \quad AX \leq b$$

$$X \geq 0$$

Where X represents the vector of variables (to be determined), C and b are vectors of (known) coefficients and A is a (known) matrix of coefficients. The expression to be maximized or minimized is called the objective function ($C^T X$ in this case). The equations $AX \leq b$ are the constraints which specify a convex polytope over which the objective function is to be optimized. (In this context, two vectors are comparable when every entry in one is less-than or equal-to the corresponding entry in the other. Otherwise, they are incomparable.)

3.1 USES OF LINEAR PROGRAMMING

Linear programming is a considerable field of optimization for several reasons. Many practical problems in operations research can be expressed as linear programming problems. Linear programming can be applied to various fields of study. It is used most extensively in business and economics, but can also be utilized for some engineering problems. Industries that use linear programming models include transportation, energy, telecommunications, and manufacturing. It has proved useful in modeling diverse types of problems in planning, routing, scheduling, assignment, and design.

3.2 INTERIOR POINT METHOD (IPM)

An interior point method is a linear or nonlinear programming method that achieves optimization by going through the interior of the solid defined by the problem rather than around its surface (Forsgren, et al. 2002). A polynomial time linear programming (LP) algorithm using an interior point method was found by Narendra Karmarkar (Forsgren et al, 2002). Interior point methods were known as early as the 1960s in the form of the barrier function methods. Narendra Karmarkar proposed a new polynomial algorithm for LP that held great promise and performed well in practice. The main idea of this algorithm is quite different from Simplex Method. Unlike Simplex Method, iterates are calculated not on the boundary, but in the interior of the feasible region. This algorithm is an iterative algorithm that makes use of projective transformations and a potential function (Karmarkar's potential function) (Forsgren et al, 2002). The current iterate is mapped to the center of the special set in the interior feasible region using a projective transformation. This set is an intersection of the standard simplex and a hyperplane obtained from the constraints. Then, the potential function is minimized over the ball inscribed in the set. The minimiser is mapped back to the original space and becomes a new iterate.

3.2.1 COMPARATIVE DISCUSSION BETWEEN SIMPLEX ALGORITHMS AND INTERIOR POINT ALGORITHMS

The simplex algorithm solve a linear programming problem by moving along the edges of the polytope defined by the constraints, from vertices to vertices with successively smaller values of the objective function, until the minimum is reached but, does not find the solution exactly, as showed in figure 3.1. In contrast to the simplex algorithm, interior point algorithms for linear programming iterate from the interior of the polytope defined by the constraints. They find the solution and get closer to it very quickly.

All forms of the Simplex method reach the optimum by traversing a series of basic solutions. Since each basic solution represents an extreme point of the feasible region, the path followed by the algorithm moves around the boundary of the feasible region. This can be inefficient since the number of extreme points can become very large. In contrast to the simplex algorithm, interior point method approaches the optimum from the interior of the feasible solution space. Only in the limit does the solution approach an optimum solution at the boundary of the feasible region. The development of the interior point methods is a very important step in the theory and practice of optimization.

Interior point algorithm is a polynomial time algorithms. This means that the time required to solve an LP problem of size n would take at most an^b where a and b are two positive numbers. On the other hand, the Simplex algorithm is an exponential time algorithm in solving LP problems (Kumar et al, accessed April, 2011). This implies that, in solving an LP problem of size n there exists a positive number such that for any of the Simplex algorithm would find its solution in a time of at most $c2^n$. For large enough n (with positive a, b and c), $c2^n > an^b$. This means that, in theory, the polynomial time algorithms are superior to exponential algorithms for large LP problems.

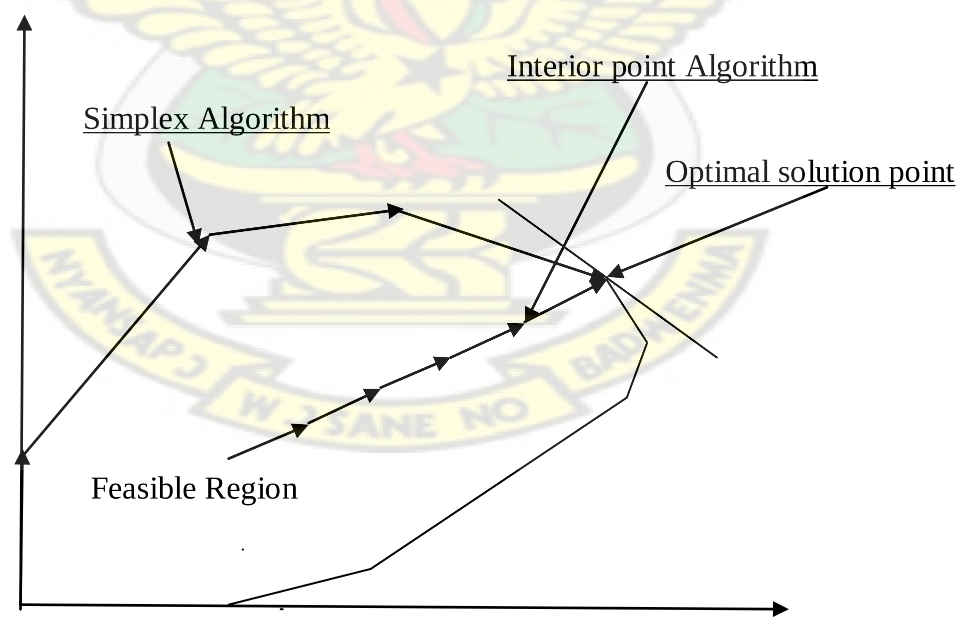


Figure 3.1; difference in optimum search path between simplex algorithm and interior point algorithm

3.3 INTERIOR-POINT METHODS FOR LINEAR PROGRAMMING (LP) PROBLEMS

3.3.1 KARUSH-KUHN-TUCKER (KKT) CONDITION FOR LINEAR PROGRAMMING PROBLEMS

Consider an LP problem in the standard form:

Given the data, vector $b \in \mathbb{R}^m$, $C \in \mathbb{R}^n$ and matrix $A \in \mathbb{R}^{m+n}$, find a vector $x \in \mathbb{R}^n$ that solves the problem;

$$\text{P:} \qquad \begin{aligned} &\text{Minimize} && C^T x \\ &\text{Subject to} && Ax = b \\ & && x \geq 0 \end{aligned} \qquad (3.1)$$

The vector $x \in \mathbb{R}^n$ is called a vector of primal variables and the set $F_p = \{x: A = xb, x \geq 0\}$ is called primer feasible region.

The corresponding dual problem is then given by:

$$\text{Maximize} \qquad b^T y$$

D: Subject to $A^T y + s = C$ (3.2)

$$y, s \geq 0$$

The vector $y \in \mathbb{R}^n$ is called a vector of dual variables and the vector $s \in \mathbb{R}^n$ is called a vector of dual slack variables. The set $F_d = \{(y, s): A^T y + s = C,\ y, s \geq 0\}$ is called a dual feasible region.

Consider now a logarithmic barrier reformulation for the primal problem (3.1).

Minimize $C^T x - \mu \sum_{i=1}^{n} \ln x_i$

P: Subject to $Ax = b$ (3.3)

$$x \geq 0$$

Problem (3.1) and (3.2) are equivalent in the sense that they have the same solution sets. The Lagrange function for the problem (3.3) is

$$L(x, y) = C^T x - \mu \sum_{i=1}^{n} \ln x_i - y^T (Ax - b), \tag{3.4}$$

from which the Karush-Kuhn-Tucker (KKT) conditions can be derived

$$\nabla_x L(x, y) = C - \mu X^{-1} e - A^T y = 0$$

$$\nabla_y L(x, y) = b - Ax = 0 \tag{3.5}$$

$$y, x > 0,$$

Where $X \in \mathbb{R}^{nxn}$ represents a diagonal matrix with the components of the vector $x \in \mathbb{R}^n$on its diagonal, $e \in \mathbb{R}^n$ is a vector of ones, and $\mu > 0$ is a parameter. Using the transformation $s = \mu X^{-1} e$, system (3.5) becomes

$$A^T y + s = C$$

$$Ax = b, \quad x > 0 \tag{3.6}$$

$$Xs = \mu e$$

The logarithmic barrier model for the dual LP problem (3.2) is

$$\text{Maximize} \quad b^T y - \mu \sum_{i=1}^{n} \ln s_i$$

$$\text{D:} \qquad \text{Subject to} \quad A^T y + s = C \tag{3.7}$$

$$y, s \geq 0$$

The KKT conditions for the above problem are

$$\nabla_x L(x, y, s) = A^T y + s - C = 0$$

$$\nabla_y L(x, y, s) = b - Ax = 0 \tag{3.8}$$

$$\nabla_s L(x, y, s) = \mu S^{-1} e - x = 0$$

$$s > 0$$

Or equivalently

$$A^T y + s - C = 0, s > 0$$

$$Ax - b = 0, x > 0 \tag{3.9}$$

$$Xs = \mu e$$

Combining the KKT conditions for the primal (3.6) and dual (3.9) barrier models we obtain primal-dual KKT conditions

$$A^T y + s - C = 0, s > 0$$

$$Ax - b = 0, x > 0 \tag{3.10}$$

$$Xs = \mu e$$

The above conditions are very similar to the original KKT conditions for LP.

$$A^T y + s - C = 0, s > 0 \qquad \leftarrow \text{Dual feasibility}$$

$$Ax - b = 0, x > 0 \qquad \leftarrow \text{Primal feasibility} \tag{3.11}$$

$$Xs = 0 \qquad \leftarrow \text{Complementarity}$$

The only difference between (3.10) and (3.11) are strict positivity of the variables and perturbation of the complementarity equation. The complementarity equation in (3.11) can be written as $x^T s = 0$, also $x^T s = b^T y - C^T x$ and therefore $x^T s$ can be viewed as a primal-dual gap between objective functions. Hence, the complementarity condition in (3.11) can be interpreted as the condition of primal-dual gap being zero.

3.3.2 CENTRAL PATH

Let $(x^* y^* s^*)$ be a solution of problem (3.11), then (x^*) is a solution of the primal LP problem (3.1) and $(y^* s^*)$ is a solution of the dual LP problem (3.2). The system (3.10) was parameterized in $\mu > 0$ if rank $(A) = m$. Therefore, the solution is denoted as $(x(\mu),\ y(\mu),\ s(\mu))$ such that $x(\mu)$ solves for (3.1) and $(y(\mu),\ s(\mu))$ solves for (3.2). The set of μ −center gives a homotopy path, which is called the central path of (3.1) and (3.2) respectively. The solution $(x^* y^* s^*)$ is being obtained when limit of the central path exist as $\mu \to 0$. Tracing the central path while reducing μ at each iteration for (3.1) and (3.2), Barrier Method was introduced to solve (3.11), (Meggido, 1989).

3.3.3 BARRIER METHOD (BM)

The generic Barrier Method can be stated as follows;

Step 1: Given μ_k, w^k, $f(w)$ solve system (3.10) by appropriate Modified Newton's Method (MNM).

Step 3: Decrease the value of $\mu_k \leftarrow \mu_{k+1}$

Step 4: Set $k \leftarrow k+1$ and go to step 1

However, tracing the central path exactly, thus, solving the system (3.10) with very high accuracy using BM would be too costly and inefficient. The preferred method of choice for finding an approximate solution of the system (3.10) is Modified (damped) Newton's Method

3.3.4 BARRIER METHOD WITH MODIFIED (DAMPED) NEWTON'S METHOD

The MNM is formalized below.

Step 1: Given μ_k, w^k, $f(w)$ solve system (3.10) by appropriate Modified Newton's Method.

Step 2a: find the search direction d_w by solving the linear system $\nabla f(w) d_w = -f(w)$.

Step 2b: Find step size α_κ.

Step 2c: Update w^k to $w^{k+1} = w^k + \alpha_\kappa d_w^k$.

Step 3: Decrease the value of $\mu_k \leftarrow \mu_{k+1}$

Step 4: Set $k \leftarrow k+1$ and go to step 1

where

$$f(w) = \begin{bmatrix} Ax - b \\ A^T y + s - C \\ Xs - \gamma\mu e \end{bmatrix} \tag{3.12}$$

and

$$\nabla f(x^k, y^k, s^k)\begin{bmatrix} d_x \\ d_y \\ d_s \end{bmatrix} = -f(x^k, y^k, s^k) \tag{3.13}$$

implies

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^k & I \\ S^k & 0 & X^k \end{bmatrix}\begin{bmatrix} d_x^k \\ d_y^k \\ d_s^k \end{bmatrix} = \begin{bmatrix} b - Ax \\ C - s^k - A^k y^k \\ \gamma\mu_k e - X^k S^k \end{bmatrix} = \begin{bmatrix} r_P^k \\ r_D^k \\ \gamma\mu_k e - X^k S^k \end{bmatrix} \tag{3.14}$$

$w = (x, y, s)$, r_P^k and r_D^k are called primal and dual residuals respectively and γ is a scaling factor. The statement that approximate solutions of (3.10), should not be "too far" from the central path is formalized by introducing the horn neighbourhood of the central path. The horn neighbourhoods of the central path can be defined using different norms.

$$N_2(\beta) = \{(x, s): \|Xs - \mu e\|_2 \leq \beta\mu\}, \tag{3.15}$$

$$N_\infty(\beta) = \{(x, s): \|Xs - \mu e\|_\infty \leq \beta\mu\}, \tag{3.16}$$

or even a pseudo norm

$$N_\infty^-(\beta) = \{(x, s): \|Xs - \mu e\|_\infty^- \leq \beta\mu\} = \{(x, s): Xs \geq (1 - \beta)\mu\}, \tag{3.17}$$

$\|Z\|_\infty^- = \|Z^-\|_\infty$ and $(Z^-)_j = min\{Z_j, 0\}$. These neighbourhoods have the following inclusion relations among them

$$\Gamma \subseteq N_2(\beta) \subseteq N_\infty(\beta) \subseteq N_\infty^-(\beta). \tag{3.18}$$

The step size (α_k) is chosen in such a way that iterates stay in one of the above horn neighbourhoods.

$$\alpha_k = \max\{\alpha' : \|\, X(\alpha)s(\alpha) - \mu(\alpha)e \,| \leq \beta\mu(\alpha) \; for \propto \in [0, \alpha']\}, \tag{3.19}$$

where

$$x(\propto) = x^k + \propto d_x$$

$$s(\propto) = s^k + \propto d_s \quad (3.20)$$

$$\mu(\alpha) = \frac{x^T(\propto)s(\propto)}{n}$$

Now, the first step of the barrier algorithm BM can be completed by calculating the new iterates

$$x^{k+1} = x^k + \alpha_k d_x^k$$

$$y^{k+1} = y^k + \alpha_k d_y^k \quad (3.21)$$

$$s^{k+1} = s^k + \alpha_k d_s^k$$

The second step of BM is the calculation of μ_{k+1} using the last equation in (3.20). It can be shown that the sequence $\{\mu_k\}$ is decreasing at least at a constant rate. An iterate (x^k, y^k, s^k) is an ε −approximate optimal solution if

$$\|Ax^k - b\| \le \varepsilon_P, \|A^k y^k + s^k - C\| \le \varepsilon_D, ((x^k)^T s^k \le \varepsilon_G \quad (3.22)$$

For a given $(\varepsilon_P, \varepsilon_D, \varepsilon_G) > 0$

3.4 INTERIOR POINT ALGORITHMS

Step 1: choose $\beta, \gamma \epsilon$ (0,1) and $(\varepsilon_P \varepsilon_D \varepsilon_G) > 0$

Choose (x^0, y^0, s^0) and such that $(y^0, s^0) > 0$ and $\|X^0 s^0 - \mu_0 e\| \le \beta\mu_0$

where $\mu_0 = \frac{(x^0)^T S^0}{n}$

Step 2: set $k = 0$

Step 3: Set $r_p^k = b - Ax^k$, $r_D^k = C - A^T y^k - s^k$, $\mu_k = \frac{(x^k)^T S^k}{n}$

Step 4: Check the termination. If $\|r_p^k\| \le \varepsilon_P, \|r_D^k\| \le \varepsilon_D, ((x^k)^T s^k \le \varepsilon_G$

Step 5: Compute the direction by soling the system

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^k & I \\ S^k & 0 & X^k \end{bmatrix} \begin{bmatrix} d_x^k \\ d_y^k \\ d_s^k \end{bmatrix} = \begin{bmatrix} r_P^k \\ r_D^k \\ \gamma\mu_k e - X^k S^k \end{bmatrix}$$

Step 6: Compute the step size

$$\alpha_k = \max\{\alpha' : \| X(\alpha)s(\alpha) - \mu(\alpha)e \| \leq \beta\mu(\alpha) \; for \propto \in [0, \alpha']\},$$

Where $x(\propto) = x^k + \propto d_x$, $s(\propto) = s^k + \propto d_s$, $\mu(\alpha) = \frac{x^T(\propto)s(\propto)}{n}$

Step 7: $x^{k+1} = x^k + \alpha_k d_x^k,\ y^{k+1} = y^k + \alpha_k d_y^k,\ s^{k+1} = s^k + \alpha_k d_s^k$

Step 8: $k \leftarrow k + 1$ and go to step 3.

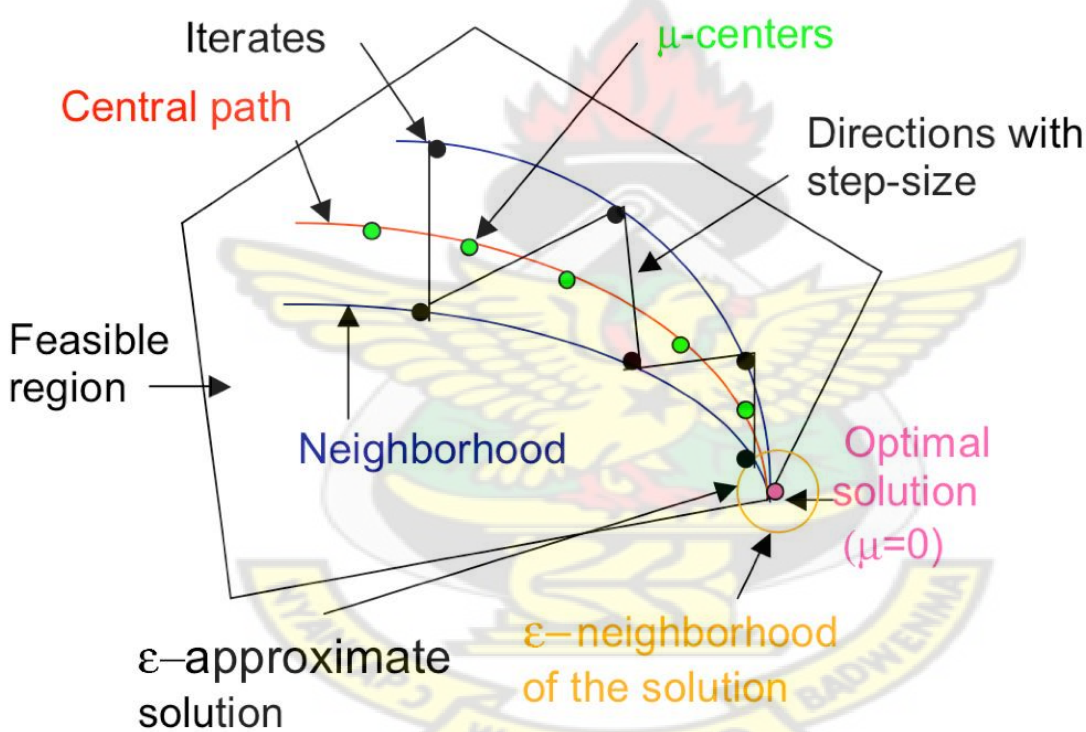


Figure 3.2; Graphical representation of the IPM algorithm: where $\varepsilon = tolerance$

There are many modifications and variations of this algorithm and it represents a broad class of algorithms. For example, we can consider different neighbourhoods of the central path. Because of the relation in (3.18), if $N_2(\beta)$ is used, IMP is called a short-step algorithm, and if $N_\infty(\beta)$ or $N_\infty^-(\beta)$ is selected, IMP is called a long-step algorithm.

The IPMs are iterative algorithms which produce only an $\varepsilon -$ approximate optimal solution of the problem. However, as in the case of the Ellipsoid and Karmarkar's algorithms, it can be shown that if the input data are rational numbers, the IPM finds the exact solution of LP in $O(\sqrt{n}L)$ iterations proving that this is the algorithm with the best known polynomial iteration complexity. Nevertheless, this can still correspond to very large number of iterations. However, it is possible to perform far less iteration and still be able to recover the exact optimal solution of the problem. *(Lesaja, 2009)*

The main idea of the method is to perform orthogonal projection of an iteration to the optimal set when the iteration is "near" the optimal set (there are several different criteria as to how to determine when the iterate is "near" the optimal set.). Another interesting fact is that in the case when LP problem has infinitely many optimal solutions, IPMs tend to find an exact optimal solution that is in the "center" of the optimal set as opposed to the Simplex method (SM) that finds the "corner" (vertex) of the optimal set. However, it is possible to recover a vertex optimal solution as well.

The IPM is also a path-following algorithm since iterates are required to stay in the horn neighbourhood of the central path. These algorithms are designed to reduce the primal-dual gap (μ) directly in each iteration. There is another group of interior-point algorithms that are designed to reduce the primal-dual gap (μ) indirectly in each iteration. This algorithm directly reduces the objective function to a constant number in each iteration and known as potential-reduction algorithm or Karmarkar’s algorithm. Iterates of these algorithms do not necessarily stay in the horn neighbourhood of the central path, *(Lesaja, 2009)*.

The interior-Point Algorithm can now be summarized in the following types of IPMs.

There are at least three major types of IPMs:

(1) the Karmarkar algorithm.

(2) the affine scaling algorithm.

(3) the primal-dual path following algorithm.

3.5 KARMARKAR’S METHOD

Karmarkar's algorithm is an algorithm introduced by Narendra Karmarkar in 1984 for solving linear programming problems. It was the first reasonably efficient algorithm that solves these problems in polynomial time, *(Wikimedia Foundation Inc., 2011)*. Karmarkar's algorithm falls within the class of interior point methods: the current guess for the solution does not follow the boundary of the feasible set as in the Simplex method, but it moves through the interior of the feasible region and reaches the optimal solution only asymptotically. We consider first the linear programming problem, which is undoubtedly the optimization problem solved most frequently in practice.

3.5.1 DESCRIPTION OF KARMARKAR’S ALOGORITHM

Unlike the Simplex method, the sequence of iterates in Karmarkar's algorithm is easy to visualize. Instead of moving sedately from one corner to an adjacent one in the feasible region where the value of the objective function is reduced, it starts from the centre of a polygonal feasible region and moves linearly in the "direction of steepest descent" toward the boundary, and arrives at the solution fairly quickly. This "direction of steepest descent" is given by the negative gradient of the objective function. Implementation problems are:

i. The determination of the "centre" of a polygonal region.

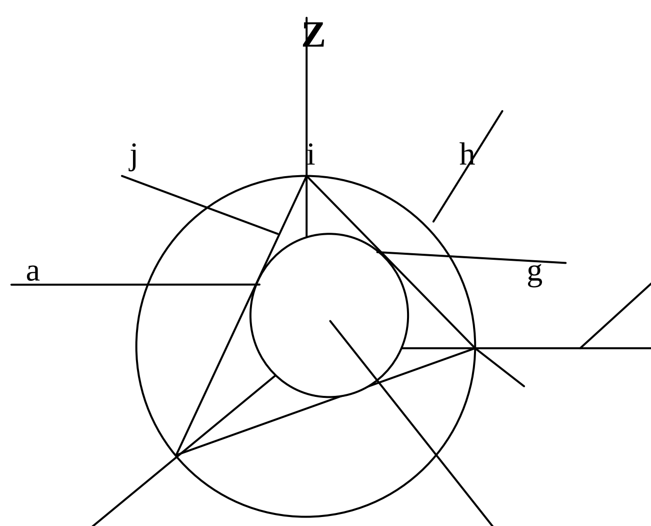
ii. Movement of the current interior point to the center of this region

Instead of treating the implementation problem, Karmarkar solves a geometric problem and generalized the result (Lemire, 1989). The simplified problem involves an (n-1)-dimensional unit simplex which is defined by

$$\Delta^{n-1} = \{ X \epsilon \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = 1, x_i \geq 0 \} = \{ X \epsilon \mathbb{R}^n \mid e^T X = 1, X \geq 0 \} \quad (3.23)$$

Where $e^T = (1, \ldots, 1)$.

Figure 3.2 comprises of a triangle (j), an internal circle (c) and external circle (h) all lie on the xyz-plane. (a, g, d) show the intersections between the edges of (j) and (c). Also (i, b, f) show the intersections between vertices of (j) and (h). The feasible region is represented by (e).



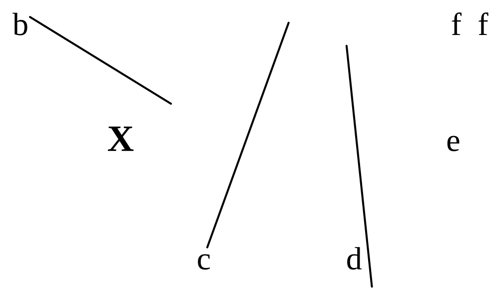


Figure 3.3 graphical representation of feasible region

where

$a = \left(\frac{1}{2}, 0\frac{1}{2}\right)^T$ $\quad$ $b = (1,0,0)^T$ $\quad$ $c = B(a_0, r)$

$d = \left(\frac{1}{2}, \frac{1}{2}, 0\right)^T$ $\quad$ $e = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^T$ $\quad$ $f = (0,1,0)^T$

$g = \left(0, \frac{1}{2}, \frac{1}{2}\right)^T$ $\quad$ $h = B(a_0, R)$ $\quad$ $i = (0,0,1)$ $\quad$ j=Δ^2

Therefore, Δ^0 consists of +1 on the $\mathbb{R}$ line. The Δ^1, is the straight line segment between $(1,0)^T$ and $(0,1)^T$. The triangular region in the first octant with corners $(1,0,0)^T$, $(0,1,0)^T$ and $(0,0,1)^T$ comprises the Δ^2. From a diagram of any of these low-dimensional simplexes (n >1), it is clear that $\frac{e}{n} \in \mathbb{R}^n$ is the centre of the unit, Δ^{n-1}. This result also holds in general. Karmarkar's idea is to transform the original problem such that it's feasible region in $\mathbb{R}^{n-1}$ is contained in Δ^{n-1}. In order to do this, he uses projective transformations. Projective transformations are non-linear transformations under which lines and subspaces are preserved but distances and angles are distorted. In Karmarkar's method, two types of projective transformations are used:

The first T_1, takes the polytope in positive orthant:

$$P+= \{x \in \mathbb{R}^{n-1} \mid x \geq 0\} \tag{3.24}$$

and "scrunches" it into Δ^{n-1}. This is type of projective transformation, it transforms the original (n-1) polytope ((n-1) -dimensional polygon) to lie within Δ^{n-1}. It maps a given initial strictly positive feasible point to the centre of Δ^{n-1}. The second, T_2 maps from Δ^{n-1} to itself and map the current interior point to the centre of Δ^2. Each time one of these transformations is used, the objective function is adjusted accordingly so that the "direction of steepest descent" changes in each iteration. To prevent the iterates of the algorithm from moving outside the boundary of the feasible region. Karmarkar use the geometry of Δ^{n-1} to solve this problem. Within an (n-1) -dimensional simplex, it is possible to inscribe an (n-1)-dimensional sphere defined by;

$$S^{n-1} = \{x \in \mathbb{R}^{n-1} \mid (x - a_0)^T(x - a_0) \leq r^2\} \tag{3.25}$$

with centre $a_0 = \frac{e}{n}$ and radius, r. From fig 3.3 of the two-dimensional Δ^2, it makes sense that this sphere should touch Δ^2 at the points: $\left(\frac{1}{2}, \frac{1}{2}, 0\right)^T$, $\left(\frac{1}{2}, 0 \frac{1}{2}\right)^T$ and $\left(0, \frac{1}{2}, \frac{1}{2}\right)^T$. In general, S^{n-1} touches Δ^{n-1} at the n points: $(0, \frac{1}{n-1}, \frac{1}{n-1}, \dots, \frac{1}{n-1})^T, \ . \ . \ . \ , (\frac{1}{n-1}, \frac{1}{n-1}, \dots, \frac{1}{n-1}, 0)^T$. By the Euclidean distance formula:

$$r = \sqrt{\frac{1}{n(n-1)}} \tag{3.26}$$

This sphere will be subsequently denoted by B(a_0, r). There also exists a smaller circumscribing (n-1)- dimensional sphere for Δ^{n-1}. In a similar, it may be observed that this sphere has centre, $\frac{e}{n}$ and points $(1,0,0, \dots, 0)^T$, $(0,1,0, \dots, 0)^T$, . . . ,$(0,0, \dots, 0,1)^T$. This implies that its radius must be

$$R = \sqrt{\frac{n-1}{n}} = (n-1)r \tag{3.27}$$

it shall be noted as $B(a_0, R)$. Karmarkar uses the sphere, $B(a_0, \alpha r)\ where\ 0 < \alpha < 1$, to ensure that all iterates are interior points thus, they are strictly positive feasible points. This is necessary in order to apply his projective transformations. It is clear that $B(a_0, \alpha r) \subset \Delta^{n-1}, \subset B(a_0, R)$, for all α. Thus, if we minimize the objective function over the inner sphere, $B(a_0, \alpha r)$, we will produce iterate that are interior points. It turns out that minimizing an objective function over an (n-1)-dimensional sphere is a much simpler task than minimizing it over Δ^{n-1}.

3.5.2 CONVERSION OF LPP STANDARD FORM INTO KARMARKAR'S STANDAND FORM

Let the LPP be given in standard form:

$$\text{P:} \qquad \begin{aligned} &\text{Minimize} \quad C^T X \\ &\text{Subject to} \quad AX = b \\ &\qquad\qquad X \geq 0 \end{aligned} \tag{3.28}$$

The following are assumptions for Karmarkar's algorithm:

i. The minimal value of the objective function is 0

ii. $\frac{e}{n}$ is a feasible point for this LP

iii. A has rank m

Converting an LP problem into the standard form of Karmarkar, the assumptions (i, ii and iii) must be satisfied. The key feature of the karmarkar's standard form is the simplex structure, which of course results in a bounded feasible region. Regularizing problem (3.28) we add a bounding constraint

$$e^T X = x_1 + x_2 + \cdots + x_n \leq Q \tag{3.29}$$

For Q is a positive integer, derived from the feasibility and optimality considerations. We can choose $Q = 2^L$, where L is the problem size (number of variables). By introducing a slack variable x_{n+1}, we have a new linear program:

$$\text{P:} \quad \begin{aligned} &\text{Minimize} \quad C^T X \\ &\text{Subject to} \quad AX = b \\ &e^T X + x_{n+1} = Q \\ &X, x_{n+1} \geq 0 \end{aligned} \tag{3.30}$$

Keeping the matrix structure of A undisturbed for sparsity manipulation, we introduce a new variable $x_{n+2} = 1$ and rewrite the constraints of (3.30) as

$$AX - bx_{n+2} = 0 \tag{3.31}$$

$$e^T X + x_{n+1} + Qx_{n+2} = 0 \tag{3.32}$$

$$e^T X + x_{n+1} + x_{n+2} = Q + 1 \tag{3.33}$$

$$X, x_{n+1}, x_{n+2} \geq 0$$

Note that the constraint $x_{n+2} = 1$ is direct consequence of (3.32) and (3.33). Normalizing (3.33) for the required simplex structure, apply the transformation:

$$x_j = (Q + 1)y_j, j = 1,2, \ldots, n + 2. \tag{3.34}$$

Now, we have an equivalent linear program

$$
\begin{aligned}
&\text{Minimize} && (Q+1)C^T Y \\
&\text{Subject to} && AY - by_{n+2} = 0 \\
& && e^T Y + y_{n+1} - Q y_{n+2} = 0 \\
& && e^T Y + y_{n+1} - y_{n+2} = 1 \\
& && Y, y_{n+1}, y_{n+2} \geq 0
\end{aligned}
\tag{3.35}
$$

The problem (3.35) is now in the standard form required by the Karmarkar agolrithm. In order to satisfy the assumption (ii), we may introduce an artificial variable y_{n+3} with a large cost coefficient M in Big-M method. Big M is a positive finite and large but not too large to produce accumulation of round off errors during iterations. Example:

$$
\begin{aligned}
& && \text{Minimize:} && 0.2x_1 + 0.5x_2 \\
&\text{P:} && \text{Subject to:} && 3x_1 + 2x_2 \geq 6 \\
& && && x_1 + 2x_2 \leq 4 \\
& && && x_1, x_2 \geq 0
\end{aligned}
\tag{3.36}
$$

With M=10, computer gives the optimal solution $x_1 = 1$ and $x_2 = 1.5$ while with M=999999 the optimal solution is $x_1 = 4$ and $x_2 = 0$. Note that first solution is correct.

Generic karmarkar's standard form

$$\begin{aligned}
\text{Minimize} \quad & (Q+1)C^TY + My_{n+3} \\
\text{Subject to} \quad & AY - by_{n+2} - [Ae - b]y_{n+3} = 0 \\
& e^TY + y_{n+1} - Qy_{n+2} - (n+1-Q)y_{n+3} = 0 \\
& e^TY + y_{n+1} + y_{n+2} + y_{n+3} = 1 \\
& Y, y_{n+1}, y_{n+2}, y_{n+3} \geq 0
\end{aligned} \tag{3.37}$$

This form (3.37) satisfies assumption (ii) as $(\frac{1}{n+3}, \frac{1}{n+3}, \dots, \frac{1}{n+3})$ is the interior point solution and its minimum value is zero (assumption (i)).

3.5.3 FORMULATION OF KARMARKAR`S ALGORITHM

Given the generic LP problem in Karmarkar form;

$$P: \quad \begin{aligned}
\text{Minimize:} \quad & C^TX \\
\text{Subject to:} \quad & AX = 0 \\
& e^TX = 1 \\
& X \geq 0
\end{aligned} \tag{3.38}$$

where

$A \in \mathbb{R}^{m+n}$

$X, e, C \in \mathbb{R}^n$

First of all, due to the last two constraints, the feasible region is contained in the simplex, Δ^{n-1}. In addition, the centre of this simplex is already an interior point due to assumption (ii). Thirdly, since the minimum value of the objective function is zero, we may terminate the algorithm when $C^T x = 0$ or when $C^T x$ is within the binary precision (i.e. $\frac{C^T x}{C^T a_0} \leq 2^{-L1}$), (Lemire, 1989).

where $L1$ is

$$L1 \coloneqq \lceil 1 + log_2 n + log_2 m + \sum_{i=1}^{i=m} \sum_{j=1}^{j=n} \left(1 + log_2\left(1 + |a_{ij}|\right)\right) + \sum_{i=1}^{i=m}\left(1 + log_2(1 + |b_i|)\right)\rceil \tag{3.39}$$

Consider the projective transformation defined by;

$$T_2(x) = \frac{D^{-1}x}{e^T D^{-1}x} \tag{3.40}$$

Where a is an interior point and $D = diag(a_1, a_2, \ldots, a_n)$

This transformation, $T_2: (\Delta^{n-1} \rightarrow \Delta^{n-1})$ is a one to one correspondence and the point a is mapped to the centre of the simplex, Δ^{n-1}.

Its inverse T_2^{-1} is given by;

$$T_2^{-1}(y) = \frac{Dy}{e^T Dy} \tag{3.41}$$

This means that T_2 has the desired properties of our second projective transformation. In order to use it to solve the original problem, we must also transform the objective function and the feasibility conditions. If $y = T_2(x)$, the feasibility constraints are transformed as:

$$\begin{gathered} Ax = 0 \Leftrightarrow AT_2^{-1}(y) = 0 \Leftrightarrow AT_2(x) = 0 \\ e^T x = 1 \Leftrightarrow e^T y = 1 \\ x \geq 0 \Leftrightarrow y \geq 0 \end{gathered} \tag{3.42}$$

The last two remain the same for y, the objective function $c^T x$ becomes $\frac{c^T Dy}{e^T Dy}$. Thus the problem becomes:

$$\text{Minimize:} \quad \frac{c^T Dy}{e^T Dy}$$

$$\text{Subject to:} \quad ADy = 0 \tag{3.43}$$

$$e^T y = 1$$

$$y \geq 0$$

However, the function, $e^T Dy$, may be approximated by a constant around the centre of the simplex, Δ^{n-1}. Therefore our new problem becomes:

$$\text{Minimize:} \quad C^T Dy$$

$$\text{Subject to:} \quad ADy = 0 \tag{3.44}$$

$$e^T y = 1$$

$$y \geq 0$$

To solve this problem we must bear in mind Karmarkar's trick of minimizing over the inscribed sphere to keep the iterates of his algorithm feasible. That implies that we are actually going to solve the problem:

$$\text{Minimize:} \quad \hat{c}^T y$$

$$\text{Subject to:} \quad ADy = 0 \tag{3.45}$$

$$e^T y = 1$$

$$(y - a_0)^T (y - a_0) \leq \propto^2 r^2$$

where $\quad \hat{c} = D^T c,\ 0 < \propto < 1,\ r = \frac{1}{\sqrt{n(n-1)}}$

The only difference here is the additional constraint, $ADy = 0$. The optimal solution becomes $a_0 - \propto r\check{c}_p$ where $\check{c}_p$ is the projection of $\hat{c}$, having unit length, onto the nullspace of B and

$$B = \begin{bmatrix} AD \\ e^T \end{bmatrix} \tag{3.46}$$

The formulae that define $\check{c}_p$ are:

$$C_p = (I - B^T(BB^T)^{-1}B)\hat{c} \tag{3.47}$$

and

$$\check{c}_p = \frac{c_p}{\|c_p\|} \tag{3.48}$$

We do not project $\hat{c}$ onto the affine space:

$$\{x \,\epsilon\, \mathbb{R}^n \mid Bx = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\} \tag{3.49}$$

This is because we want the optimal solution:

$$g = a_0 - \propto r\check{c}_p \tag{3.50}$$

To retain the property, $e^T x = 1$, in order to remain inside Δ^{n-1}. Since $e^T a_0 = 1$ already, we only require that $e^T \check{c}_p = 0$. The projection matrix is $I - B^T(BB^T)^{-1}B$ and it is not the standard projection matrix. This is due to the fact that the null space is orthogonal to the row space, not the column space. B is a full rank, $m + 1$, due to assumption (iii) which ensures A is of rank m.

3.5.4 KARMARKAR'S ALGORITHM

Step 1

Preliminary

$k = 0$

$x_k = (\frac{1}{n}, \dots, \frac{1}{n})^T$

$r = 1/\sqrt{n(n-1)}$

$\propto = \frac{n-1}{3n}$

Step 2

Iteration k

(a) Define the following:

$Y_k = X_k$

$D_k = diag(X_k)$

$P = \begin{bmatrix} AD_k \\ 1 \end{bmatrix}$

$\hat{c} = C^T D_k$

(b) Compute the following:

$C_p = [I - P^T(PP^T)^{-1}P]\hat{c}^T$

If $c_p = 0$, any feasible solution becomes an optimal solution. Stop

otherwise

$Y_{new} = Y_k - \propto r \frac{C_p}{\|C_p\|}$

$X_{k+1} = \frac{D_k Y_{new}}{e D_k Y_{new}}$

$Z = C^T X_{k+1}$

$k = k + 1$

Repeat iteration k until the objective function (z) value is less than or equal to zero

EXAMPLE

Carry out the first three iterations of Karmarkar's algorithm for the following problem;

Minimize: Z= $2x_1 + x_2 + 2x_3 - 2x_4$

Subject to $2x_1 + x_2 + 2x_3 - 2x_4 - 3x_5 = 0$

$$2x_1 + 0x_2 - x_3 + x_4 - 2x_5 = 0$$

$$x_1 + x_2 + x_3 + x_4 + x_5 = 1$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0;$$

SOLUTION

Preliminary Step:

$k = 0$

$$X_O = \left[\frac{1}{n}, \ldots, \frac{1}{n}\right]^T = \left[\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right]^T$$

$$r = \frac{1}{\sqrt{n(n-1)}} = \frac{1}{\sqrt{5(5-1)}} = \frac{1}{\sqrt{20}}$$

$$\propto = \frac{(n-1)}{3n} = \frac{(5-1)}{3*5} = \frac{4}{15}$$

Iteration 0;

$Y_O = X_O$

$$D_O = \mathrm{diag}(X_O) = diag\left[\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right]$$

$$A = \begin{bmatrix} 2 & 1 & 2 & -2 & -3 \\ 2 & 0 & -1 & 1 & -2 \end{bmatrix}$$

$$\mathrm{C}^{\mathrm{T}} = [2 \quad 1 \quad 2 \quad -2 \quad 0]$$

$$AD_O = \begin{bmatrix} 0.4 & 0.2 & 0.4 & -0.4 & -0.6 \\ 0.4 & 0 & -0.2 & 0.2 & -0.4 \end{bmatrix}$$

$$\hat{c} = \mathrm{C}^{\mathrm{T}} D_O = [0.4 \quad 0.2 \quad 0.4 \quad -0.4 \quad 0]$$

$$P = \begin{bmatrix} AD_O \\ 1 \end{bmatrix} = \begin{bmatrix} 0.4 & 0.2 & 0.4 & -0.4 & -0.6 \\ 0.4 & 0 & -0.2 & 0.2 & -0.4 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$P^T = \begin{bmatrix} 0.4 & 0.4 & 1 \\ 0.2 & 0 & 1 \\ 0.4 & -0.2 & 1 \\ -0.4 & 0.2 & 1 \\ -0.6 & -0.4 & 1 \end{bmatrix}$$

$$PP^T = \begin{bmatrix} 0.88 & 0.24 & 0 \\ 0.24 & 0.4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$(PP^T)^{-1} = \begin{bmatrix} 1.3587 & -0.8152 & 0 \\ -0.8152 & 2.9891 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}$$

$$P^T(PP^T)^{-1}P = \begin{bmatrix} 0.3652 & -0.2435 & -0.1130 & -0.2870 & 0.2783 \\ -0.2435 & 0.7454 & -0.3413 & -0.0587 & -0.1022 \\ -0.1130 & -0.3413 & 0.3326 & 0.2674 & -0.1457 \\ -0.2870 & -0.0587 & 0.2674 & 0.3326 & -0.2543 \\ 0.2783 & -0.1022 & -0.1457 & -0.2543 & 0.2239 \end{bmatrix}$$

$$C_P = [I - P^T(PP^T)^{-1}P]\hat{c}^T = [0.167 \quad -0.0613 \quad -0.0874 \quad -0.1526 \quad 0.1343]^T$$

$$\|C_P\| = \sqrt{0.167^2 + (-0.0613)^2 + (-0.0874)^2 + (-0.1526)^2 + 0.1343^2} = 0.2374$$

$$Y_{new} = Y_O - \propto r\frac{C_P}{\|C_P\|} = \left[\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5}\right]^T$$

$$-\frac{\left(\frac{4}{15}\right) * \left(\frac{1}{\sqrt{20}}\right) * [0.167 \quad -0.0613 \quad -0.0874 \quad -0.1526 \quad 0.1343]^T}{0.2374}$$

$$Y_{new} = [0.1649 \quad 0.2129 \quad 0.2184 \quad 0.2321 \quad 0.1718]^T$$

$$X_1 = Y_{new} = [0.1649 \quad 0.2129 \quad 0.2184 \quad 0.2321 \quad 0.1718]^T$$

$$Z = \mathrm{C}^{\mathrm{T}}X_1 = [2 \quad 1 \quad 2 \quad -2 \quad 0] * [0.1649 \quad 0.2129 \quad 0.2184 \quad 0.2321 \quad 0.1718]^T = 0.5154$$

k=0+1=1

Since $\mathrm{C}^{\mathrm{T}}X_1 > 0$, continue

Iteration 1;

$$D_1 = diag(X_1) = diag[0.1649 \quad 0.2129 \quad 0.2184 \quad 0.2321 \quad 0.1718]$$

$$\check{c} = C^T D_1 = [2 \quad 1 \quad 2 \quad -2 \quad 0] * diag[0.1649 \quad 0.2129 \quad 0.2184 \quad 0.2321 \quad 0.1718]$$

$$\check{c} = [0.3298 \quad 0.2129 \quad 0.4368 \quad -0.4642 \quad 0]$$

$$AD_1 = \begin{bmatrix} 2 & 1 & 2 & -2 & -3 \\ 2 & 0 & -1 & 1 & -2 \end{bmatrix} * diag[0.1649 \quad 0.2129 \quad 0.2184 \quad 0.2321 \quad 0.1718]$$

$$AD_1 = \begin{bmatrix} 0.3298 & 0.2129 & 0.4368 & -0.4642 & -0.5154 \\ 0.3298 & 0 & -0.2194 & 0.2321 & -0.3436 \end{bmatrix}$$

$$P = \begin{bmatrix} AD_1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.3298 & 0.2129 & 0.4368 & -0.4642 & -0.5154 \\ 0.3298 & 0 & -0.2321 & 0.2321 & -0.3436 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$P^T = \begin{bmatrix} 0.3298 & 0.3298 & 1 \\ 0.2129 & 0 & 1 \\ 0.4368 & -0.2321 & 1 \\ -0.4642 & 0.2321 & 1 \\ -0.5154 & -0.3436 & 1 \end{bmatrix}$$

$$PP^T = \begin{bmatrix} 0.826 & 0.0827 & 0 \\ 0.0827 & 0.3284 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$(PP^T)^{-1} = \begin{bmatrix} 1.242 & -0.3128 & 0 \\ -0.3128 & 3.1239 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}$$

$$P^T(\,PP^T)^{-1}P = \begin{bmatrix} 0.6069 & -0.2652 & -0.1134 & -0.2729 & 0.2765 \\ -0.2652 & 0.7437 & -0.3301 & -0.0618 & -0.0866 \\ -0.1314 & -0.3301 & 0.3544 & 0.2736 & -0.1665 \\ -0.2729 & -0.0618 & 0.2736 & 0.2967 & -0.2356 \\ 0.2765 & -0.0866 & -0.1665 & -0.2356 & 0.2122 \end{bmatrix}$$

$$C_P = [I - P^T(\,PP^T)^{-1}P]\hat{c}^T = [0.1425 \quad -0.0446 \quad -0.0858 \quad -0.1214 \quad 0.1093]^T$$

$$\|C_P\| = \sqrt{0.1425^2 + (-0.0446)^2 + \,(-0.0858)^2 + (-0.1214)^2 + 0.1093^2} = 0.2374$$

$$Y_{new} = Y_O - \propto r\frac{C_P}{\|C_P\|} = \left[\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5}\right]^T$$

$$-\frac{\left(\frac{4}{15}\right) * \left(\frac{1}{\sqrt{20}}\right) * [0.1425 \quad -0.0446 \quad -0.0858 \quad -0.1214 \quad 0.1093]^T}{0.2374}$$

$$Y_{new} = \ [0.1642 \quad 0.2112 \quad 0.2216 \quad 0.2305 \quad 0.1725]^T$$

$$D_1Y_{new} = \ diag[0.1649 \quad 0.2129 \quad 0.2184 \quad 0.2321 \quad 0.1718]$$
$$* \, [0.1642 \quad 0.2112 \quad 0.2216 \quad 0.2305 \quad 0.1725]^T$$

$$D_1Y_{new} = \ [0.0271 \quad 0.0450 \quad 0.0484 \quad 0.0535 \quad 0.0296]^T$$

$$1D_1Y_{new} = [1 \quad 1 \quad 1 \quad 1 \quad 1] * [0.0271 \quad 0.0450 \quad 0.0484 \quad 0.0535 \quad 0.0296]^T = 0.2036$$

$$X_2 = \frac{D_1Y_{new}}{1D_1Y_{new}} = \frac{[0.0271 \quad 0.0450 \quad 0.0484 \quad 0.0535 \quad 0.0296]^T}{0.2036}$$

$X_2 = [0.1330 \quad 0.2209 \quad 0.2377 \quad 0.2628 \quad 0.1456]^T$

$Z = \mathrm{C}^{\mathrm{T}} X_2 = [2 \quad 1 \quad 2 \quad -2 \quad 0] * [0.1330 \quad 0.2209 \quad 0.2377 \quad 0.2628 \quad 0.1456]^T = 0.4367$

k=1+1=2

Since $\mathrm{C}^{\mathrm{T}} X_1 > 0$, continue

Iteration 2:

$D_1 = diag(X_1) = diag[0.1331 \quad 0.2209 \quad 0.2377 \quad 0.2628 \quad 0.1456]$

$\check{c} = \mathrm{C}^{\mathrm{T}} D_1 = [2 \quad 1 \quad 2 \quad -2 \quad 0] * diag[0.1330 \quad 0.2209 \quad 0.2377 \quad 0.2628 \quad 0.1456]$

$\check{c} = [0.266 \quad 0.2209 \quad 0.4754 \quad -0.5256 \quad 0]$

$$AD_1 = \begin{bmatrix} 2 & 1 & 2 & -2 & -3 \\ 2 & 0 & -1 & 1 & -2 \end{bmatrix} * diag[0.1330 \quad 0.2209 \quad 0.2377 \quad 0.2628 \quad 0.1456]$$

$$AD_1 = \begin{bmatrix} 0.2661 & 0.2209 & 0.47543 & -0.5255 & -0.4368 \\ 0.2661 & 0 & -0.2377 & 0.2628 & -0.2912 \end{bmatrix}$$

$$P = \begin{bmatrix} AD_1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.2661 & 0.2209 & 0.4753 & -0.5255 & -0.4368 \\ 0.2661 & 0 & -0.2377 & 0.2628 & -0.2912 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$P^T = \begin{bmatrix} 0.2661 & 0.2661 & 1 \\ 0.2209 & 0 & 1 \\ 0.4753 & -0.2377 & 1 \\ -0.5255 & 0.2628 & 1 \\ -0.4368 & -0.2912 & 1 \end{bmatrix}$$

$$PP^T = \begin{bmatrix} 0.8126 & 0.0530 & 0 \\ -0.0530 & 0.2812 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$(PP^T)^{-1} = \begin{bmatrix} 1.2460 & 0.2357 & 0 \\ 0.2351 & 3.6010 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}$$

$$P^T(\ PP^T)^{-1}P = \begin{bmatrix} 0.4234 & -0.2871 & -0.1447 & -0.2611 & 0.3695 \\ -02871 & 0.7392 & -0.3185 & -0.0690 & -0.0647 \\ -0.1447 & -0.3185 & 0.3682 & 0.2774 & -0.1824 \\ -0.2611 & -0.0690 & 0.2774 & 0.2721 & -0.2195 \\ 0.2695 & -0.0647 & -0.1824 & -0.2195 & 0.1971 \end{bmatrix}$$

$$C_P = [I - P^T(\ PP^T)^{-1}P]\hat{c}^T = [0.1177 \quad -0.0282 \quad -0.0797 \quad -0.0959 \quad 0.0861]^T$$

$$\|C_P\| = \sqrt{0.1177^2 + (-0.0282)^2 + \ (-0.0797)^2 + (-0.0959)^2 + 0.0861^2} = 0.1939$$

$$Y_{new} = Y_O - \propto r\frac{C_P}{\|C_P\|} = \left[\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right]^T$$

$$- \frac{\left(\frac{4}{15}\right) * \left(\frac{1}{\sqrt{20}}\right) * [0.1177 \quad -0.0282 \quad -0.0797 \quad -0.0959 \quad 0.0861]^T}{0.1939}$$

$$Y_{new} = \ [0.1638 \quad 0.2087 \quad 0.2245 \quad 0.2295 \quad 0.1735]^T$$

$$D_1 Y_{new} = \ diag[0.1330 \quad 0.2209 \quad 0.2377 \quad 0.2628 \quad 0.1456]$$

$$* [0.1638 \quad 0.2087 \quad 0.2245 \quad 0.2295 \quad 0.1735]^T$$

$D_1 Y_{new} = [0.0218 \quad 0.0461 \quad 0.0534 \quad 0.0603 \quad 0.0253]^T$

$1D_1 Y_{new} = [1 \quad 1 \quad 1 \quad 1 \quad 1] * [0.0218 \quad 0.0461 \quad 0.0534 \quad 0.0603 \quad 0.0253]^T$

$1D_1 Y_{new} = 0.2068$

$$X_2 = \frac{D_1 Y_{new}}{1D_1 Y_{new}} = \frac{[0.0218 \quad 0.0461 \quad 0.0534 \quad 0.0603 \quad 0.0253]^T}{0.2068}$$

$X_2 = [0.1054 \quad 0.2229 \quad 0.258 \quad 0.2916 \quad 0.1222]^T$

$Z = \mathrm{C}^{\mathrm{T}} X_2 = [2 \quad 1 \quad 2 \quad -2 \quad 0] * [0.1054 \quad 0.2229 \quad 0.258 \quad 0.2916 \quad 0.1222]^T = 0.3665$

k=2+1=3

Since $\mathrm{C}^{\mathrm{T}} X_1 > 0$, continue

3.6 AFFINE SCALING METHOD

The affine-scaling algorithm was originally proposed for LP problem by Dikin and independently rediscovered by Barnes, Vanderbei, Meketon and Freedman, after Karmarkar proposed the first polynomial-time interior-point method. This algorithm is often called the primal (or dual) affine-scaling algorithm because the algorithm is based on the primal (or dual) problem. There is also a primal-dual affine-scaling algorithm. It is sometimes called the Dikin-type primal-dual affine scaling algorithm. *(Muramatsu, 1998)*

3.6.1 PRIMAL AFFINE SCALING ALGORITHM

We present a generic feasible interior point algorithm for the LP problem. Consider an LP problem in the standard form: given the data, vector $x, b \in \mathbb{R}^n$, $C \in \mathbb{R}^n$ and matrix $A \in \mathbb{R}^{m+n}$;

$$\text{P:} \qquad \begin{array}{ll} \text{Minimize} & C^T x \\ \text{Subject to} & Ax = b \\ & x \geq 0 \end{array} \qquad (3.51)$$

The following algorithm is used to solve LP problems

Step 1

Set $k = 0$, $\varepsilon > 0, 0 < \propto < 1$

Find $x^k > 0$ and $Ab = b$

Step 2 Compute

$p^k = (\, AX_k^2 A^T)^{-1} AX_k^2 C$

$r^k = C - A^T p^k$

If $r^k \geq 0$, and $e^T X_k r^k \leq \varepsilon$

Then STOP! $X^* \leftarrow X^k, p^* \leftarrow p^k$

Otherwise,

Step 3 Compute

$d_x^k = -X_k r^k$

If $d_x^k > 0$, then STOP! Unbounded

If $d_x^k = 0$, then STOP! $X^* \leftarrow X^k$

Otherwise,

Step 4 find

$\alpha_k = \min_i \{\frac{\alpha}{(d_x^k)_i} \mid (d_x^k)_i < 0\}$

$X^{k+1} = X^k + \alpha_k X_k d_x^k$

$k \leftarrow k + 1$ d rediscovered by Barnes [4], a

Go to step 2

1.6.2 DUAL AFFINE SCALING ALGORITHM

We present a generic infeasible interior point algorithm for the LP problem. Consider an LP problem in the standard form: given the data, vectors $x, y, b, s \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $C \in \mathbb{R}^n$ and matrix $A \in \mathbb{R}^{m+n}$;

$$\text{P:} \quad \begin{aligned} \text{Minimum:} \quad & C^T x \\ \text{Subject to:} \quad & Ax = b \\ & x \geq 0 \end{aligned} \tag{3.52}$$

The corresponding dual problem is then given by:

$$\text{D:} \quad \begin{aligned} \text{Maximum:} \quad & b^T y \\ \text{Subject to:} \quad & A^T y + s = C \\ & s \geq 0 \end{aligned} \tag{3.53}$$

The following algorithm is used to solve LP problems

Step 1:

Set $k = 0$ and find (y^k, s^k)

$A^T y^k + s^k = C, \qquad s^k > 0$

Step 2:

Set $S_k = diag(s^k)$

Compute $\qquad d_y^k = (A S_k^{-2} A^T)^{-1} b$

$$d_s^k = -A^T d_y^k$$

Step 3:

If $d_s^k = 0$, STOP! $y^k \leftarrow y^*, s^k \leftarrow s^*$

If $d_s^k > 0$, STOP! (D) is unbounded

Step 4: Compute

$X^k = -S_k^{-2} d_s^k$

If $X^k \geq 0$ and $C^T X^k - b^T y^k \leq \varepsilon$

STOP! $y^k \leftarrow y^*, s^k \leftarrow x^k \leftarrow x^*$

Step 5: compute

$\beta_k = \min_i \{\frac{\alpha s_i^k}{(d_s^k)_i} \mid (d_s^k)_i < 0\}$

$Y^{k+1} = X^k + \beta_k d_y^k$

$S^{k+1} = S^k + \beta_k d_s^k$

$k \leftarrow k + 1$

Go to step 2

3.6.3 PRIMAL-DUAL AFFINE SCALING ALGORITHM

We present a generic feasible interior point algorithm for the LP problem. Consider an LP problem in the standard form: given the data, vectors $x, y, b, s \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $C \in \mathbb{R}^n$ and matrix $A \in \mathbb{R}^{m+n}$;

$$\text{P:} \quad \begin{array}{ll} \text{Minimize:} & C^T x \\ \text{Subject to:} & Ax = b \\ & x \geq 0 \end{array} \tag{3.54}$$

The corresponding dual problem is then given by:

$$\text{D:} \quad \begin{array}{ll} \text{Maximum} & b^T y \\ \text{Subject to} & A^T y + s = C \\ & y, s \geq 0 \end{array} \tag{3.55}$$

The following algorithm is used to solve LP problems

Step 1

Initialize with $x^k > 0$ feasible in (P) and (y^k, s^k) feasible in (D) with $s^k > 0$. Let $k = 0$.

Step 2

Let $\mu^k = \frac{(x^k)^T s^k}{n^2}$. (Other scaling is possible.)

Step 3 Calculate

$$D^k = \begin{bmatrix} \sqrt{\frac{x_1^k}{s_1^k}} & 0 & \cdots & 0 \\ 0 & \sqrt{\frac{x_2^k}{s_2^k}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sqrt{\frac{x_n^k}{s_n^k}} \end{bmatrix}$$

$A^k = AD^k$

Step 4 Calculate

$\Delta x = -D^k p_k D^k (s - \mu X^{-1} e)$

$\Delta y = (A(D^k)^2 A^T)^{-1} A(D^k)^2 (s - \mu X^{-1} e)$

$\Delta s = -A^T \Delta y$

Step 5 Calculate primal and dual step lengths $\propto_p$ and $\propto_D$,

ensuring $x^k + \propto_p \Delta x > 0$ and $s^k + \propto_D \Delta s > 0$.

Step 6 Update

$x^{k+1} = x^k + \propto_p \Delta x$

$y^{k+1} = y^k + \propto_D \Delta y$

$s^{k+1} = s^k + \propto_D \Delta s$

Let $k \leftarrow k + 1$. Calculate $(x^k)^T s^k$. If small enough, STOP. Else return to Step 2.

3.5 PRIMAL – DUAL PATH-FOLLOWING ALGORITHM

We present a generic infeasible interior point algorithm for the LP problem. Consider an LP problem in the standard form: given the data, $y \in \mathbb{R}^m$, $x, b, s \in \mathbb{R}^n$, $C \in \mathbb{R}^n$ and matrix $A \in \mathbb{R}^{m+n}$;

$$\text{P:} \quad \begin{aligned} &\text{Minimize:} && C^T x \\ &\text{Subject to:} && Ax = b \\ & && x \geq 0 \end{aligned} \tag{3.56}$$

The corresponding dual problem is then given by:

$$\text{D:} \quad \begin{aligned} &\text{Maximum} && b^T y \\ &\text{Subject to} && A^T y + s = C \\ & && y, s \geq 0 \end{aligned} \tag{3.57}$$

The following algorithm is used to solve LP problems

Step 1

Initialize with $x^k > 0$ feasible in (P) and (y^k, s^k) feasible in (D) with $s^k > 0$. Let $k = 0$.

Step 2

Let $\mu^k = \frac{(x^k)^T s^k}{n^2}$. (Other scaling is possible.)

Step 3 Calculate

$$D^k = \begin{bmatrix} \sqrt{\frac{x_1^k}{s_1^k}} & 0 & \cdots & 0 \\ 0 & \sqrt{\frac{x_2^k}{s_2^k}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sqrt{\frac{x_n^k}{s_n^k}} \end{bmatrix}$$

$A^k = AD^k$

Step 4 Calculate

$\triangle x = -D^k p_k D^k (s - \mu X^{-1} e)$

$\triangle y = (A(D^k)^2 A^T)^{-1} A(D^k)^2 (s - \mu X^{-1} e)$

$\triangle s = -A^T \triangle y$

Step 5 Calculate primal and dual step lengths $\propto_p$ and $\propto_D$,

ensuring $x^k + \propto_p \Delta x > 0$ and $s^k + \propto_D \Delta s > 0$.

Step 6 Update

$x^{k+1} = x^k + \propto_p \Delta x$

$y^{k+1} = y^k + \propto_D \Delta y$

$s^{k+1} = s^k + \propto_D \Delta s$

Let $k \leftarrow k + 1$. Calculate $(x^k)^T s^k$. If small enough, STOP. Else return to Step 2.

CHAPTER 4

DATA COLLECTION AND MODELING

4.0 DATA

Unibank is in the process of formulating a loan policy involving a 60% of its internal and external generated income which total up to GH¢400,000,000.00. Being a full-service facility, the bank gives loans to different clientele. The following Table 4.1 provides the types of loans, the interest rate charged by the bank, and the probability of bad debt as estimated from previous years. Budget for loans = 0.6* GH¢400,000,000.00 = GH¢240,000,000.00

TABLE 4.1; LOANS AVAILABLE TO UNIBANK

TYPES OF LOANS	INTEREST RATE	PROBABILITY OF BAD BEDT
PERSONAL	0.24	0.02
SMALL AND MEDIUM ENTERPRISES (SME)	0.03	0.01
AGRICULTURE	0.12	0.10
CONSTRUCTION	0.24	0.04
CAR	0.20	0.07
INDUSTRY	0.24	0.06

Source: Unibank (loan officer, 2011)

Bad debts are assumed irretrievable and hence produce no principal or interest revenue. Competition with other banking institutions in the area requires that the bank apply the following conditions:

Condition (i), the sum of personal loan, SME loan, agricultural loan, construction loan, car loan and industrial loan must be equal to the total funds available.

Condition (ii), the sum of personal loan and SME loan must be equal to agricultural loan, car loan, industrial loan and construction loan 45%.

Condition (iii), allocate 55% of the total funds to personal loan and SME loan.

Condition (iv), allocate 10% of the total funds to agricultural loan.

Condition (v), to assist the construction firms and industry in the region, construction loan and industrial loan must be equal 50% of car loan and SME loan.

Condition (vi),assign 65% of the total fund to SME loans.

Condition (vii), the bank also has a stated policy specifying that the overall ration for bad debts on all loans may be equal to 5%.

The objective of Unibank is to optimize its net return which comprises of the difference between the revenue from interest and lost funds due to bad debts.

Source: Unibank (loan officer)

4.1 FORMULATIOM OF LPP MODEL INSTANCE

The variables of the model can be defined as follows;

x_1 = personal loan (in millions of Ghana cedis)

x_2 = small and medium enterprise loans

x_3 = agricultural loans

x_4 = construction loans

x_5 = car loans

x_6 = industrial loans

Thus the objective function is;

$\boldsymbol{Z}$ = 0.24(0.98x_1) + .03(0.99x_2) + 0.12(0.90x_3) + 0.24(0.93x_4) + 0.20(0.93x_5) + 0.24(0.94x_6)

- (0.02x_1 +0.01 x_2 +0.10 x_3 +0.04 x_4 +0.07x_5 +0.06x_6)

$z = 0.2152x_1 + 0.0197x_2 + 0.008x_3 + 0.1904x_4 + 0.116x_5 + 0.1656x_6$

Subject to the constraints:

1. From condition (i), limit on total funds available : $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 240$
2. From condition (ii), limit on personal loan and SME loan compare to agricultural loan, car loan, industrial loan and construction loan:

$x_1 + x_2 = 0.45(x_3 + x_4 + x_5 + x_6)$

$x_1 + x_2 - 0.45x_3 - 0.45x_4 - 0.45x_5 - 0.45x_6 = 0$

3. From condition (iii), limit on personal loan and SME loan:

$x_1 + x_2 = 0.55 * 240$

$x_1 + x_2 = 132$

4. From condition (iv), limit on agricultural loan :

x_3 = 0.10 *240

x_3 = 24

5. From condition (v), limit on construction loan and industrial loan :

$x_4 + x_6 = 0.5\,(x_2 + x_4 + x_5 + x_6)$

$-0.5x_2 + 0.5x_4 - 0.5x_5 + 0.5x_6 = 0$

6. From condition (vi), limit on SME loan:

x_2 = 0.65*240

x_2 =156

7. From condition (vii), limit on bad debts:

$$\frac{0.02x_1 + 0.01x_2 + 0.10x_3 + 0.04x_4 + 0.07x_5 + 0.06x_6}{x_1 + x_2 + x_3 + x_4 + x_5 + x_6} = 0.05$$

$-0.03x_1 - 0.04x_2 - 0.5x_3 - 0.01x_4 - 0.02x_5 - 0.01x_6 = 0$

8. Non- negativity: $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$, $x_4 \geq 0$, $x_5 \geq 0$ and $x_6 \geq 0$;

4.2 SUMMARY OF LP PROBLEM

Minimize

$$z = 0.2152x_1 + 0.0197x_2 + 0.008x_3 + 0.1904x_4 + 0.116x_5 + 0.1656x_6$$

Subject to the constraints:

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 240$$

$$x_1 + x_2 - 0.45x_3 - 0.45x_4 - 0.45x_5 - 0.45x_6 = 0$$

$$x_1 + x_2 + 0x_3 + 0x_4 + 0x_5 + 0x_6 = 132$$

$$0x_1 + 0x_2 + x_3 + 0x_4 + 0x_5 + 0x_6 = 24$$

$$0x_1 - 0.5x_2 + 0x_3 + 0.5x_4 - 0.5x_5 + 0.5x_6 = 0$$

$$0x_1 + x_2 + 0x_3 + 0x_4 + 0x_5 + 0x_6 = 156$$

$$-0.03x_1 - 0.04x_2 - 0.5x_3 - 0.01x_4 - 0.02x_5 - 0.01x_6 = 0$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

4.3 CONVERTING LPP STANDARD FORM INTO KARMARKAR STANDARD FORM

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -0.45 & -0.45 & -0.45 & -0.45 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -0.5 & 0 & -0.5 & -0.5 & 0.5 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -0.03 & -0.04 & -0.5 & -0.01 & -0.02 & -0.01 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.2152 \\ 0.0197 \\ 0.0080 \\ 0.1904 \\ 0.1160 \\ 0.1656 \end{bmatrix} \qquad b = \begin{bmatrix} 240 \\ 0 \\ 132 \\ 24 \\ 0 \\ 156 \\ 0 \end{bmatrix}$$

$n = 6$ $\quad Q = 2^6 = 64$ $\quad e^T = [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1]$ $\quad M = 20$

Minimize $Z = (Q+1)C^T X + 20x_{n+3}$

Minimize Z= (64+1)* $[0.2152 \quad 0.0197 \quad 0.0080 \quad 0.1904 \quad 0.1160 \quad 0.1656]$

$*[x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6]^T + 20x_9$

Minimize $Z = 13.988x_1 + 1.2805x_2 + 0.52x_3 + 12.376x_4 + 7.54x_5 + 42.64x_6 + 20x_9$

Subject to the constraints:

$$AX - bx_{n+2} - [Ae - b]x_{n+3} = 0$$

$$\left[\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -0.45 & -0.45 & -0.45 & -0.45 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -0.5 & 0 & -0.5 & -0.5 & 0.5 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -0.03 & -0.04 & -0.5 & -0.01 & -0.02 & -0.01 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}\right] - x_8 \begin{bmatrix} 240 \\ 0 \\ 132 \\ 24 \\ 0 \\ 156 \\ 0 \end{bmatrix}$$

$$-x_9 \left[\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -0.45 & -0.45 & -0.45 & -0.45 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -0.5 & 0 & -0.5 & -0.5 & 0.5 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -0.03 & -0.04 & -0.5 & -0.01 & -0.02 & -0.01 \end{bmatrix} * \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 240 \\ 0 \\ 132 \\ 24 \\ 0 \\ 156 \\ 0 \end{bmatrix}\right] = 0$$

$x_1 + x_2 + x_3 + x_4 + x_5 + x_6$ -240x_8+ 234x_9 = 0

$x_1 + x_2$ - 0.45x_3 - 0.45x_4 - 0.45x_5 - 0.45x_6 + 0x_8- 0.2x_9 = 0

$x_1 + x_2$ + 0x_3 + 0x_4 + 0x_5 + 0x_6 -132x_8+ 130x_9 = 0

0x_1 + 0x_2 + x_3 + 0x_4 + 0x_5 + 0x_6-24x_8+ 23x_9 = 0

0x_1 - 0.5x_2+0x_1 + 0.5x_4 -0.5x_5 + 0.5x_6 - 0x_8+ 0x_9 = 0

0x_1 +x_2 +0x_3 + 0x_4 + 0x_5 + 0x_6 - 156x_8+ 155x_9 = 0

-0.03x_1 - 0.04x_2 - 0.5x_3 - 0.01x_4 - 0.02x_5 - 0.01x_6 + 0x_8+ 0.61x_9 = 0

$$e^T X + x_{n+1} - Q x_{n+2} - (n + 1 - Q) x_{n+3} = 0$$

$$[x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6] * \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + x_7 - 64 x_8 - (6 + 1 - 64) x_9 = 0$$

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 - 64 x_8 + 57 x_9 = 0$$

$$e^T X + x_{n+1} + x_{n+2} + x_{n+3} = 1$$

$$[x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6] * \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + x_7 + x_8 + x_9 = 1$$

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 = 1$$

4.4 KARMARKAR'S STANDARD FORM

Minimize

$$Z = 13.988x_1 + 1.2805x_2 + 0.52x_3 + 12.376x_4 + 7.54x_5 + 42.64x_6 + 0x_7 + 0x_8 + 20x_9$$

Subject to the constraints

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 - 240x_8 + 234x_9 = 0$$

$$x_1 + x_2 - 0.45x_3 - 0.45x_4 - 0.45x_5 - 0.45x_6 + 0x_8 - 0.2x_9 = 0$$

$$x_1 + x_2 + 0x_3 + 0x_4 + 0x_5 + 0x_6 - 132x_8 + 130x_9 = 0$$

$$0x_1 + 0x_2 + x_3 + 0x_4 + 0x_5 + 0x_6 - 24x_8 + 23x_9 = 0$$

$$0x_1 - 0.5x_2 + 0x_1 + 0.5x_4 - 0.5x_5 + 0.5x_6 - 0x_8 + 0x_9 = 0$$

$$0x_1 + x_2 + 0x_3 + 0x_4 + 0x_5 + 0x_6 - 156x_8 + 155x_9 = 0$$

$$-0.03x_1 - 0.04x_2 - 0.5x_3 - 0.01x_4 - 0.02x_5 - 0.01x_6 + 0x_8 + 0.61x_9 = 0$$

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 - 64x_8 + 57x_9 = 0$$

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 = 1$$

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9 \geq 0$$

4.5 SUMMARY OF MODEL INSTANCE

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & -240 & 234 \\ 1 & 1 & -0.45 & -0.45 & -0.45 & -0.45 & 0 & 0 & -0.2 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & -132 & 130 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -24 & 23 \\ 0 & -0.5 & 0 & 0.5 & -0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -156 & 155 \\ -0.03 & 0.04 & -0.5 & -0.01 & -0.02 & -0.01 & 0 & 0 & 0.61 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & -64 & 57 \end{bmatrix}$$

$$C = \begin{bmatrix} 13.988 \\ 1.2805 \\ 0.52 \\ 12.376 \\ 7.54 \\ 42.64 \\ 0 \\ 0 \\ 20 \end{bmatrix}$$

$E = [1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1]$

$N = [1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1]$

4.6 COMPUTATION PROCEDURE

The computer brand used in running the programming code was LG with 150GB capacity hard disk drive, processing speed of 1.8GHz and random access memory (RAM) of 1GB. The programming code was written in matlab to run data: A, C, N and E . The programming code can be found in the appendix. It was run up to maximum of 1000 iterations but converges at iteration 939 to produce the final results. Ten runs produced the same results as shown below;

4.7 RESULTS

$Z = 17.2860$, $x_1 = 0.1138$, $x_2 = 0.1459$, $x_3 = 0.0999$, $x_4 = 0.0000$, $x_5 = 0.1424$

$x_6 = 0.2882$, $x_7 = 0.0000$, $x_8 = 0.1052$ and $x_9 = 0.1046$

Where, Z = objective function, $x_i \; for \; i = 1, \dots, 6$ = basic variables and $x_i \; for \; i = 7, \dots, 9$ = nonbasic variables.

CHAPTER 5

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

5.1 SUMMARY

The computational results presented in this thesis illustrate the use of Karmarkar interior point methods for linear programming. There are several things to observe about the output data. The iteration converges at when tel= 939 and the outcomes are $Z = 17.2860$ for the objective function and X =[0.1138, 0.1459, 0.0999, 0.0000, 0.1424, 0.2882] for the basic variables.

The optimal solution converts x_1 to GH¢113,800.00 for personal loan, x_2 to GH¢145,900.00 for small and medium enterprise loans, x_3 to GH¢99,900.00 for agricultural loans, no amount allocated to x_4 for construction loans, x_5 to GH¢142,400.00 for car loans and x_6 to GH¢288,200.00 for industrial loans show that the bank should allocate funds to all the types of loans accept construction loan and this would yield a minimum profit of GH¢17,286,000.00.

The solution meets the policy of loan preference of Unibank since the result shows industrial loan takes the lead followed by small and medium enterprise loans then car loans then personal loan finally agricultural loan then.

5.2 CONCLUSIONS

The data taken from Unibank was formulated into a linear programming problem. Karmarkar interior point method which approaches the optimum a polynomial time algorithms was used in the computation.

Furthermore, optimizing the disbursement of the funds available for loans from Unibank would result in the appropriate allocation of funds to their customers. The computation shows that Unibank would be able to make a minimum profit of GH¢17,286,000.00 on loans alone.

Concluding, the solution meets the policy of loan preference of Unibank since the result shows industrial loan takes the lead followed by small and medium enterprise loans then car loans then personal loan finally agricultural loan then.

5.3 RECOMMENDATIONS

Recalling from summary, the model proposed that funds should be allocated in the following other: industrial loan, small and medium enterprises (SME) loan, personal loan, agricultural loan and car loan. It is therefore recommended that Unibank may adopt this proposed model as one of its research methods, since the output conforms to the loan policy of the bank.

Lastly, it is also recommended that interior point method should further be researched by students, since it is a polynomial time algorithm of solving linear programming problems.

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APPENDIX

% Assign the data (A,C,E and N) to MATLAB variable

% N and E is the number basic variables represented in ones

% “tel” is the number of iterations

```
mn =size(A)
m=mn(1)
n=mn(2)
X1= ([N]/n)'
I=eye (n)
r=1/sqrt(n*(n-1))
v= (n-1)/(3*n)
X= ([N]/n)'
tel=0
while(C'*X>0)&(tel<500)
T=diag(X)
AT=A*T
P= [AT; E]
CT=C'*T
CP= (I-P'/(P*P' )*P)*CT'
CPN=norm (CP)
```

```
Y=X1-(r*v)*(CP/CPN)

X= (T*Y)/(E*T*Y)

Z=C'*X

tel=tel+1

end
```