KWAME NKRUMAH UNIVERSITY OF SCIENCE AND TECHNOLOGY

Some Applications of Max-plus Algebra

By

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Declaration

I hereby declare that this submission is my own work towards the award of the MPhil degree and that, to the best of my knowledge, it contains no material previously published by another person nor material which had been accepted for the award of any other degree of the university, except where due acknowledgement had been made in the text.

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Dedication

This work is dedicated to my mother, Mary Ama Serwaah and my sister, Mrs. Agnes Marfo.

Abstract

Max-plus algebra is an analogue of conventional linear algebra developed on the operations $\oplus$ and $\otimes$. The algebraic structure is a semi-ring whose elements are the usual real numbers along with $\epsilon = -\infty$ and $e = 0$, where $\oplus$ represents taking the maximum and $\otimes$ is the standard addition. In this thesis we use the discrepancy method of max-plus to solve $n \times n$ and $m \times n$ system of linear equations where $m < n$. We apply the above concept to solve a real-life problem in a synchronised event. We also apply max-plus algebra in solving linear programming problem involving linear equations and inequalities.

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To my siblings, I thank you all for the love you have shown me through my entire studies.

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Chapter 1

Introduction

1.1 Background

It is seen that a lot of attention has been given to the study of simple systems of linear equations in the form $A \otimes x = b$, where A is a matrix, b and x are vectors of suitable dimensions. This has led us to present a work that solves linear systems of equations in max-plus algebra. We develop a method called the discrepancy to solve such a system. This discrepancy method solves linear systems of equations in just a three steps irrespective of the size of the equations. This method is able to determine the nature of solution to any system of equations by what we call the reduced discrepancy matrix. This reduced discrepancy matrix is developed from the discrepancy matrix. Unique solution, infinitely many solutions or no solution is the nature of solution that a system of linear equations in max-plus can have.

Systems of linear equations over max-plus algebra are used in several branches of applied mathematics. These can assist in modeling and analysis of discrete event systems. We present an application of max-plus to a real-life problem in a synchronise event which deals with the preparation of a shop before sales. Max-plus algebra has been applied to a lot of real-life problems, e.g., a large scale model of Dutch railway network or synchronizing traffic lights in Delft by Olsder et al. (1998).

We present a system of inequalities. An algorithm is then used to solve a max-linear program involving linear equations and inequalities. Max-plus algebra is a semi-ring which is the set $\mathbb{R}_{max} = \{-\infty\} \bigcup \mathbb{R}$ together with the operations $a \oplus b = max(a, b)$ and $a \otimes b = a + b$. The additive and multiplicative identities are taken to be $\in = -\infty$ and $e = 0$ respectively. Its operations are associative, commutative and distributive as in conventional algebra. This has made it useful in various areas.

Cunningham-Green (1979) showed that the problem $A \otimes x = b$ can be solved using residuation. That is the equality in $A \otimes x = b$ be relaxed so that the set of its sub-solutions is studied. It was shown that the greatest solution of $A \otimes x \leq b$ is given by $\bar{x}$ where

$$\bar{x}_j = min_{i \in M}(b_i \otimes a_{ij}^{-1}) forall j \in \mathbb{N}$$

The equation $A \otimes x = b$ is also solved using the above results as follows: the equation $A \otimes x = b$ has solution if and only if $A \otimes \bar{x} = b$.

Zimmermann (1976) developed a method for solving $A \otimes x = b$ by set covering and also presented an algorithm for solving max-linear programs with one sided constraints. This method is proved to has a computational complexity of O(mn), where m and n are the number of rows and columns of input matrices respectively. Akian et al. (2005) extended Zimmermann's solution method by set covering to the case of functional Galois connections.

Butkovic (2010) developed a max-algebraic method for finding all solutions to a system of inequalities $x_i - x_j > b_{ij}$, $i, j = 1, ..., n$ using n generators. Using this method he developed a pseudopolynomial algorithm which either finds a bounded mixed-integer solution or decides that no solution exists.

Cechla'rova' and Diko (1999) also proposed a method for resolving infeasibility of the system $A \otimes x = b$. The techniques presented in his method are to modify the right-hand side as little as possible or to omit some equations. It was shown that the problem of finding the minimum number of those equations is NP-complete.

1.2 Problem Statement

There has been severals ways of finding solutions to $A \otimes x = b$ in max-plus algebra. One of such ways of solving is the discrepancy method. This discrepancy method has only been used to solve a system of $m \times n$ equations (where $m > n$). We seek to find out if this discrepancy method can be used to solve an $n \times n$ system and also a system of $m \times n$ equations (where $m < n$). We will find out if the discrepancy method over max-plus algebra can be applied to a real-life problem in a synchronised event. Linear programming problems are solved by converting inequalities to equations which increase the number of variables or constraints. This therefore increases the computational complexity. We seek to find out if there is a method that will not require any new variables or constraints.

1.3 Objectives

The main objectives of this thesis is to:

1. use the discrepancy method to solve an $n \times n$ systems of linear equations and $m \times n$ systems of linear equations where $m < n$.

2. use max-plus algebra to solve a synchronised event problem.

3. solve a linear programming problem involving a linear equations and inequalities.

1.4 Methodology

Discrepancy method was modified to solve $n \times n$ and $m \times n$ (where $m < n$) system of max-plus equations. A reduced discrepancy, $R_{A,b}$, is develop from the discrepancy matrix to determine whether the system has a unique solution, infinitely many solutions or no solution. This method is use to solve a synchronised event problem.

An algorithm is use to solve a linear programming problem consisting of linear equations and inequalities.

1.5 Justification

Discrepancy method was proposed to solve $m \times n$ systems of linear equations where $m > n$ in max-plus algebra. There is the need to modify the discrepancy method to solve other systems such as $n \times n$ and $m \times n$ (where $m < n$). This is because the method solves linear systems of equations in only three steps irrespective of the size of the systems. This is the only method that determines the nature of solution to a system of equations in max-plus.

Systems of linear equations over max-plus can assist in modeling and analysis of discrete event systems. We solve a real-life problem in a synchronised event where much attention is not given.

We also present a polynomial algorithm which solves a linear programming problem whose constraints are linear equations and inequalities. This algorithm avoids the situation where you will require new constraints or variables which increases the computational complexity.

1.6 Structure of the Thesis

This thesis is composed of five chapters. The first chapter is an introduction, dealing with the background of the work, statement of problem, objectives of the work, methodology, justification of the problem especially to the benefit of the society. Chapter two is the literature review.

Chapter three is where theorems are explained in a way that can easily be understood. Here, we were able to explain how we arrived at a simplified matrix called the discrepancy matrix ($D_{A.b}$) which is used to solve a system of max-plus equations. Applications of the theorems were done in chapter four. Finally, chapter five comprises of discussion of results and conclusion.

Chapter 2

Literature Review

The first use of the max-plus semiring can be traced back at least to the late fifties and it grew in the sixties, with works of Cunninghame-Green, Vorobgev, Romanovskii, and more generally of the Operations Research community (on path algebra). The first enterprise of systematic study of this algebra seems to be seminal "Minimax algebra" by Cunningham-Green (1979). The theory of linear independence using bideterminants, which is the ancestor of symmetrization was initiated by Gordran and Minoux (following Kuutzmann). The last chapter of "Operatorial Methods" of Maslov (1987) inaugurated the max-plus operator and measure theory (motivated by semiclassical asymptotics). There is an "extremal algebra" tradition, mostly in East Europe, oriented towards algorithms and computational complexity. Results in this spirit can be found in the book of U. Zimmermann (1981). This tradition has been pursued by Butkovic (1994).

In max-plus algebra we work with the max-plus semi-ring which is the set $\mathbb{R}_{max} = \{-\infty\} \bigcup \mathbb{R}$ together with the operations $a \oplus b = max(a, b)$ and $a \otimes b = a + b$. The additive and multiplicative identities are taken to be $\in = -\infty$ and $e = 0$ respectively. Its operations are associative, commutative and distributive as in conventional algebra, Farlow (2009).

Max-plus algebra emerged in the late 1950s, soon after the field of Operations Research began to developed. The field of Operations Research is a scientific approach to decision making. Most problems in Operations Research involve a "search for optimality", by Andersen (2002). Many problems that arise in the field of Operation Research have been solved by the development of algorithmic procedures that lead to optimal solutions.

Max-plus algebra is one of many idempotent semi-rings which have been considered in various fields of mathematics. It first appeared in 1956 in Kleene's paper on nerve sets and automata. It has found applications in many areas such as combinatorics, optimization, mathematical physics and algebraic geometry, Halburd and Southall (2007). It is also used in control theory, machine scheduling, discrete event processes, manufacturing systems, telecommunication networks, parallel processing systems and traffic control. Many equations that are used to describe the behaviour of these applications are nonlinear in conventional algebra but become linear in max-plus algebra. This is a primary reason for its utility in various areas, Shutter (2000).

Many of the theorems and techniques we use in classical linear algebra have analogues in the max-plus semi-ring. Cunninghame-Green, Gaubert, Gondran and Minoux are among the researchers who have devoted a lot of time creating much of the max-plus linear algebra theory we have today, Farlow (2009). Although many individuals some of which mentioned above have researched into some of the possible uses and theories regarding max-plus, the first attempt of a complete study, Minimax Algebra, by Cunninghame-Green, was not published until 1979. Many of these initial studies were limited to what are now called path algebras. More recently, the usage of max-plus has been extended to consider Discrete Event Systems and Dynamic Programming, Gaubert (1997).

To illustrate the usefulness of max-plus algebra in a simple example, let's look at a railroad network between two cities. A similar example can be found in Heidergott et al. (2006). This is an example of how max-plus algebra can be applied to a discrete event system. Assume we have two cities such that S_1 is the station in the first city, and S_2 is the station in the second city. This system contains 4 trains. The time it takes a train to go from S_1 to S_2 is 3 hours where the train travels along track 1. It takes 2 hours to go from S_2 to S_1 where the train travels along track 2. These tracks can be referred to as the long distance tracks. There are two more tracks in this network, one which runs through city 1 and one which runs through city 2. We can refer to these as the inner city tracks. Call them tracks 3 and 4 respectively. We can picture track 3 as a loop beginning and ending at S_1. Similarly track 4 starts and ends at S_2. The time it takes to traverse the loop on track 3 is 2 hours. The times it takes to travel from S_2 to S_2 on track 4 is 4 hours. Track 3 and track 4 each contain a train. There are trains that circulate along the two long distance tracks. In this network we also have the following criteria:

1. The travel times along each track indicated above are fixed.
2. The frequency of the trains must be the same on all four tracks.
3. Two trains must leave a station simultaneously in order to wait for the change over of passengers.
4. the two $(k+1)^{st}$ trains leaving S_i can not leave until the k^{th} train that left the other station arrives at S_i.

$x_i(k-1)$ will denote the k^{th} departure time for the two trains from station i. Therefore $x_1(k)$ denotes the departure time of the pair of $k+1$ trains from S_1 and S_2. So $x_1(0)$ denotes the departure time of the first pair of trains from station 1 and likewise $x_2(0)$ denotes the departure time of the first pair of trains from station 2.

Let's say we want to determine the departure time of the k^{th} trains from station 1. We can see that

$$x_1(k+1) \geq x_1(k) + a_{11} + \delta$$

and

$$x_1(k+1) \geq x_2(k) + a_{12} + \delta$$

where a_{ij} denotes the travel time from station j to station i and δ is the time allowed for the passengers to get on and off the train. So in our situation we have $a_{12} = 2$, $a_{11} = 2$, $a_{22} = 4$ and $a_{21} = 3$. We will assume $\delta = 0$ in this example. So it follows that

$$x_1(k+1) = max\{x_1(k) + a_{11}, x_2(k) + a_{12}\}$$

Similarly we can see that

$$x_2(k+1) = max\{x_1(k) + a_{21}, x_2(k) + a_{22}\}.$$

In conventional algebra we would determine successive departure times by iterating the nonlinear system

$$x_i(k+1) = max_{j=1,2,...,n}\{a_{ij} + x_j(k)\}.$$

In max-plus we would express this as

$$x_i(k+1) = \bigoplus_{j=1}^{n}(a_{ij} \otimes x_j(k)), i = 1, 2, ..., n.$$

where

$$\bigoplus_{j=1}^{n}(a_{ij} \otimes x_j) = (a_{i1} \otimes x_i) \oplus (a_{i2} \otimes x_2) \oplus ...(a_{in} \otimes x_n)$$

for $i = 1, 2, ..., n$.

In the example we have $x_1(1) = 0 \oplus 2 = 2$ and $x_2(1) = 1 \oplus 4 = 4$ provided we are given $x_1(0) = -2$ and $x_2(0) = 0$.

We can create a matrix A using the values a_{ij} such that $A = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}$ and $x(0) = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$.

We can also express this system using matrices and vectors such that

$x(k) = A \otimes x(k-1)$:

$x(1) = A \otimes x(0)$,

$x(2) = A \otimes x(1) = A \otimes A \otimes x(0) = A^{\otimes k} \otimes x(0)$.

Continuing in this fashion we see that $x(k) = A^{\otimes k} \otimes x(0)$. This gives us a simple example of how a system of equations which is not linear in conventional algebra becomes linear in max-plus algebra.

Exotic semiring such as the (max, +) semiring $(\mathbb{R} \bigcup \{-\infty\}, max, +)$, or the tropical semiring $(N \bigcup \{+\infty\}, min, +)$ have been invented and reinvented many times since the late fifties. This is in relation to various fields such as performance evaluation of manufacturing systems and discrete event system theory, graph theories (path algebra) and Markov decision processes, Hamilton-Jacobi theory, asymptotic analysis (low temperature asymptotics in statistical physics, large deviations, WKB Method); language theory(automata with multiplicities), Gaubert (2007).

Despite this apparent profusion, there is a small set of common, non-naive, basic results and problems in general not known outside the max-plus community which seem to be useful in most applications. Gaubert (2007) therefore presented on what is believe to be the minimal core of max-plus results , and to illustrate their result by typical applications at the frontier of language theory, control, and operations research (performance evaluation of discrete event systems, analysis of Markov decision processes with average cost). He used basic techniques such as solving all kinds of systems of linear equations, sometimes with exotic symmetrization and determinant techniques; using the max-plus Perron-Frobenius theory to study the dynamics of max-plus linear maps.

The Incline algebra introduced by Cao et al. (1984) are idempotent semirings in which $a \oplus ab = a$. Since the beginning of the eighties, Discrete Event Systems, which were previously considered by distinct communities (queuing networks scheduling), have been gathered into a common algebraic frame. "Synchronization and Linearity" by Baccelli et al. (1992) gives a comprehensive account of deterministic and stochastic max-plus linear discrete event systems together with recent algebraic results (such as symmetrization). Another recent text is the collection of articles edited by Maslov and Samborskii (1992) which is only the most visible part of the (considerable) work of the Idempotent Analysis School. A theory of probabilities in max-plus algebra motivated by dynamic programming and large deviations has been developed by Akian, Quadrat and Viot; and by Moral and Salut (1988). Recently, the max-plus semiring has attracted attention from the linear algebra community, Bapat et al. (1995).

There are a lot of concepts in conventional algebra that have been dealt with in max-plus algebra. Some of such concepts are solving systems of linear equations, the eigenvalue problem and linear independence. Applications of max-plus in infinite dimensional settings is an emerging area of research. Although this work is limited to finite dimensional settings, we want to indicate what some of the infinite dimensional issues are. Instead of vectors $v \in \mathbb{R}_{max}$ this will involve problems for $\mathbb{R}_{max}$- valued functions $\phi : x \longrightarrow \mathbb{R}_{max}$ defined on some domain. We might call this max-plus functional analysis. Just as many finite dimensional optimization problems become linear from the max-plus perspective, the nonlinear equations of continuous state optimization problems (such as optimal control) likewise become linear in a max-plus context.

Imagine that we are considering optimization problems for a controlled system of ordinary differential equations,

$$\dot{x}(t) = f(x(t), u(t)),$$

where the control function u(t) takes values in some prescribed control set U. Typical optimization problems involve some sort of running cost function L(x,u). For instance a finite horizon problem would have a specified terminal time T and terminal cost function $\phi(\cdot)$. For a given initial state $x(t) = x$, $t<T$ the goal would be to maximize

$$J(x, t, u(\cdot)) = \int_t^T L(x(s), u(s))ds + \phi(x(T))$$

for a given intial condition $x = x(t)$ over all allowed control functions $u(\cdot)$. In other words we would want

$$V(x) = S_T[\phi](x),$$

where

$$S_T[\phi](x) = sup_{u(\cdot)} \int_0^T L(x(s), u(s))ds + \phi(x(T))$$

So S_T is the solution operator. In other problems, like the nonlinear H_∞ problem of McEneaney (2008), the desired solution turns to be a fixed point: $W = S_T[W]$.

In the conventional sense S_T is a nonlinear operator (In fact $S_T : T \geqslant 0$ forms a nonlinear semigroup). However it is linear in the max-plus sense:

$$S_T[c \otimes \phi] = S_T[c + \phi] = c + S_T[\phi] = c \otimes S_T[\phi]$$

and

$$S_T[\phi \otimes \psi] = S_T[max(\phi, \psi)] = max(S_T[\phi], S_T[\psi]) = S_T[\phi] \oplus S_T[\psi].$$

With this observation one naturally asks if it is possible to develop max-plus analogues of eigenfunction expansions and something like the method of separation of variables in the context of these nonlinear problems. The idea would be to make an appropriate choice of basis functions: $\psi_i{:}X \longrightarrow \mathbb{R}_{max}$, use an approximation

$$\psi(x) \approx \bigoplus_{1}^{N} a_i \otimes \psi_i(x),$$

and then take advantage of the max-plus linearity to write

$$S_T[\psi](x) \approx \bigoplus_{1}^{N} a_i \otimes S_T[\psi_i](x).$$

If the ψ_i are chosen so that the expansion $S_T[\psi_i] \approx \oplus_j b_{ij}\psi_j$ can be worked out, then an approximation to the finite time optimization problems $S_T[\phi] \asymp \oplus_1^N c_i \otimes \psi_i$ where $\phi = \oplus a_j \psi_i$ would be given by a max-plus matrix product:

$$[c_i] = B \otimes [a_j].$$

To do all this carefully one must choose the appropriate function spaces in which to work, and carry out some sort of error analysis. This has in fact been done by W.M. McEneaney for the H_∞ problem in McEneaney (2008). Moreover methods of this type offer the prospect of avoiding the so-called "curse of dimensionality". But there are many questions about how to do this effectively in general cases. For instance, what basis functions should one use? At present relatively little research has been done in this direction, aside from the papers of McEneaney cited.

More than sixteen years after the beginning of a linear theory for certain discrete event systems in which max-plus algebra and similar algebraic tools play a central role, some papers attempt to summarize some of the main achievements in an informal style based on examples. By comparison with classical linear system theory, there are areas which are practically untouched, mostly because the corresponding mathematical tools are yet to be fabricated. This is the case of the geometric approach of systems which is known, in the classical theory, to provide another important insight to system-theoretic and control-synthesis problems, beside the algebraic machinery.

For what later became the Max-Plus working group at INRIA, the story about discrete event systems (DES) and max-plus algebra began in August 1981, that is more than sixteen and a half years ago, at the time of Cohen, Gaubert, Quadrat (1998) was written. Actually, speaking of 'discrete event systems' is somewhat anachronistic for that time when this terminology was not even in use. Sixteen years is not a short period of time as compared to the time it took for classical linear system theory to emerge as a solid piece of science. On the other hand, those who have been working in the field of max-plus linear systems have benefited from the guidelines and concepts provided by that classical theory. On the other hand, the number of researchers involved in this new area of system theory for DES has remained rather small when compared with the hundreds of their colleagues who contributed to the classical theory. In addition, while this classical theory was based on relatively well established mathematical tools, and in particular linear algebra and vector spaces, the situation is quite different with max-plus algebra. This algebra and similar other algebraic structures sometimes referred to as 'semirings' or 'dioids', were already studied by several researchers when we started to base our system-theoretic work upon such tools. Today, a very basic understanding of some fundamental mathematical issues in this area is still lacking, which certainly contribute to slow down the progress in system theory itself. This is why an account of the present situation in the field can hardly separate the system-theoretic issues from the purely mathematical questions.

Indeed, the models and equations involved are not restricted to DES: connections with other fields (optimization and optimal decision processes, asymptotic in probability theory, to quote but a few) have been established since then, and this has contributed to create a fruitful synergy in this area of mathematics. Yet, some papers concentrate on DES applications. To be more specific, while classical system theory deals with systems which evolve in time according to various physical, chemical or biological phenomena which are described by ordinary or partial differential equations (or their discrete-time counterparts). Discrete Event Systems are referred to as 'man-made' systems. The importance of which has been constantly increasing with the emergence of new technologies. Computers, computer networks, telecommunication network, modern manufacturing systems and transportation system are typical examples of Discrete Event Systems. Among the basic phenomena that characterize their dynamics, one may quote synchronization and competition in the use of common resources. Competitions basically call for decisions in order to solve the conflicts (whether at the design stage or on line, through priority and scheduling policies). Through 'classical' glasses, synchronization looks like a very nonlinear and nonsmooth phenomenon. This is probably why DES have been, for a long time, left apart by classical system and control theory; they were considered rather in the realm of operations research or computer science, although they are truly dynamical systems.

Linear models are the simplest abstraction (or ideal model) upon which a large part of classical system and control theory have been based until the late sixties. To handle more complex models, say, with smooth nonlinearities, it was necessary to adapt the mathematical tools while keeping most of the concepts provided by earlier developments. Differential geometry, power series in noncommutative variables, differential algebra have been used to develop such models. This has raised essential questions such as controllability and observability, stabilization and feedback synthesis, etc., to be revisited. Max-plus, min-plus and other idempotent semiring structures turn out to be the right mathematical tools to bring back linearity, in the best case, or at least a certain suitability with the nature of phenomena to be described, in this field of DES.

Cohen, Gaubert, Quadrat (1998) came out with a work to summarise some of the most basic achievements in the last sixteen years in this new area of system theory which turned towards DES performance related issues (as opposed to logical aspects considered in the theory of Ramadge and Wonham (1989)). A lot of works rely upon several surveys already devoted to the subject (Cohen et al., 1989a; Cohen,1994; Quadrat and Max Plus, 1995; Gaubert and Max Plus, 1997) in addition to the book (Baccelli et al., 1992). On the other hand, the other works try to suggest new directions of developments. This essentially concerns the understanding of geometric aspects of system theory in the max-plus algebra.

In many applications the models use operations of maximum and minimum together with further arithmetical operations. The max-plus algebra is useful for investigations of discrete event systems and the sequence of states in discrete time corresponds to powers of matrices in max-plus algebra. A typical application of discrete events systems are production lines, where every machine must wait while starting a new operation until the operations on other machines are completed. The eigen problem for max-plus matrices describes the steady state of a system. For special types of matrices such as Circulant, Toeplitz, Hankel or Monge matrix, the computation can often be performed in the simpler way than in the general case. Hence the investigation of special cases is important from the computational point of view. In Tomášková (2011) the eigenspace structure for a special case of so-called circulant matrices was studied. Circulant matrices arises for example, in applications involving the discrete Fourier transform and study of cyclic codes for error corrections.

Max-plus algebra is used in some special problems of the Operational Research such as dealing with dynamic programming, finding ways to traffic problems, special problem of planning, etc. Among the known applications, Max-plus algebra can be classified as "JobShop Scheduling", that is the determination of steady state behavior of a set of machines. This problem is often reduced to only finding eigenvectors. With suitability of max-plus algebra to solve various problems that occur or are expressed in discrete time such as synchronization or scheduling, it is sometimes referred to as "schedule algebra", Tomášková (2011).

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Chapter 3

Preliminary Concepts of Max-plus Algebra

3.1 Max-Plus Algebra

Myśková (2009) defined max-plus algebra as the algebraic structure in which classical addition and multiplication are replaced by $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$, respectively. Each system of linear equations in max-plus algebra can be written in the matrix form $A \otimes x = b$, where A is a matrix and b is a vector of suitable size.

3.2 Definitions and Basic properties of Max-plus Algebra

We look at the algebraic properties of max-plus algebra in this section. We recall that in max-plus algebra for a, b $\in R_{\max} = \{-\infty\} \cup \mathbb{R}$, we define the two operations $\oplus$ and $\otimes$ by equation $a \oplus b = \max(a, b)$; $a \otimes b = a + b$.

Multiplicative identity in max-plus algebra is $e = 0$:

$$a \otimes e = e \otimes a = a + 0 = a$$

for all $a \in \mathbb{R}_{\max}$

The additive identity is $\epsilon = -\infty$:

$$a \oplus \epsilon = \epsilon \oplus a = \max(a, -\infty) = a$$

for all $a \in \mathbb{R}_{\max}$.

This shows that $\oplus$ and $\otimes$ are commutative which also satisfy other properties similar to + and x in conventional algebra.

$\otimes$ distributes over $\oplus$ for all $a, b, c \in \mathbb{R}_{\max}$:

$$a \otimes (b \oplus c) = a + \max(b, c) = \max(a + b, a + c) = (a \otimes b) \oplus (a \otimes c)$$

$$(a \oplus b) \otimes c = \max(a, b) + c = \max(a + c, b + c) = (a \otimes c) \oplus (b \otimes c)$$

The following lemma contains the other basic properties.

Lemma 3.1 *For all $x, y, z \in \mathbb{R}_{\max}$*

1. Commutative

   $x \oplus y = y \oplus x$ and $x \otimes y = y \otimes x$

2. Associative

   $x \oplus (y \oplus z) = (x \oplus y) \oplus z$ and $x \otimes (y \otimes z) = (x \otimes y) \otimes z$

3. Distributive

   $x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$

4. Zero Element

   $x \oplus \epsilon = \epsilon \oplus x = x$

5. Unit Element

   $x \otimes e = e \otimes x = x$

6. Multiplicative Inverse

   if $x \neq \epsilon$ then there exists a unique y such that $x \otimes y = e$

7. Absorbing Element

   $x \otimes \epsilon = \epsilon \otimes x = \epsilon$

8. Idempotency of Addition

   $x \oplus x = x$

Definition 1 *For $x \in \mathbb{R}_{max}$ and $n \in \mathbb{N}$*

$x^{\otimes n} = \underbrace{x \otimes x \otimes x \otimes ... \otimes x}_{n-times}.$

In the max-plus algebra exponentiation reduces to the conventional multiplication, $x^{\otimes n} = nx$. It would be natural to extend max-plus exponentiation to more general exponents as follows :

- if $x \neq \epsilon$, $x^{\otimes 0} = e = 0$
- if $\alpha \in \mathbb{R}, x^{\otimes \alpha} = \alpha x$
- if $k > 0$ then $\epsilon^{\otimes k} = \epsilon$ ($k \leq 0$ is not defined)

Here are the laws of exponents in max-plus.

Lemma 3.2 *For $m, n \in N$, $x \in \mathbb{R}_{max}$*

1. $x^{\otimes m} \otimes x^{\otimes n} = mx + nx = (m+n)x = x^{\otimes(m \otimes n)}$
2. $x^{m \otimes n} = (mx)^{\otimes n} = nmx = x^{\otimes(m^{\otimes n})}$
3. $x^{\otimes 1} = 1x = x$
4. $x^{\otimes m} \otimes y^{\otimes m} = (x \otimes y)^{\otimes m}$

Using $\oplus$ we can define the existence of order in the max-plus semi-ring.

Definition 2 *We say that $a \leq b$ if $a \oplus b = b$, by Farlow(2009).*

Definition 3 *A monoid is a closed set under an associative binary operations which has a multiplicative identity.*

Definition 4 *Upper bound is an element greater than or equal to all the elements in a given set: 3 and 4 are upper bound of the set consisting of 1, 2 and 3.*

Definition 5 *A ring is an algebraic structure on which are defined two binary operations which satisfy the following conditions:*

1. *closure under addition*

2. *associativity of addition*

3. *commutativity of addition*

4. *identity element for addition, i.e. the zero element*

5. *inverse elements for addition, i.e. negative elements*

6. *closure under multiplication*

7. *associativity of multiplication*

8. *multiplication is distributive over addition*

These criteria are called the ring axioms.

Definition 6 *A semi-ring is a set S equipped with two internal composition laws, called addition and multiplication, respectively, that satisfy the following axioms:*

1. *S is a commutative monoid for addition*

2. *S is a monoid for multiplication*

3. *multiplication distributes over addition*

4. *the neutral element for addition is absorbing for multiplication.*

Two important aspects of max-plus algebra are that it does not have additive inverses and it is idempotent. This is why max-plus algebra is considered a semi-ring and not a ring.
All idempotent semi-rings have been generalised by the following Lemma by Heidergott et al. (2006)

Lemma 3.3 *The idempotency of $\oplus$ in the max-plus semi-ring implies that for every $a \in \mathbb{R}_{\max} \backslash \{\epsilon\}$, a does not have an additive inverse.*

Proof

Suppose $a \in \mathbb{R}_{\max}$ such that $a \neq \epsilon$ has an inverse with respect to $\oplus$. Let b be the inverse of a, then we would have

$$a \oplus b = \epsilon$$

By adding a to the left of both sides of the equation we get

$$a \oplus (a \oplus b) = a \oplus \epsilon = a$$

Using the associativity property and the idempotency property of $\oplus$,

$$a = a \oplus (a \oplus b) = (a \oplus a) \oplus b = a \oplus b = \epsilon$$

which is a contradiction since we assumed $a \neq \epsilon$.

3.3 Matrices in Max-Plus Algebra

Definition 7 *A matrix (designated by an uppercase boldface letter) is a rectangular array of elements arranged in horizontal rows and vertical columns.*

Matrix addition in max-plus can only be performed on matrices of the same dimensions. The results from the matrix sum $A \oplus B$ is the maximum from the corresponding entries.

In max-plus, multiplication of matrices is defined as
$(A \otimes B)_{ij} = \bigoplus_{k=1}^{n} A_{ik} \otimes B_{kj}$, for all $(\mathbb{R}_{\max})^{nxn}$ in $\mathbb{R}_{\max}$.
Whilst the scalar multiplication of a matrix in max-plus is where each entry of the matrix is increased by the scalar.
A zero matrix is a matrix that has all the entries being $-\infty$, which is being denoted by ϵ.
An identity matrix has its diagonal as a 0 and the other entries being $-\infty$. For

any matrix A and I of the same dimensions $I \otimes A = A \otimes I$

3.3.1 Numerical examples of operations on matrices

Let $X = \begin{bmatrix} 3 & 0 \\ -2 & 4 \end{bmatrix}$, $Y = \begin{bmatrix} 6 & 1 \\ -\infty & 9 \end{bmatrix}$ and $\alpha = 2$, where $X, Y \in \mathbb{R}_{\max}^{n \times n}$

3.3.2 Matrix Addition

$$X \oplus Y = \begin{bmatrix} 3 & 0 \\ -2 & 4 \end{bmatrix} \oplus \begin{bmatrix} 6 & 1 \\ -\infty & 9 \end{bmatrix}$$

$$X \oplus Y = \begin{bmatrix} 3 \oplus 6 & 0 \oplus 1 \\ -2 \oplus -\infty & 4 \oplus 9 \end{bmatrix}$$

$$X \oplus Y = \begin{bmatrix} 6 & 1 \\ -2 & 9 \end{bmatrix}$$

3.3.3 Scalar Multiplication

$$\alpha \otimes X = 2 \otimes \begin{bmatrix} 3 & 0 \\ -2 & 4 \end{bmatrix}$$

$$\alpha \otimes X = \begin{bmatrix} 2 \otimes 3 & 2 \otimes 0 \\ 2 \otimes -2 & 2 \otimes 4 \end{bmatrix}$$

$$\alpha \otimes X = \begin{bmatrix} 5 & 2 \\ 0 & 6 \end{bmatrix}$$

3.3.4 Multiplication of Two Matrices

$$X \otimes Y = \begin{bmatrix} 3 & 0 \\ -2 & 4 \end{bmatrix} \otimes \begin{bmatrix} 6 & 1 \\ -\infty & 9 \end{bmatrix}$$

$$X \otimes Y = \begin{bmatrix} (3 \otimes 6) \oplus (0 \otimes -\infty) & (3 \otimes 1) \oplus (0 \otimes 9) \\ (-2 \otimes 6) \oplus (4 \otimes -\infty) & (-2 \otimes 1) \oplus (4 \otimes 9) \end{bmatrix}$$

$$X \otimes Y = \begin{bmatrix} 9 \oplus -\infty & 4 \oplus 9 \\ 4 \oplus -\infty & -1 \oplus 13 \end{bmatrix}$$

$$X \otimes Y = \begin{bmatrix} 9 & 9 \\ 4 & 13 \end{bmatrix}$$

Multiplication of matrices in $(\mathbb{R}_{\max}, \oplus, \otimes)$ is associative, that is, A $\otimes$ (B $\otimes$ C)= (A $\otimes$ B) $\otimes$ C but not commutative , thus, A $\otimes$ B $\neq$ B $\otimes$ A. It is only commutative when A $=$ B or when one of them is a unit matrix. This is where A, B and C are matrices with entries from $\mathbb{R}_{max}$.

3.4 Solving Systems of Equations in Max-plus Algebra

Definition 8 *Simultaneous equations is a set of independent equations in two or more variables. Example is*

$$\begin{array}{ccccccc} 5 \otimes x_1 & \oplus & -4 \otimes x_2 & \oplus & 3 \otimes x_3 & = & -4 \\ -10 \otimes x_1 & \oplus & 8 \otimes x_2 & \oplus & -6 \otimes x_3 & = & 8 \\ 15 \otimes x_1 & \oplus & -12 \otimes x_2 & \oplus & 9 \otimes x_3 & = & -12 \end{array}$$

We develop the theory of linear systems of equations for max-plus in this section. We consider the solution to the matrix equation $A \otimes x = b$, in general, where A is an $n \times n$ matrix, x is an $n \times 1$ vector and b is an $n \times 1$ vector. To first get an idea for how to go about solving the system of equations, it will be helpful if we look at the equivalent system in the usual arithmetic. $Ax = b$ can be rewritten as the following detailed matrix equation and then the equivalent system of max-plus equations:

$$A \otimes x = b$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$\begin{array}{ccccccccc} (a_{11} \otimes x_1) & \oplus & (a_{12} \otimes x_2) & \oplus & \cdots & \oplus & (a_{1n} \otimes x_n) & = & b_1 \\ (a_{21} \otimes x_1) & \oplus & (a_{22} \otimes x_2) & \oplus & \cdots & \oplus & (a_{2n} \otimes x_n) & = & b_2 \\ \vdots & \vdots & \vdots & \vdots & & & & \vdots & \\ (a_{n1} \otimes x_1) & \oplus & (a_{n2} \otimes x_2) & \oplus & \cdots & \oplus & (a_{nn} \otimes x_n) & = & b_n \end{array}$$

We have

$$\max\{(a_{11} + x_1),(a_{12} + x_2), \cdots, (a_{1n} + x_n)\} = b_1$$
$$\max\{(a_{21} + x_1),(a_{22} + x_2), \cdots, (a_{2n} + x_n)\} = b_2$$
$$\vdots$$
$$\max\{(a_{n1} + x_1),(a_{n2} + x_2), \cdots, (a_{nn} + x_n)\} = b_n$$

We first consider the case that a solution exists and some of the entries of b is $-\infty$. Without loss of generality, we can reorder the equations so that the finite entries of b occur first:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_k \\ -\infty \\ \vdots \\ -\infty \end{pmatrix}$$

Which is

$$
\begin{aligned}
&max(a_{11}+x_1, a_{12}+x_2, \cdots, a_{1n}+x_n) = b_1\\
&\quad\vdots\\
&max(a_{k1}+x_1, a_{k2}+x_2, \cdots, a_{kn}+x_n) = b_k\\
&max(a_{k+1,1}+x_1, a_{k+1,2}+x_2, \cdots, a_{k+1,n}+x_n) = -\infty\\
&\quad\vdots\\
&max(a_{n1}+x_1, a_{n2}+x_2, \cdots, a_{nn}+x_n) = -\infty
\end{aligned}
$$

We let the finite part of A be A_1 with dimensions $k \times l$, that of b be $b' = \begin{pmatrix} b_1 \\ \vdots \\ b_k \end{pmatrix}$

and that of x be $x' = \begin{pmatrix} x_1 \\ \vdots \\ x_l \end{pmatrix}$

You have to note that if $A \otimes x = b$ has a solution , then $x_{k+1} = x_n = -\infty$, and $A \otimes x' = b'$. Thus, $A \otimes x = b$ has a solution if and only if x' is a solution to $A_1 \otimes x' = b'$ and solutions to $A \otimes x = b$ are $x = \begin{pmatrix} x' \\ -\infty \\ \vdots \\ -\infty \end{pmatrix}$

The solvability of a system with infinite entries in b can therefore be reduced to that of a system where all the entries in b are finite. Therefore our attention will be restricted to systems $A \otimes x = b$ where all the entries of b are finite. If there is to be a solution to the system of max-plus equations, then $a_{ij} + x_j \leq b$ for all $i \in \{1, ..., n\}$ and $j \in \{1, ..., n\}$. To find a solution to the system, we first consider each component of x separately, when we consider x_1 for example. If there is a solution to the system, then $a_{i1} + x_1 \leq b_i$ for $i = 1, 2, 3, ..., n$. Thus $x_1 \leq b_i - a_{i1}$

for each i leading us to the following system of upper bounds on x_1:

$$\begin{array}{ccccc} x_1 & \leq & b_1 & - & a_{11} \\ x_1 & \leq & b_2 & - & a_{21} \\ & \vdots & & & \\ x_1 & \leq & b_n & - & a_{n1} \end{array}$$

If this system of inequalities has a solution, then it satisfies

$$x_1 \leq min\{(b_1 - a_{11}), (b_2 - a_{21}), ..., (b_n - a_{n1})\}$$

Similarly, we can find the possible solutions for $x_2, \cdots, x_n$, giving us the following system of inequalities on the entries of x:

$$\begin{gathered} x_1 \leq min\{(b_1 - a_{11}), (b_2 - a_{21}), \cdots, (b_n - a_{n1})\} \\ x_2 \leq min\{(b_1 - a_{12}), (b_2 - a_{22}), \cdots, (b_n - a_{n2})\} \\ \vdots \\ x_n \leq min\{(b_1 - a_{1n}), (b_2 - a_{2n}), \cdots, (b_n - a_{nn})\} \end{gathered}$$

This leads us to the candidate for the solution to $A \otimes x = b$, which we will denote by x'.

$$x' = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad \text{where} \qquad \begin{gathered} x_1 \leq \min\{(b_1 - a_{11}), (b_2 - a_{21}), ..., (b_n - a_{n1})\} \\ x_2 \leq \min\{(b_1 - a_{12}), (b_2 - a_{22}), ..., (b_n - a_{n2})\} \\ \vdots \\ x_n \leq min\{(b_1 - a_{1n}), (b_2 - a_{2n}), ..., (b_n - a_{nn})\} \end{gathered}$$

Let us introduce another matrix to simplify the process of solving a system of max-plus equations. We define the discrepancy matrix, $D_{A.b}$ as follows:

$$\begin{pmatrix} b_1 - a_{11} & b_1 - a_{12} & \cdots & b_1 - a_{1n} \\ b_2 - a_{21} & b_2 - a_{22} & \cdots & b_2 - a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_n - a_{n1} & b_n - a_{n2} & \cdots & b_n - a_{nn} \end{pmatrix}$$

Note that $D_{A.b}$ is simply a matrix with all the upper bounds of the x_j's and that each x_j can be found by taking the minimum of the jth column of $D_{A.b}$.

Another matrix is formed from $D_{A.b}$ called reduced discrepancy matrix, $R_{A.b}$: $R_{A.b} = (r_{ij})$ where

$$r_{ij} = \begin{cases} 1 & \text{if } d_{ij}\text{=mininum of column j} \\ 0 & \text{otherwise} \end{cases}$$

$R_{A.b}$ is useful in predicting the number of solutions to the matrix equation $A \otimes x = b$.

Theorem 3.4.1 *Let $A \otimes x = b$ be a matrix equation in $(\mathbb{R}_{max}, \oplus, \otimes)$ where A is an $n \times n$ matrix, and b is a $n \times 1$ vector with all entries finite.*

1. If there is a zero-row in the reduced discrepancy matrix, $R_{A.b}$, then there is no solution to the matrix equation.

2. If there is at least one (1) in each row of the reduced discrepancy matrix, $R_{A.b}$, then x' is a solution $A \otimes x = b$.

Proof

1. Without loss of generality, denote the zero-row of $R_{A.b}$ by row k. Suppose

to the contrary that x̃ is a solution of $A \otimes x = b$. Then

$$\tilde{x_j} \leq \min_l (b_l - a_{lj}) < b_k - a_{kj}.$$

Thus $\tilde{x_j} + a_{kj} < b_k$ for all j. Hence, x̃ does not satisfy the kth equation and is not a solution to $A \otimes x = b$.

2. We prove the contrapositive. Suppose x' is not a solution to the matrix equation. By definition, $x'_j \leq b_k - a_{kj}$ for all j,k. Hence $\max_j(a_{kj} + x'_j) \leq b_k$ and if x' is not a solution then there is a k with $\max_j(a_{kj} + x'_j) < b_k$. This is equivalent to $x'_j < b_k - a_{kj}$ for all j. Since $x'_j = min(b_l - a_{il})$ for some l, there is no entry in row k of $R_{A.b}$ that is 1.

Now, provided we know that a solution to $A \otimes x = b$ exists, how can we tell the number of solutions to this equation? We need to define the concept of fixed entries in $R_{A.b}$.

Definition 9 *The 1 in a row of $R_{A.b}$ is a variable-fixing entry if either*

1. *It is the only 1 in that row (a lone-one), or*
2. *It is in the same column as a lone-one.*
   The remaining 1's are called slack entries.

A 1 in the jth column of $R_{A.b}$ signifies the minimum of all the upper bounds for x_j. If there are no other ones in the row where a one occur, then the only way that the equation corresponding to that row can be solved is to have x_j achieve the bound. This causes the value of x_j to be fixed at a specific value, i.e. it is a variable-fixing entry.
There must be a lone-one in each column for $A \otimes x = b$ to have a unique solution. The following theorem shows that in order for $A \otimes x = b$ to have a unique solution, each component of x must be fixed, i.e. there can be no slack entries (a slack entry can only exist if there are no variable-fixing entries in that column).

Theorem 3.4.2 *Let $A \otimes x = b$ be a matrix equation in $(\mathbb{R}_{max}, \oplus, \otimes)$ where A is an $n \times n$ matrix, b is an $n \times 1$ vector with finite entries, and a solution to the equation exists.*

1. *If each column of $R_{A.b}$ has a lone-one, then the solution to the matrix equation is unique.*

2. *If there are slack entries in $R_{A.b}$, then there are infinite solutions to the matrix equation.*

Proof

1. If there is a lone-one in each column of $R_{A.b}$, then there is a variable-fixing entry in each column of $R_{A.b}$. There can be no slack entries since all the columns contain a variable-fixing entry. All the components of x are fixed and thus the solution is unique.

2. Let r_{ij} be one of the slack entries in $R_{A.b}$ and let $\tilde{x}$ be a solution to the equation $A \otimes x = b$. Since r_{ij} is not fixed, then there are no fixed entries in the jth column of $R_{A.b}$. Thus, equality can be achieved for each row equation without using the $\tilde{x}_j$ component. Thus, while the value of $\tilde{x}_j$ does indicate the maximum value possible for this component, any smaller value will not alter the existence of equalities in the row equations.

3.5 System of inequalities

In this part of our work, we are going to show how a one-sided system of inequalities can be solved.

A system of the form

$$A \otimes x \leq b$$

where $A = (a_{ij}) \in \mathbb{R}^{m \times n}$ and $b = (b_1, ..., b_m)^T \in \mathbb{R}$, is called a one-sided max-linear system of inequalities or in short as one-sided system of inequalities. Here, we will only present a result which shows that the principal solution, $\bar{x}(A, b)$ is the greatest solution to the system. Here we mean that if $A \otimes x \leq b$ has a solution then $\bar{x}(A, b)$ is the greatest of all the solutions. Let as denote the solution set of the system by $S(A, b, \leq)$, where

$$S(A, b, \leq) = \{x \in \mathbb{R}^n; A \otimes x \leq b\}$$

Theorem 3.5.1 *$x \in S(A, b, \leq)$ if and only if $x \leq \bar{x}(A, b)$*

Proof

Suppose $x \in S(A, b, \leq)$. Then we have

$$A \otimes x \leq b$$

$$max_j(a_{ij} + x_j) \leq b_i, forall i$$

$$a_{ij} + x_j \leq b_i, forall i, j$$

$$x_j \leq b_i \otimes a_{ij}^{-1}, forall i, j$$

$$x_j \leq min_i(b_i \otimes a_{ij}^{-1}), forall i, j$$

$$x \leq \bar{x}(A, b)$$

Hence the proof.

A system of inequalities

$$A \otimes x \leq b$$

$$C \otimes x \geq d$$

Which is discussed in Cechla'rova' (2001) where the following result was presented.

Lemma 3.4 *A system of inequalities*

$$A \otimes x \leq b$$

$$C \otimes x \geq d$$

has a solution if and only if $C \otimes \bar{x}(A, b) \geq d$.

3.6 A system which contains both equations and inequalities

In this part of our work, we will consider a system containing both equations and inequalities where the results were taken from Aminu (2011). Let $A = (a_{ij}) \in \mathbb{R}^{k\times n}$, $C = (c_{ij}) \in \mathbb{R}^{r\times n}$, $b = (b_1, ..., b_k)^T \in \mathbb{R}^k$ and $d = (d_1, ..., d_r)^T \in \mathbb{R}^r$.
A one-sided max-linear system containing both equations and inequalities is of the form:

$$A \otimes x = b$$

$$C \otimes x \leq d$$

These are the notations we shall use throughout our work:

$$R = \{1, 2, ..., r\}$$

$$S(A, C, b, d) = \{x \in \mathbb{R}^n, A \otimes x = b and C \otimes x \leq d\}$$

$$S(C, d, \leq) = \{x \in \mathbb{R}^n, C \otimes x \leq d\}$$

$$\bar{x}_j(C, d) = min_{i\in R}(d_i \otimes C_{ij}^{-1}) for all j \in \mathbb{N}$$

$$K = \{1, 2, ..., k\}$$

$$K_j = \{k \in K; b_k \otimes a_{kj}^{-1} = min_{i\in k}(b_i \otimes a_{ij}^{-1})\} for all j \in \mathbb{N}$$

$$\bar{x}_j(A, b) = min_{i\in K}(b_i \otimes a_{ij}^{-1}) for all j \in \mathbb{N}$$

$$\bar{x} = (\bar{x_1}, ..., \bar{x_n})^T$$

$$J = \{j \in \mathbb{N}; \bar{x_j}(C, d) \geq \bar{x_j}(A, b)\} and$$

$$L = \mathbb{N} \setminus J$$

We also defined the vector $\hat{x} = (\hat{x}_1, \hat{x}_2, ..., \hat{x}_n)^T$, where

$$\hat{x}_j(A, C, b, d) = \begin{cases} \bar{x}_j(A, b) & \text{if } j \in J \\ \bar{x}_j(C, d) & \text{if } j \in L \end{cases}$$

and $\mathbb{N}_{\hat{x}} = \{j \in \mathbb{N}; \hat{x}_j = \bar{x}_j\}$

Theorem 3.6.1 *Let $A = (a_{ij}) \in \mathbb{R}^{k\times n}$, $C = (c_{ij}) \in \mathbb{R}^{r\times n}$, $b = (b_1, ..., b_k)^T \in \mathbb{R}^k$ and $d = (d_1, ..., d_r)^T \in \mathbb{R}^r$. Then the following three statements are equivalent:*

i. $S(A, C, b, d) \neq \emptyset$

ii. $\hat{x}(A, C, b, d) \in S(A, C, b, d)$

iii. $\bigcup_{j\in J} K_j = K$

Proof:

$(i) \implies (ii)$. Let $x \in S(A, C, b, d)$, therefore $x \in S(A, b)$ and $x \in S(C, d, \leq)$. Since $x \in S(C, d, \leq)$, it follows from Theorem 3.5.1 that $x \leq \bar{x}(C, d)$. Now that $x \in S(A, b)$ and also $x \in S(C, d, \leq)$, we need to show that $\bar{x}_j(C, d) \geq \bar{x}_j(A, b)$ for all $j \in N_x$ (that is $\mathbb{N}_x \subseteq J$). Let $j \in N_x$ then $x_j = \bar{x}_j(A, b)$. Since $x \in S(C, d, \leq)$ we have $x \leq \bar{x}(C, d)$ and therefore $\bar{x}_j(A, b) \leq \bar{x}_j(C, d)$ thus $j \in J$. Hence, $\mathbb{N}_x \subseteq J$ and therefore $\bigcup_{j\in J} K_j = K$. This proves $(i) \implies (iii)$.

$(iii) \implies (i)$. Suppose $\bigcup_{j\in J} K_j = K$. Since $\hat{x}(A, C, b, d) \leq \bar{x}(C, d)$ we have $\hat{x}(A, C, b, d) \in S(C, d, \leq)$. Also $\hat{x}(A, C, b, d) \leq \bar{x}(A, b)$ and $\mathbb{N}_{\hat{x}} \supseteq J$ gives $\bigcup_{j\in \mathbb{N}_{\hat{x}}(A,C,b,d)} K_j \supseteq \bigcup_{j\in J} K_j = K$, Hence $\bigcup_{j\in \mathbb{N}_{\hat{x}}(A,C,b,d)} K_j = K$, therefore $\hat{x}(A, C, b, d) \in S(A, b)$ and $\hat{x}(A, C, b, d) \in S(C, d, \leq)$. Hence $\hat{x}(A, C, b, d) \in S(A, C, b, d)$ (that is $S(A, C, b, d) \neq \emptyset$) and this proves $(iii) \implies (ii)$.

Theorem 3.6.2 *Let $A = (a_{ij}) \in \mathbb{R}^{k\times n}$, $C = (c_{ij}) \in \mathbb{R}^{r\times n}$, $b = (b_1, ..., b_k)^T \in \mathbb{R}^k$ and $d = (d_1, ..., d_r)^T \in \mathbb{R}^r$. Then $x \in S(A, C, b, d)$ if and only if*

i. $x \in \hat{x}(A, C, b, d)$ *and*

ii. $\bigcup_{j\in \mathbb{N}_x} K_j = K$ *where* $\mathbb{N}_x = \{j \in \mathbb{N}; x_j = \bar{x}_j(A, b)\}$

Proof:

$(\Longrightarrow)$ Let $x \in \hat{x}(A, C, b, d)$, then $x \leq \hat{x}(A, b)$ and $x \leq \hat{x}(C, d)$. Since $\hat{x}(A, C, b, d) =$

$\bar{x}(A,b) \oplus' \bar{x}(C,d)$ we have $x \leq \hat{x}(A,C,b,d)$. Also, $x \in S(A,C,b,d)$ implies that $x \in S(C,d,\leq)$. It follows from Theorem 3.6.1 that $\bigcup_{j \in \mathbb{N}_x} K_j = K$.
($\Longleftarrow$) Suppose that $x \in \hat{x}(A,C,b,d) = \bar{x}(A,b) \oplus' \bar{x}(C,d)$ and $\bigcup_{j \in \mathbb{N}_x} K_j = K$. It follows from Theorem 3.6.1 that $x \in S(A,b)$, also by Theorem 3.6.1 $x \in S(C,d,\leq)$. Thus $x \in S(A,b) \bigcap S(C,d,\leq) = S(A,C,b,d)$.
The vector $\hat{x}(A,C,b,d)$ plays an important role in the solution of one-sided system containing both equations and inequalities.

3.7 Max-linear program with equation and inequality constraints

Let $f = (f_1, f_2, ..., f_n)^T \in \mathbb{R}^n$ be a given vector. The tasks of minimising [maximising] the function $f(x) = f^T \otimes x = max(f_1 + x_1, f_2 + x_2, ..., f_n + x_n)$
subject to

$$A \otimes x = b$$

$$C \otimes x \leq d$$

is called max-linear program with one-sided equations and inequalities which is denoted by $MLP_{\leq}^{min}$ and $MLP_{\leq}^{max}$. Let $S^{min}(A,C,b,d)$ and $S^{max}(A,C,b,d)$ be the set of optimal solutions respectively.

Lemma 3.5 *(Aminu (2009)) Suppose $f \in \mathbb{R}^n$ and let $f(x) = f^T \otimes x$ be defined on $\mathbb{R}^n$. Then,*

1. *f(x) is max-linear, i.e. $f(\alpha \otimes x \oplus \beta \otimes y) = \alpha \otimes f(x) \oplus \beta \otimes f(y)$ for every $x, y \in \mathbb{R}^n$, .*

2. *f(x) is isotone, i.e. $f(x) \leq f(y)$ for every $x, y \in \mathbb{R}^n$, $x \leq y$.*

Proof:

1. Let $\alpha, \beta \in \mathbb{R}$. Then we have

$$\begin{aligned} f(\alpha \otimes x \oplus \beta \otimes y) &= f^T \otimes \alpha \otimes x \oplus f^T \otimes \beta \otimes y \\ &= \alpha \otimes f^T \otimes x \oplus \beta \otimes f^T \otimes y \\ &= \alpha \otimes f(x) \oplus \beta \otimes f(y) \end{aligned}$$

2. Let $x, y \in \mathbb{R}^n$ such that $x \leq y$. Since $x \leq y$, we have

$$max(x) \leq max(y)$$

$$f^T \otimes x \leq f^T \otimes y, for any f \in \mathbb{R}^n$$

$$f(x) \leq f(y)$$

It is possible to convert equations to inequalities and otherwise, but this is going to increase the number of constraints or variables and therefore increases the computational complexity. The method we present here does not require any new constraints or variables.

It follows from Theorem 3.6.2 and Lemma 3.5 that $\hat{x} \in S^{max}(A, C, b, d)$. Aminu (2009) present a polynomial algorithm which finds $x \in S^{min}(A, C, b, d)$ or recognises that $S^{min}(A, C, b, d) = \emptyset$. From Theorem 3.6.1 either $\hat{x} = S(A, C, b, d)$ or $S(A, C, b, d) = \emptyset$. It was therefore assumed in the algorithm below that $S(A, C, b, d) \neq \emptyset$ and also $S^{min}(A, C, b, d) \neq \emptyset$.

Given $f = (f_1, f_2, ..., f_n)^T \in \mathbb{R}^n$, $b = (b_1, b_2, ..., b_k)^T \in \mathbb{R}^k$, $d = (d_1, d_2, ..., d_r)^T \in \mathbb{R}^r$, $A = (a_{ij}) \in \mathbb{R}^{k \times n}$ and $C = (c_{ij}) \in \mathbb{R}^{r \times n}$.

1. Find $\bar{x}(A, b)$, $\bar{x}(C, d)$, $\hat{x}(A, C, b, d)$, K_j, $j \in J$; $J = \{j \in \mathbb{N}; \bar{x}_j(C, d) \geq \bar{x}_j(A, b)\}$

2. $x := \hat{x}(A, C, b, d)$

3. $H(x) := \{j \in \mathbb{N}; f_j + x_j = f(x)\}$

4. $J := J \setminus H(x)$

5. If

$$\bigcup_{j \in J} K_j \neq K$$

then stop ($x \in S^{min}(A, C, b, d)$)

6. set x_j small enough (so that it not active on any equation or inequality) for every $j \in H(x)$

7. Go to 3

Chapter 4

Some Max-plus Applications

4.1 Solutions to Systems of Equations in Max-algebra

Example 4.1.1 *Max-plus system with unique solution*

To solve $A \otimes x = b$*, where* $A = \begin{bmatrix} 1 & -9 & 4 \\ -4 & 18 & -8 \\ 2 & 1 & -4 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, *and* $b = \begin{bmatrix} 1 \\ -6 \\ -3 \end{bmatrix}$

Calculate the discrepancy matrix: $D_{A.b} = \begin{bmatrix} 0 & 10 & -3 \\ -2 & -24 & 2 \\ -5 & -4 & 1 \end{bmatrix}$

Taking the minimum of each column of $D_{A.b}$ gives the solution

$x_1' = min(0, -2, -5) = -5$

$x_2' = min(10, -24, -4) = -24$

$x_3' = min(-3, 2, 1) = -3$

The candidate solution to $A \otimes x = b$ becomes $x' = (-5, -24, -3)^T$. We can verify that this is the only solution to $A \otimes x = b$ by substituting it back in:

$$\begin{bmatrix} 1 & -9 & 4 \\ -4 & 18 & -8 \\ 2 & 1 & -4 \end{bmatrix} \otimes \begin{bmatrix} -5 \\ -24 \\ -3 \end{bmatrix} = \begin{bmatrix} \max(-4, -33, 1) \\ \max(-9, -6, -11) \\ max(-3, -23, -7) \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \\ -3 \end{bmatrix}$$

This will therefore be the only solution to the matrix equation as we show later.

Example 4.1.2 *Max-plus system with infinitely many solutions*

To solve $A \otimes x = b$, *where* $A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & -1 & 0 \\ 4 & 0 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, *and* $b = \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}$

Calculate the discrepancy matrix: $D_{A.b} = \begin{bmatrix} 5 & 5 & 3 \\ 1 & 4 & 3 \\ -2 & 2 & 3 \end{bmatrix}$

Taking the minimum of each column of $D_{A.b}$ gives the solution

$x_1' = min(5, 1, -2) = -2$

$x_2' = min(5, 4, 2) = 2$

$x_3' = min(3, 3, 3) = 3$

The candidate solution to $A \otimes x = b$ becomes $x' = (-2, 2, 3)^T$. This solution can be verified by substituting it back in

$$\begin{bmatrix} 1 & 1 & 3 \\ 2 & -1 & 0 \\ 4 & 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} max(-1, 3, 6) \\ max(0, 1, 3) \\ max(2, 2, 2) \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}$$

X' this therefore a solution to the given matrix equation. It can be seen that there are other solution that also work. Any x of the form $\{x : x = (u, v, 3)^T$ where $u \leq -2$ and $v \leq 2\}$ is also a solution to the given matrix equation.

Example 4.1.3 *Max-plus system with no solution*

To solve $A \otimes x = b$, *where* $A = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 4 & 3 \\ 1 & 2 & 0 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, *and* $b = \begin{bmatrix} 5 \\ 5 \\ 7 \end{bmatrix}$

The discrepancy matrix: $D_{A.b} = \begin{bmatrix} 3 & 6 & 6 \\ 5 & 1 & 2 \\ 6 & 5 & 7 \end{bmatrix}$

Which gives the solution of $x' = (3,1,2)^T$.

x' is verified to see that it is not a solution.

$$\begin{bmatrix} 2 & -1 & -1 \\ 0 & 4 & 3 \\ 1 & 2 & 0 \end{bmatrix} \otimes \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} max(5,0,1) \\ max(3,5,5) \\ max(4,3,2) \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ \underline{4} \end{bmatrix} \neq b = \begin{bmatrix} 5 \\ 5 \\ 7 \end{bmatrix}$$

It is observed that the underlined entry does not correspond the entry of b. We know that a solution x must satisfy $x_1 \leq 3$, $x_2 \leq 1$, and $x_3 \leq 2$ because the components of x' are the upper bounds. It is seen from the third row that $max(x_1 + 1, x_2 + 2, x_3 + 0) \leq 4 < 7$.

A reduced discrepancy matrix $R_{A.b}$ is use to predict the number of solutions to the matrix equation $A \otimes x = b$. The table below shows the various examples and their $D_{A.b}$ and $R_{A.b}$. Where the minimum occurs in each column of $D_{A.b}$ has been underlined for each entries. Note that they are the 'one' entries of each correspond $R_{A.b}$.

Example	$D_{A.b}$	$R_{A.b}$
Unique Solution	$\begin{bmatrix} 0 & 10 & \underline{-3} \\ -2 & \underline{-24} & 2 \\ \underline{-5} & -4 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & \underline{1} \\ 0 & \underline{1} & 0 \\ \underline{1} & 0 & 0 \end{bmatrix}$
Infinitely Many Solutions	$\begin{bmatrix} 5 & 5 & \underline{3} \\ 1 & 44 & \underline{3} \\ \underline{-2} & \underline{2} & \underline{3} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & \underline{1} \\ 0 & 0 & \underline{1} \\ \underline{1} & \underline{1} & \underline{1} \end{bmatrix}$
No Solution	$\begin{bmatrix} \underline{3} & 6 & 6 \\ 5 & \underline{1} & \underline{2} \\ 6 & 5 & 7 \end{bmatrix}$	$\begin{bmatrix} \underline{1} & 0 & 0 \\ 0 & \underline{1} & \underline{1} \\ 0 & 0 & 0 \end{bmatrix}$

The minimum entry of the column, j, in the $D_{A.b}$ matrix is the maximum solution to the system of inequalities for x_j. To change this system of inequalities to a system of equalities, we must have equality in each row inequality, i.e. there must be at least one minimum in each row of $D_{A.b}$ i.e. there must be at least one (1) in each row of $R_{A.b}$ for a solution to exist.

A one (1) in the jth column of $R_{A.b}$ signifies the minimum of the upper bounds for x_j. If there are no other ones in the row where a 1 occurs, the only way that the equation corresponding to that row can be solved is to have x_j achieve the bound. This causes the value of x_j to be fixed at a specific value, making it a variable-fixing entry. We illustrate this by underlining the variable-fixing entries for the examples in the table below.

Example	$R_{A.b}$
Unique Solution	$\begin{bmatrix} 0 & 0 & \underline{1} \\ 0 & \underline{1} & 0 \\ \underline{1} & 0 & 0 \end{bmatrix}$
Infinitely Many Solutions	$\begin{bmatrix} 0 & 0 & \underline{1} \\ 0 & 0 & \underline{1} \\ 1 & 1 & \underline{1} \end{bmatrix}$
No Solution	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

From the table above, to consider the $R_{A.b}$ for One Solution, all of the non-zero entries are variable-fixing entries. The first row equation fixes the x_3 component where $x_3 = -3$. The second row equation fixes the x_2 component where $x_3 = -24$. Finally, the third row equation fixes the x_1 component where $x_3 = -5$ making all the components of x to be fixed.

There are slack entries in $R_{A.b}$ for Infintely Many Solutions. The first row equation fixes the x_3 component, $x_3 = 3$. The component solution to the second row equation has already been fixed by the first row equation. In the third row equation, there are three possible ways for equality to be achieved, is either $x_1 = -2$, $x_2 = 2$ or $x_3 = 3$. But x_3 which is 3, has already been fixed. As long as $x_1 \leq -2$ and $x_2 \leq 2$, no problem can be caused.

For $R_{A.b}$ in the example of No Solution, because there is a third row of $R_{A.b}$ containing zeros (or no 1's), there is no solution for the system of equations which does not satisfy the condition that there must be at least one minimum in each row of $D_{A.b}$, i.e. there must be at least a one (1) in each row of $R_{A.b}$ for a solution to exist.

We did not only apply the discrepancy method to a 3-by-3 system of equations but also to a 4-by-4 systems. This is to further explain that the method works for all $n \times n$ system of equations.

Example 4.1.4 *Max-plus systems with unique solution*

To solve $Ax = b$, *where* $A = \begin{bmatrix} 5 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, *and* $b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

The discrepancy matrix: $D_{A.b} = \begin{bmatrix} -4 & 2 & 2 & 2 \\ 2 & -3 & 2 & 2 \\ 2 & 2 & -2 & 2 \\ 2 & 2 & 2 & -3 \end{bmatrix}$

Taking the minimum of each column of $D_{A.b}$ gives the solution

$x'_1 = min(-4, 2, 2, 2) = -4$

$x'_2 = min(2, -3, 2, 2) = -3$

$x'_3 = min(2, 2, -2, 2) = -2$

$x'_4 = min(2, 2, 2, -3) = -3$

The candidate solution to $Ax = b$ becomes $x' = (-4, -3, -2, -3)^T$.

This solution can be verified by substituting it back in

$$\begin{bmatrix} 5 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix} \otimes \begin{bmatrix} -4 \\ -3 \\ -2 \\ -3 \end{bmatrix} = \begin{bmatrix} max(1,-4,-3,-4) \\ max(-5,1,-3,-4) \\ max(-5,-4,1,-4) \\ max(-5,-4,-3,1) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

This will be the only solution to the matrix equation which we will show later by the reduced discrepancy ($R_{A.b}$).

Example 4.1.5 *Max-plus system with infinitely many solutions*

To solve $Ax = b$, *where* $A = \begin{bmatrix} 4 & -1 & 1 & 1 \\ -1 & 3 & -1 & -1 \\ -1 & 0 & -3 & -1 \\ 1 & 1 & 0 & -2 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, *and* $b = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$

Calculate the discrepancy matrix: $D_{A.b} = \begin{bmatrix} -3 & 2 & 0 & 0 \\ 1 & -3 & 1 & 1 \\ 0 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{bmatrix}$

which gives the solution of $x' = (-3,-3,0,0)^T$. This solution can be verified by substituting it back in

$$\begin{bmatrix} 4 & -1 & 1 & 1 \\ -1 & 3 & -1 & -1 \\ -1 & 0 & -3 & -1 \\ 1 & 1 & 0 & -2 \end{bmatrix} \otimes \begin{bmatrix} -3 \\ -3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} max(1,-4,1,1) \\ max(-4,0,-1,-1) \\ max(-4,-3,-3,-1) \\ max(-2,-2,0,-2) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

x' is therefore a solution to the given matrix equation. There are other solutions that also work. Any x of the form $\{x : x = (u,-3,0,0)^T$, where $u \leq -3\}$ is also a solution to the given matrix equation.

Example 4.1.6 *Max-plus system with no solution*

To solve $Ax = b$, *where* $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 4 \\ -1 & 1 & -1 & -1 \\ -1 & 3 & 1 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, *and* $b = \begin{bmatrix} 2 \\ 1 \\ -6 \\ -2 \end{bmatrix}$

The discrepancy matrix: $D_{A.b} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & -3 \\ -5 & -7 & -5 & -5 \\ -1 & -5 & -3 & -1 \end{bmatrix}$ which gives the solution of $x' = (-5, -7, -5, -5)^T$.

x' is verified to see that it is not a solution

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 4 \\ -1 & 1 & -1 & -1 \\ -1 & 3 & 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} -5 \\ -7 \\ -5 \\ -5 \end{bmatrix} = \begin{bmatrix} max(-4, -6, -4, -4) \\ max(-4, -5, -3, -1) \\ max(-6, -6, -6, -6) \\ max(-6, -4, -4, -6) \end{bmatrix} = \begin{bmatrix} \underline{-1} \\ \underline{-1} \\ -6 \\ \underline{-4} \end{bmatrix} \neq b = \begin{bmatrix} 2 \\ 1 \\ -6 \\ -2 \end{bmatrix}$$

It is observed that the underlined entries do not correspond the entry of b. We know that a solution X must satisfy $x_1 \leq -5$, $x_2 \leq -7$, $x_3 \leq -5$ and $x_4 \leq -5$ because the components of X' are the upper bounds. It is seen from the first, second and fourth row that $max(x_1 + 1, x_2 + 1, x_3 + 1, x_4 + 1) \leq -1 < 2$, $max(x_1+1, x_2+2, x_3+2, x_4+4) \leq -1 < 1$, and $max(x_1-1, x_2+3, x_3+1, x_4-1) \leq -4 < -2$ respectively. The matrix equation has therefore no solution.

The table below shows the various examples and their $D_{A.b}$ and $R_{A.b}$.

Example	$D_{A.b}$	$R_{A.b}$
Unique Solution	$\begin{bmatrix} \underline{-4} & 2 & 2 & 2 \\ 2 & \underline{-3} & 2 & 2 \\ 2 & 2 & \underline{-2} & 2 \\ 2 & 2 & 2 & \underline{-3} \end{bmatrix}$	$\begin{bmatrix} \underline{1} & 0 & 0 & 0 \\ 0 & \underline{1} & 0 & 0 \\ 0 & 0 & \underline{1} & 0 \\ 0 & 0 & 0 & \underline{1} \end{bmatrix}$
Infinitely Many Solutions	$\begin{bmatrix} \underline{-3} & 2 & \underline{0} & \underline{0} \\ 1 & \underline{-3} & 1 & 1 \\ 0 & -1 & 2 & \underline{0} \\ -1 & -1 & \underline{0} & 2 \end{bmatrix}$	$\begin{bmatrix} \underline{1} & 0 & \underline{1} & \underline{1} \\ 0 & \underline{1} & 0 & 0 \\ 0 & 0 & 0 & \underline{1} \\ 0 & 0 & \underline{1} & 0 \end{bmatrix}$
No Solution	$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & -3 \\ \underline{-5} & \underline{-7} & \underline{-5} & \underline{-5} \\ -1 & -5 & -3 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \underline{1} & \underline{1} & \underline{1} & \underline{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$

To consider $R_{A.b}$ for Unique Solution, all the non-zero entries are variable-fixing entries. The first row equation fixes the x_1 component where $x_1 = -4$. The second row equation also fixes x_2, where $x_2 = -3$. The third row also fixes x_3, where $x_3 = -2$. Finally, the fourth row equation fixes x_4, where $x_4 = -3$. This has made all the x components to be fixed.

There are slack entries in $R_{A.b}$ for Infinitely Many Solutions. The first row has three possible solutions to achieve equality, is either $x_1 = -3$, $x_3 = 0$ or $x_4 = 0$. We choose x_3 component, where $x_3 = 0$. The second row fixes x_2, where $x_2 = -3$. The third row equation fixes the x_4 component, where $x_4 = 0$. The component solution to the fourth row equation has already been fixed by the first row.

For the $R_{A.b}$ in No Solution, there are three rows i.e., the first, second and fourth containing zeros (no 1's). This does not satisfy the condition for a solution to exist. Therefore the system of equations has no solution.

We also applied this discrepancy method to a system of $m \times n$ equations where $m < n$. Examples of such a system that we considered were 3-by-4 and 4-by-6 systems of equations.

Example 4.1.7 *Max-plus system with infinitely Many solutions*

To solve $Ax = b$, *where* $A = \begin{bmatrix} 0 & -1 & 1 & 1 \\ 1 & 2 & 2 & -1 \\ 1 & 3 & 1 & 0 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, *and* $b = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$

The discrepancy matrix: $D_{A.b} = \begin{bmatrix} 2 & 3 & 1 & 1 \\ 2 & 1 & 1 & 4 \\ 1 & -1 & 1 & 2 \end{bmatrix}$ which gives the solution of $x' = (1, -1, 1, 1)^T$.

X' is therefore a solution to the given matrix equation.

$$\begin{bmatrix} 0 & -1 & 1 & 1 \\ 1 & 2 & 2 & -1 \\ 1 & 3 & 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} max(1, -2, 2, 2) \\ max(2, 1, 3, 0) \\ max(2, 2, 2, 1) \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$$

It can be seen that there are other solutions that also work. Any x of the form $\{x : x = (u, v, 1, w)^T$, where $u \leq 1$, $v \leq -1$, and $w \leq 1\}$ is also a solution to the given matrix equation.

Example 4.1.8 *Max-plus system with no solution*

To solve $Ax = b$*, where* $A = \begin{bmatrix} -3 & 2 & 1 & 4 \\ 0 & 5 & 3 & -1 \\ -1 & 6 & 0 & 8 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, *and* $b = \begin{bmatrix} 10 \\ 14 \\ 9 \end{bmatrix}$

The discrepancy matrix: $D_{A.b} = \begin{bmatrix} 13 & 8 & 9 & 6 \\ 14 & 9 & 11 & 15 \\ 10 & 3 & 9 & 1 \end{bmatrix}$ which gives the solution of

$x' = (10, 3, 9, 1)^T$. x' is verified to show that it is not a solution

$$\begin{bmatrix} -3 & 2 & 1 & 4 \\ 0 & 5 & 3 & -1 \\ -1 & 6 & 0 & 8 \end{bmatrix} \otimes \begin{bmatrix} 10 \\ 3 \\ 9 \\ 1 \end{bmatrix} = \begin{bmatrix} max(7,5,10,5) \\ max(10,8,12,0) \\ max(9,9,9,9) \end{bmatrix} = \begin{bmatrix} 10 \\ \underline{12} \\ 9 \end{bmatrix} \neq b = \begin{bmatrix} 10 \\ 14 \\ 9 \end{bmatrix}$$

The underlined entry does not correspond the entry of b. A solution x must satisfy $x_1 \leq 10$, $x_2 \leq 3$, $x_3 \leq 9$, and $x_4 \leq 1$ since the components of x' are the upper bounds. From the second row $max(x_1 + 0, x_2 + 5, x_3 + 3, x_4 - 1) \leq 12 < 14$. This makes the matrix equation to have no solution.

The table below shows the various examples and their $D_{A.b}$ and $R_{A.b}$.

Example	$D_{A.b}$	$R_{A.b}$
Infinitely Many Solutions	$\begin{bmatrix} 2 & 3 & \underline{1} & \underline{1} \\ 2 & 1 & \underline{1} & 4 \\ \underline{1} & \underline{-1} & \underline{1} & 2 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & \underline{1} & \underline{1} \\ 0 & 0 & \underline{1} & 0 \\ \underline{1} & \underline{1} & \underline{1} & 0 \end{bmatrix}$
No Solution	$\begin{bmatrix} 13 & 8 & \underline{9} & 6 \\ 14 & 9 & 11 & 15 \\ \underline{10} & \underline{3} & \underline{9} & \underline{1} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & \underline{1} & 0 \\ 0 & 0 & 0 & 0 \\ \underline{1} & \underline{1} & \underline{1} & \underline{1} \end{bmatrix}$

From the $R_{A.b}$ in the Infinitely Many Solutions, there are slack entries. In the first row equation, there are two possible ways for equality to be achieved, is either $x_3 = 1$ or $x_4 = 1$. We fix the x_3 component, where $x_3 = 1$ for the first row. The second row has already been fixed by the first row. The third row also has either $x_1 = 1$, $x_2 = -1$ or $x_3 = 1$ for equality to be achieved. But x_3 which is 1 has already been fixed.

To consider $R_{A.b}$ in No Solution, there is the second row which is having all zeros (no 1's). Therefore the system of equations has no solution.

Example 4.1.9 *Max-plus system with infinitely many solutions for a 4-by-6 equations.*

$To\ solve\ Ax = b,\ where\ A = \begin{bmatrix} 1 & 6 & 3 & 1 & -2 & 0 \\ -1 & -1 & -3 & 1 & 2 & 3 \\ -9 & -5 & 2 & 2 & -8 & -7 \\ 4 & 1 & -2 & 1 & 3 & 2 \end{bmatrix},\ X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix},\ and$

$$b = \begin{bmatrix} 0 \\ -2 \\ -11 \\ 1 \end{bmatrix}$$

The discrepancy matrix: $D_{A.b} = \begin{bmatrix} -1 & -6 & -3 & -1 & 2 & 0 \\ -1 & -1 & 1 & -3 & -4 & -5 \\ -2 & -6 & -13 & -13 & -3 & -4 \\ -3 & 0 & 3 & 0 & -2 & -1 \end{bmatrix}$

which gives the solution of $x' = (-3, -6, -13, -13, -4, -5)^T$.

X' is therefore a solution to the given matrix equation.

$$\begin{bmatrix} 1 & 6 & 3 & 1 & -2 & 0 \\ -1 & -1 & -3 & 1 & 2 & 3 \\ -9 & -5 & 2 & 2 & -8 & -7 \\ 4 & 1 & -2 & 1 & 3 & 2 \end{bmatrix} \otimes \begin{bmatrix} -3 \\ -6 \\ -13 \\ -13 \\ -4 \\ -5 \end{bmatrix} = \begin{bmatrix} max(-2, 0, -10, -12, -6, -5) \\ max(-4, -7, -16, -12, -2, -2) \\ max(-12, -11, -11, -11, -12, -12) \\ max(1, -5, -15, -12, -1, -3) \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ -11 \\ 1 \end{bmatrix}$$

It can be seen that there are other solutions that also work. Any x of the form $\{x : x = (-3, -6, p, q, r, s)^T$, where $p \leq -13$, $q \leq -13$, $r \leq -4$, and $s \leq -5\}$ is also a solution to the given matrix equation.

Example 4.1.10 *Max-plus system with no solution for a 4-by-6 equations.*

To solve $Ax = b$, *where* $A = \begin{bmatrix} 1 & 1 & -2 & -1 & 3 & 1 \\ 1 & -1 & 1 & 2 & 0 & 4 \\ 2 & 1 & -1 & -1 & 4 & 6 \\ 1 & -1 & 2 & 1 & 1 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$, *and* $b = \begin{bmatrix} 4 \\ 3 \\ 7 \\ 2 \end{bmatrix}$

The discrepancy matrix: $D_{A.b} = \begin{bmatrix} 3 & 3 & 6 & 5 & 1 & 3 \\ 2 & 4 & 2 & 1 & 3 & -1 \\ 5 & 6 & 8 & 8 & 3 & 1 \\ 1 & 3 & 0 & 1 & 1 & 0 \end{bmatrix}$ which gives the solution of $x' = (1, 3, 0, 1, 1, -1)^T$.

X' is therefore a solution to the given matrix equation.

$$\begin{bmatrix} 1 & 1 & -2 & -1 & 3 & 1 \\ 1 & -1 & 1 & 2 & 0 & 4 \\ 2 & 1 & -1 & -1 & 4 & 6 \\ 1 & -1 & 2 & 1 & 1 & 2 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} max(2,4,-2,0,4,0) \\ max(2,2,1,3,1,3) \\ max(3,4,-1,0,5,5) \\ max(2,2,2,2,2,1) \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ \underline{5} \\ 2 \end{bmatrix} \neq b = \begin{bmatrix} 4 \\ 3 \\ 7 \\ 2 \end{bmatrix}$$

The underlined entry does not correspond the entry of b. A solution x must satisfy $x_1 \leq 1$, $x_2 \leq 3$, $x_3 \leq 0$, $x_4 \leq 1$, $x_5 \leq 1$ and $x_6 \leq -1$ since the components of x' are the upper bounds. From the third row $max(x_1 + 2, x_2 + 1, x_3 - 1, x_4 - 1, x_5 + 4, x_6 + 6) \leq 5 < 7$. This makes the matrix equation to have no solution.

The table below shows the various examples and their $D_{A.b}$ and $R_{A.b}$.

Example	$D_{A.b}$	$R_{A.b}$
Infinitely Many Solutions	$\begin{bmatrix} -1 & \underline{-6} & -3 & -1 & 2 & 0 \\ -1 & -1 & 1 & -3 & \underline{-4} & \underline{-5} \\ -2 & \underline{-6} & \underline{-13} & \underline{-13} & -3 & -4 \\ \underline{-3} & 0 & 3 & 0 & -2 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$
No Solution	$\begin{bmatrix} 3 & \underline{3} & 6 & 5 & \underline{1} & 3 \\ 2 & 4 & 2 & \underline{1} & 3 & \underline{-1} \\ 5 & 6 & 8 & 8 & 3 & 1 \\ \underline{1} & \underline{3} & \underline{0} & \underline{1} & \underline{1} & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$

From the $R_{A.b}$ for Infinitely Many Solutions, there are slack entries. In the first row we fix the x_2 component, where $x_2 = -6$. There are two possible ways to achieve equality in the second row, is either $x_5 = -4$ or $x_6 = -5$. We fix the x_5 component, where $x_5 = -4$ for the second row. In the third row equation, there are three possible ways to achieve equality, is either $x_2 = -6$, $x_3 = -13$, or $x_4 = -13$. But third row has already been fixed by the first row. The fourth row fixes the x_1 component, where $x_1 = -3$

$R_{A.b}$ for No Solution in the table above, there is no one(1) in the third row (i.e. all zeros). The system of equations has therefore no solution.

It is seen from the various examples in max-plus algebra that $n \times n$ systems of linear equations have three nature of solutions. They are either a unique solution, infinitely many solutions or no solution. Examples for $m \times n$ systems in max-plus also have two nature of solutions. It is either infinitely many solutions or no solution.

The same applies to systems of linear equations in conventional algebra.

4.2 Synchronised Event Problem

Synchronised Event Problem is a problem in which an event is schedule to meet a deadline. There are two aspects of this problem:

1. the events run simultaneously
2. the completion of the longest event must occur exactly at the deadline

These are events that often occur with a very time-sensitive deadlines. Examples for such events are the preparation of a plane for a set takeoff time, the coordination of system checks for a Space Shuttle Launch, the preparation of an athlete before an Olympic Event, or the preparation of a shop before sales.

Four shops which are within the same market but are located at some metres from each other were studied. The four shops find out that customers start buying at 7:00am. They all decided to open their shops for customers at exactly 7:00am. Since the shops want to meet that deadline, the Sale Representatives (reps) for each product for each shop are to start restocking before the set time. This will enable the shops to serve their customers on time and other consumers to make more profit because of the competitions. The four shops A, B, C, and D sell four mineral water products Voltic (V), Special Ice (S), Acquafresh (A), and Mobile (M). The shops work six days within the week, that is from Monday to Saturday. For the shops to avoid losses, each product has one Representative. The time available to the Sale Representatives to restock the shops depend on when the shop are opened to them before the set time 7am. The time available for reps and the time each rep spent on each product were taken on each of the 6days for each of the four shops. The average time of the 6days for each shop was taken.

Suppose we only coordinate the events of a single deadline, then we can find the latest start times by taking the difference of the finish time and individual event duration times. If we are to take shop A for example, when the shop is opened to the reps for V, S, A, and M. The time each rep took was 31 min, 32 min, 33 min, and 32 min respectively, where they were to finish within 48 min. Taking the difference shows that the latest starting time for each event is 17 min, 16 min, 15 min, and 16 min respectively. After considering the events of all the four shops, we will get a multiple deadlines. When we consider the case where we have four shops, each shop will have different time available to the reps for their respective products. This will depend on the size of the shop, quantity of products available to the reps to restock, time the reps report at work, and also the time the shops are opened to the reps to start restock. Below is the table for the various shops.

Table 4.1: SHOP A

	Voltic	*Special Ice*	*Aquafresh*	*Mobile*	*Time Available*
Mon	25	30	20	25	40
Tue	36	40	35	30	45
Wed	15	25	30	28	50
Thur	32	28	40	34	50
Fri	45	40	35	40	55
Sat	35	30	40	35	50
Total	188	193	200	192	290
Average Time	31	32	33	32	48

Table 4.2: SHOP B

	Voltic	*Special Ice*	*Aquafresh*	*Mobile*	*Time Available*
Mon	35	20	40	35	50
Tue	40	30	32	30	45
Wed	25	35	30	38	40
Thur	30	40	35	25	50
Fri	28	45	34	40	55
Sat	35	25	45	30	60
Total	193	195	216	198	300
Average Time	32	33	36	33	50

Table 4.3: SHOP C

	Voltic	*Special Ice*	*Aquafresh*	*Mobile*	*Time Available*
Mon	45	30	50	35	55
Tue	30	40	35	25	50
Wed	32	35	45	50	60
Thur	25	38	42	40	55
Fri	35	34	32	38	45
Sat	50	45	38	30	50
Total	217	222	242	218	315
Average Time	36	37	40	36	53

Table 4.4: SHOP D

	Voltic	*Special Ice*	*Aquafresh*	*Mobile*	*Time Available*
Mon	30	25	20	35	45
Tue	20	32	15	30	40
Wed	15	30	34	25	50
Thur	25	35	25	20	45
Fri	28	32	35	40	55
Sat	35	38	40	35	50
Total	153	192	169	185	285
Average Time	26	32	28	31	48

Below is a matrix to show the preparation before the shops are opened to customers (event times are in minutes).

$$\begin{array}{cc} & \begin{array}{cccc} V & S & A & M \end{array} \\ \begin{array}{c} \text{Shop A} \\ \text{Shop B} \\ \text{Shop C} \\ \text{Shop D} \end{array} & \begin{pmatrix} 31 & 32 & 33 & 32 \\ 32 & 33 & 36 & 33 \\ 36 & 37 & 40 & 36 \\ 26 & 32 & 28 & 31 \end{pmatrix} \end{array}$$

and the corresponding vector of the time available to the shops is $\begin{pmatrix} 48 \\ 50 \\ 53 \\ 48 \end{pmatrix}$

Example 4.2.1 *We want to find the latest starting times for the various products Voltic, Special Ice, Aquafresh, and Mobile where the events are completed at 7am when the Shops are opened to customers. The problem is formulated as a max-plus* *matrix equation, where we solve for t:*

$$\begin{bmatrix} 31 & 32 & 33 & 32 \\ 32 & 33 & 36 & 33 \\ 36 & 37 & 40 & 36 \\ 26 & 32 & 28 & 31 \end{bmatrix} \otimes \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} = \begin{bmatrix} 48 \\ 50 \\ 53 \\ 48 \end{bmatrix}$$

The Discrepancy matrix, D_s is calculated:

$$D_s = \begin{bmatrix} 17 & 16 & 15 & 16 \\ 18 & 17 & 14 & 17 \\ 17 & 16 & 13 & 17 \\ 22 & 16 & 20 & 17 \end{bmatrix}$$

The candidate solution t' is calculated:

$$t' = \begin{bmatrix} 17 \\ 16 \\ 13 \\ 16 \end{bmatrix}$$

The Reduced discrepancy matrix, R_s is also calculated:

$$R_s = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

The R_s shows that there is no solution to the problem because of the all-zero of the second row. We can indeed verify that t' is not a solution:

$$\begin{bmatrix} 31 & 32 & 33 & 32 \\ 32 & 33 & 36 & 33 \\ 36 & 37 & 40 & 36 \\ 26 & 32 & 28 & 31 \end{bmatrix} \otimes \begin{bmatrix} 17 \\ 16 \\ 13 \\ 16 \end{bmatrix} = \begin{bmatrix} 48 \\ \underline{49} \\ 53 \\ 48 \end{bmatrix}$$

The solution failed because of the underlined entry. As t' is not a strict solution to the system of equations, it does not result in a delay of the deadline of 7am. This shows that rep at Shop B finished their work earlier than expected.

When a candidate solution is not a strict solution to a system of equations, but it does not result in a delay of any of the deadline it is refer to as a non-ideal solution.

These are the daily data taken for the various Shops for the second week:

Table 4.5: SHOP A

	Voltic	*Special Ice*	*Aquafresh*	*Mobile*	*Time Available*
Mon	15	25	20	35	45
Tue	30	34	42	28	50
Wed	28	36	25	40	42
Thur	35	40	42	38	55
Fri	20	30	15	18	45
Sat	40	35	28	25	48
Total	168	200	172	184	285
Average Time	28	33	29	31	48

Table 4.6: SHOP B

	Voltic	*Special Ice*	*Aquafresh*	*Mobile*	*Time Available*
Mon	45	30	38	40	50
Tue	28	35	40	30	45
Wed	30	20	35	28	40
Thur	20	30	30	25	45
Fri	40	36	25	30	55
Sat	35	25	30	34	40
Total	198	176	198	187	275
Average Time	33	29	33	31	46

Table 4.7: SHOP C

	Voltic	*Special Ice*	*Aquafresh*	*Mobile*	*Time Available*
Mon	35	25	30	24	40
Tue	40	30	28	35	50
Wed	25	32	36	40	55
Thur	30	35	15	25	45
Fri	20	18	30	34	40
Sat	15	30	20	25	45
Total	165	170	159	183	275
Average Time	26	28	27	31	46

Table 4.8: SHOP D

	Voltic	*Special Ice*	*Aquafresh*	*Mobile*	*Time Available*
Mon	20	32	38	35	45
Tue	25	35	30	22	50
Wed	38	20	35	25	48
Thur	22	34	28	36	40
Fri	35	30	20	25	45
Sat	15	22	26	30	35
Total	155	173	177	173	263
Average Time	26	29	30	29	44

Below is a matrix formed from the averages to show the preparation before a shop is opened to customers and the vector of the time available to the shops:

$$\begin{array}{ll} & \begin{array}{cccc} V & S & A & M \end{array} \\ \begin{array}{l} \text{Shop A} \\ \text{Shop B} \\ \text{Shop C} \\ \text{Shop D} \end{array} & \begin{pmatrix} 28 & 33 & 29 & 31 \\ 33 & 29 & 33 & 31 \\ 26 & 28 & 27 & 31 \\ 26 & 29 & 30 & 29 \end{pmatrix} \end{array}$$

and the corresponding vector of the time available to the shops is

$$\begin{pmatrix} 48 \\ 46 \\ 46 \\ 44 \end{pmatrix}$$

Example 4.2.2 *At the end of the second week, these were the systems of equations formed from the averages of the daily data collected. We have also calculated for the discrepancy matrix, reduced discrepancy matrix, and the candidate solu-*

tion:

$$\begin{bmatrix} 28 & 33 & 29 & 31 \\ 33 & 29 & 33 & 31 \\ 26 & 28 & 27 & 31 \\ 26 & 29 & 30 & 29 \end{bmatrix} \otimes \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} = \begin{bmatrix} 48 \\ 46 \\ 46 \\ 44 \end{bmatrix}$$

$$D_s = \begin{bmatrix} 20 & 15 & 19 & 17 \\ 13 & 17 & 13 & 15 \\ 20 & 16 & 19 & 15 \\ 18 & 15 & 14 & 15 \end{bmatrix}, R_s = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \text{ and } t' = \begin{bmatrix} 13 \\ 15 \\ 13 \\ 15 \end{bmatrix}$$

The reduced discrepancy matrix shows that the candidate solution, t', is a solution to the system of equations.

The times available to Voltic and Aquafresh reps could be earlier without affecting the strict solution. We can also tell from the second and fourth row in R_s that the preparation for the products except Special Ice in Shop B ended simultaneously. Also the preparation for Special Ice and Mobile Water in Shop D ended simultaneously. This is because of the presence of many ones in the second and fourth rows.

4.3 Examples on a system of Max-linear Program

Example 4.3.1 *Given a system of max-linear program in which* $f = (9, 5, 2, 7)^T$ *for*

$$A \otimes x = b$$

$$C \otimes x \leq d$$

where

$$A = \begin{pmatrix} 0 & -1 & 1 & 1 \\ 1 & 2 & 2 & -1 \\ 1 & 3 & 1 & 0 \end{pmatrix}, b = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$$

$$C = \begin{pmatrix} -3 & 2 & 1 & 4 \\ 0 & 5 & 3 & -1 \\ -1 & 6 & 0 & 8 \end{pmatrix}, d = \begin{pmatrix} 10 \\ 14 \\ 9 \end{pmatrix}$$

This is the solution:

$$D_{A.b} = \begin{pmatrix} 2 & 3 & 1 & 1 \\ 2 & 1 & 1 & 4 \\ 1 & -1 & 1 & 2 \end{pmatrix}$$

$$\bar{x}(A,b) = \begin{pmatrix} min & (2,2,1) \\ min & (3,1,-1) \\ min & (1,1,1) \\ min & (1,4,2) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

$$D_{C.d} = \begin{pmatrix} 13 & 8 & 9 & 6 \\ 14 & 9 & 11 & 15 \\ 10 & 3 & 9 & 1 \end{pmatrix}$$

$$\bar{x}(C,d) = \begin{pmatrix} min & (13,14,10) \\ min & (8,9,3) \\ min & (9,11,9) \\ min & (6,15,1) \end{pmatrix} = \begin{pmatrix} 10 \\ 3 \\ 9 \\ 1 \end{pmatrix}$$

Compare $\bar{x}(A,b)$ and $\bar{x}(C,d)$, and pick the least corresponding elements to form $\hat{x}(A,C,b,d)$

$$\hat{x}(A,C,b,d) = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

We let

$$x = \hat{x}(A,C,b,d) = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

Compare the corresponding elements of $\bar{x}(A,b)$ and $\bar{x}(C,d)$ that satisfy $\bar{x}_j(C,d) \geq \bar{x}_j(A,b)$ and pick their positions (in the row), making $J = \{1,2,3,4\}$

From K_j, where $j \in J$

$K_1 = \{3\}$,

$K_2 = \{3\}$,

$K_3 = \{1,2,3\}$, and

$K_4 = \{1\}$

$$\begin{aligned} f(x) &= f_j + x_j \\ &= ((9+1),(5+-1),(2+1),(7+1)) \\ &= (10,4,3,8)^T \end{aligned}$$

$$H(x) = \{1\}$$

$$J : J \setminus H(x) = \{2,3,4\}$$

$$K = \{1,2,3\}$$

$$K_2 \cup K_3 \cup K_4 = \{1,2,3\} = K$$

set $x_1 = 10^{-5}$ (say), we get a new $x = (10^{-5}, -1, 1, 1)^T$

Going for a new $H(x)$

$$\begin{aligned} f(x) &= f_j + x_j \\ &= ((9 + 10^{-5}), (5 + -1), (2 + 1), (7 + 1)) \\ &= (9.00001, 4, 3, 8)^T \end{aligned}$$

$$H(x) = \{4\}$$

$$J : J \setminus H(x) = \{2, 3\}$$

$$K_2 \cup K_3 = \{1, 2, 3\} = K$$

set $x_4 = 10^{-5}$ (say), we get a new $x = (10^{-5}, -1, 1, 10^{-5})^T$

Going for a new $H(x)$

$$\begin{aligned} f(x) &= f_j + x_j \\ &= ((9 + 10^{-5}), (5 + 10^{-5}), (2 + 1), (7 + 10^{-5})) \\ &= (9.00001, 5.00001, 3, 7.00001)^T \end{aligned}$$

$$H(x) = \{3\}$$

$$J : J \setminus H(x) = \{2\}$$

$$K_2 \neq K$$

We stop, the optimal solution is $x = (10^{-5}, 10^{-5}, 1, 10^{-5})^T$

$f^{min} = minf(x) = min(9.00001, 5.00001, 3, 7.00001)^T$

Therefore $f^{min} = 3$

We went further to solved another example:

Example 4.3.2 *Given a system of max-linear program in which* $f = (5, 6, 1, 4, -1)^T$

for

$$A \otimes x = b$$

$$C \otimes x \leq d$$

where

$$A = \begin{pmatrix} 3 & 8 & 4 & 0 & 1 \\ 0 & 6 & 2 & 2 & 1 \\ 0 & 1 & -2 & 4 & 8 \end{pmatrix}, b = \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix}$$

$$C = \begin{pmatrix} -1 & 2 & -3 & 0 & 6 \\ 3 & 4 & -2 & 2 & 1 \\ 1 & 3 & -2 & 3 & 4 \end{pmatrix}, d = \begin{pmatrix} 5 \\ 5 \\ 6 \end{pmatrix}$$

This is the solution:

$$D_{A.b} = \begin{pmatrix} 4 & -1 & 3 & 7 & 6 \\ 5 & -1 & 3 & 3 & 4 \\ 7 & 6 & 9 & 3 & -1 \end{pmatrix}$$

$$\bar{x}(A,b) = \begin{pmatrix} min & (4,5,7) \\ min & (-1,-1,6) \\ min & (3,3,9) \\ min & (7,3,3) \\ min & (6,4,-1) \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ 3 \\ 3 \\ -1 \end{pmatrix}$$

$$D_{C.d} = \begin{pmatrix} 6 & 3 & 8 & 5 & -1 \\ 2 & 1 & 7 & 3 & 4 \\ 5 & 3 & 8 & 3 & 2 \end{pmatrix}$$

$$\bar{x}(C,d) = \begin{pmatrix} min & (6,2,5) \\ min & (3,1,3) \\ min & (8,7,8) \\ min & (5,3,3) \\ min & (-1,4,2) \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 7 \\ 3 \\ -1 \end{pmatrix}$$

Compare $\bar{x}(A,b)$ and $\bar{x}(C,d)$, and pick the least corresponding elements to form $\hat{x}(A,C,b,d)$

$$\hat{x}(A,C,b,d) = \begin{pmatrix} 2 \\ -1 \\ 3 \\ 3 \\ -1 \end{pmatrix}$$

We let

$$x = \hat{x}(A,C,b,d) = \begin{pmatrix} 2 \\ -1 \\ 3 \\ 3 \\ -1 \end{pmatrix}$$

Compare the corresponding elements of $\bar{x}(A,b)$ and $\bar{x}(C,d)$ that satisfy $\bar{x}_j(C,d) \geq \bar{x}_j(A,b)$ and pick their positions (in the row), making $J = \{2,3,4,5\}$

From K_j, where $j \in J$

$K_2 = \{1,2\}$,

$K_3 = \{1,2\}$,

$K_4 = \{2,3\}$ and

$K_5 = \{3\}$

$$\begin{aligned} f(x) &= f_j + x_j \\ &= ((5+2),(6+-1),(1+3),(4+3)),(-1+-1) \\ &= (7,5,4,7,-2)^T \end{aligned}$$

$$H(x) = \{1,4\}$$

$$J : J \setminus H(x) = \{2,3,5\}$$

$$K = \{1,2,3\}$$

$$K_2 \cup K_3 \cup K_5 = \{1,2,3\} = K$$

set $x_1 = x_4 = 10^{-4}$ (say), we get a new $x = (10^{-4}, -1, 3, 10^{-4}, -1)^T$

Going for a new $H(x)$

$$\begin{aligned} f(x) &= f_j + x_j \\ &= ((5+10^{-4}),(6+-1),(1+3),(4+10^{-4})),(-1+-1) \\ &= (5.0001,5,4,4.0001,-2)^T \end{aligned}$$

$$H(x) = \{2\}$$

$$J : J \setminus H(x) = \{3,5\}$$

$$K_3 \cup K_5 = \{1,2,3\} = K$$

set $x_2 = 10^{-4}$ (say), we get a new $x = (10^{-4}, 10^{-4}, 3, 10^{-4}, -1)^T$

Going for a new $H(x)$

$$\begin{aligned} f(x) &= f_j + x_j \\ &= ((5+10^{-4}),(6+10^{-4}),(1+3),(4+10^{-4})),(-1+-1) \\ &= (5.0001,6.0001,4,4.0001,-2)^T \end{aligned}$$

$$H(x) = \{3\}$$

$$J : J \setminus H(x) = \{5\}$$

$$K_3 \cup K_5 = \{1, 2, 3\} = K$$

since $K_5 = \{3\} \neq K$, we stop and the optimal solution is $x = (10^{-4}, 10^{-4}, 3, 10^{-4}, -1)^T$
$f^{min} = minf(x) = min(5.0001, 6.0001, 4, 4.0001, -2)^T$
Therefore $f^{min} = 4$.

Chapter 5

Conclusion

We have seen from our work that the discrepancy method of max-plus can be used to solve an $n \times n$ and $m \times n$ system of linear equations and a real-life problem in a synchronised event.

It is also interesting to note that an $n \times n$ system of linear equations has either a unique solution, an infinitely many solutions or no solution whiles $m \times n$ system of linear equations (where $m < n$) has either an infinitely many solutions or no solution in $(\mathbb{R}_{max}, \oplus, \otimes)$.

Linear programming problem involving linear equations and inequalities has also been solved in max-plus.

From the applications, we have seen that max-plus provide interesting tools that can be used to formulate and solve many problems of optimization.

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Appendix A

QUESTIONNAIRE FOR SALE REPRESENTATIVES OF A SHOP

Dear sale representatives, this questionnaire is designed to take the time various representative spend on restocking before sales begin and also the time available for restocking.

Your response will be treated confidential and used for academic purposes. I am a Master of Philosophy student of Kwame Nkrumah University of Science and Technology.

Please fill in the spaces provided below.

Table 5.1: NAME OF THE SHOP ....

	Voltic	*Special Ice*	*Aquafresh*	*Mobile*	*Time Available*
Mon					
Tue					
Wed					
Thur					
Fri					
Sat					
Total					
Average Time					