KWAME NKRUMAH UNIVERSITY OF SCIENCE AND TECHNOLOGY, KUMASI

NUMERICAL VALUATION OF SURRENDER OPTIONS

BY

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DECLARATION

I hereby declare that this submission is my own work towards the award of the M. Phil degree and that, to the best of my knowledge, it contains no material previously published by another person nor material which had been accepted for the award of any other degree of the university, except where due acknowledgment had been made in the text.

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DEDICATION

I dedicate my work firstly to the Almighty God who helped me to come all this far. Had it not been His mercies and kindness, I would not have gotten to this stage in life. Secondly to my two elder sisters and brother for their support and kindness and words of encouragement which motivated me to reach this far in my academics.

ABSTRACT

Embedded in Life insurance contracts are surrender options and also path dependency. Surrender option stems from many reasons. Multi morbidity increases the rate of mortality and a variety of adverse health outcomes which may lead to surrendering. In Ghana, poverty levels coupled with social burdens can inform a multi-morbid person to surrender a life policy contract. The study seeks to incorporate the multi-morbid survival rate of a policy holder in the Black-Scholes model for option pricing. The solution to this model come along with its own complexities. Therefore the need to resort to numerical solutions for the option valuation. Further, a comparison is made of two finite difference algorithms in solving the proposed Black-Scholes equation ;the Crank-Nicolson method and the Implicit method. In line with these objectives, simulations of survival times were performed to compute the survival rate and the stability, consistency and convergence of these algorithms were investigated. It was observed that the algorithms were stable, consistent and converges to the exact solution. However the Explicit method of the finite difference approximation is found to be conditionally stable. Numerical solution to the Black -Scholes model and the proposed model indicates that the Crank-Nicolson method converges faster than the Implicit method for the Black-Scholes while the Implicit method converges faster than the CrankNicolson method. Finally it is observed that the Implicit method converges faster as the multi-morbid survival rate decreases below the short rate of the Black-Scholes model.

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CONTENTS

LIST OF ABBREVIATION

LICs ....Life Insurance Contracts AIS ....International Accounting Standards

PDE .... Partial Differential Equations IRDA .... Insurance Regulatory and Development Authority

ALM ....Asset Liability Management GMB

.... Geometric Brownian Motion

SDE ....Stochastic Differential Equation

ODE ....Ordinary Differential Equation

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CHAPTER 1

INTRODUCTION

1.1 Background of the Study

In this study, one seeks to incorporating the multi-morbid survival rate of a policy holder in the Black Scholes model for option pricing. Pricing actuarial and life contingent insurance reserves comprises of the computation of statistics regarding the occurrences and value of prospective cash flows. For instance, the premium of an insurance policy is regarded as the expected amount of the future cash flow distribution which is computed at t=0 given the structure of the interest rate. The probabilities of the prospective benefits cash flow depend on the occurrences of the policy holder's life events(life contingencies). That is, being multi morbid or co-morbid person as years go by. The present value of the future cash flows are computed using the theory of interest. Hence demography and theory of interest are the two main concepts used in life insurance mathematics.

1.1.1 Types of Insurance Policy

Whole life Insurance

This is a type of insurance policy which is for the policyholder's whole life and it requires premium every year.Is a life insurance policy which is guaranteed to remain in force for the insured's entire lifetime. Premiums are fixed, based on the age issue, and usually do not increase with age. The insured party normally pays premiums until death, except for limited pay policies which may be paid up in 10 years, 20 years, or at age 65. Whole life insurance belongs to the cash value category of life insurance, which also includes universal life, variable life, and endowment policies.

Single Premium

This is a form of insurance contract which requires a large premium upfront. This type of policies always have fees in the beginning of the policy years if the insured cash it in.

Interest Sensitive

This is a type of insurance policy where there is either current assumption whole life or excess interest. The contracts are combination of both universal and whole life insurance. With whole life, benefits to be given during death remain the same for life, for universal life the payment of premium could change, but not exceeding the maximum of the premium guaranteed within the policy.

Life Insurance

Life insurance policy is a contract between the policyholder and the insurer. The policyholder pays premium to the insurer and in return the insurer pays benefit to the policyholder during death or when the policyholder survives the maturity date. Other events such as chronic illness and severe illness could also trigger payment(surrendering). Depending on the type of policy, policyholders are required to pay a regular premium or a lump sum premium . Life insurance contracts are legal contracts and the limitations of the insured events are based on the terms and conditions of the contract. The liabilities of the insurer are limited by writing specific exclusion into the contact; examples are catastrophic events. Life based contracts tend to fall into two major categories;

- Protection policies - Designed to provide a benefit in the event of specific event, typically a lump sum payment. A common form of this design is term insurance.
- Investment policies - Where the main objective is to facilitate the growth of capital by regular or single premiums. Common forms are whole life, universal life and variable life policies.

Life Insurance Liabilities

The correct assessment of convexity and duration of the equity and liabilities measure is crucial because they consist of the primary features of any sound assetliability management approach.. , Policyholders are the first claim when it comes to the company's asset. The holders of equity have limited liability; interest rate guarantees are common elements of LICs; and LICs according to the so called contribution principal (which states that if a risk is insured by multiple carriers, and one carrier has paid out a claim, that is entitled to collect proportionate coverage from other carriers) are entitled to received a fair share of any investment surplus. Risk-taking initially occurs on the liability side of the balance sheet. Underwriters issue insurance policies which are transformed into liabilities. Because of the time indemnity outflow,reserves are always invested on the financial market place and the portfolio of the company's assets is generated.

1.1.2 Co-morbidity and Multi-morbidity

These terms are often used by healthcare professionals in clinical practice and in health policy documents. Used in medical settings, morbidity means illness or disease and is not to be confused with mortality, which means death, and is frequently used in statistical reports. Co-morbidity simply means more than one illness or disease occurring in the same person at the same time and multi-morbidity means more than two illnesses or diseases occurring in the same person at the same time. Due to an ageing population and improved detection and treatment of disease, many older people now have more than one illness. Common co-morbid conditions in older people include heart disease, hypertension, respiratory disease, mental health problems(including dementia), cerebrovascular disease, joint disease, diabetes and sensory impairment. Alongside co-morbidity and multi-morbidity comes poly pharmacy, or the prescription of many medications.

1.1.3 Multi-morbidity and Life insurance

One of the biggest myths that aggressive life insurance agents perpetuate is that, "insurance is harder to qualify for as you age, so you better get it while you are young". To put it bluntly, insurance companies make money by betting on how long you will live. When you are young, your premiums will be relatively cheap. If you die suddenly and the company has to pay out, you were a bad bet. Fortunately, many young people survive to old age, paying higher and higher premiums as they age(the increase risk of dying due to multi-morbidity makes the odd less attractive). On the other hand, someone can purchase life insurance policy with a good health records but may be multi-morbid as age goes by when the policy is still in place, such insured may not allow the contact to mature and would like to surrender the contract or policy(that is, by selling it back to the insurer at a surrender value).

Since most insurance contracts in Ghana contains surrender options, which is an American style of put option that gives the holder the right to sell the contract to the issuer at a surrender value.

1.1.4 Types of Life Insurance Contracts

Insurance contracts come in different forms. The popular among them are the *European-style and the American style* contracts. Options of these types and all other types where the pay off has similar calculations are called *vanilla options*. Similarly options with different methods of calculating pay off are grouped as *exotic options*. Exotic options have sophisticated methods in terms of valuation and hedging.

The exercise style of an option governs the time at which exercise can occur.

European-style option pays off simply the future benefit at the expiration date. The price the buyer pays for the asset when the option or contract is exercised is called the "exercise price or strike price". The date or the last day by which the option must either be exercise or it becomes worthless is called the "expiration date or maturity date".

American-style option pays off simply the future benefit at any time before the expiration date. Since the American-style option provides an investor with a greater degree of flexibility than the European style option, the premium for the American style option is at least equal to or higher than the premium for an European-style option which otherwise has all the same features. For both, the payoff - when it occurs is given as:

$max[(K - S),0]$, for a put option. Where K is the *strike price* (The price the buyer pays for the asset when the option or contract is exercised for the underlying commodity or asset, that is, in the case of a put option.) and S is the *spot price* of the underlying asset.

1.1.5 Numerical Methods

Valuation of life insurance contracts can pose sophisticated mathematical problems and consist of path dependence derivative and in most cases analytical approach to the valuation problems do not exist. Therefore the need for numerical approach is highly recommended in valuing the contract. Among the numerical approaches for solving such problems is the Monte Carlo Simulations which is used for the valuation of the insurance contract provided that the policy holders cannot change or partially surrender the contract during its term - European contracts.

Option pricing is a critical issue in financial institutions. The appropriate tool for pricing various types of option is the Black-Schole model. The model gives a direct calculation which requires high effort of computations.

1.2 Statement of the Problem

A surrender option is an American-style put option that entitles its owner (the *policyholder*) to sell back the contract to the issuer (the *insurer*) at (the *surrender value*). The fair valuation of such an option, as well as an accurate assessment of the surrender values, are clearly crucial topics in the management of a life insurance company, both on the solvency and on the competitiveness side. According to (Broadie and Detemple

(1996)), life insurance contracts and pension plans are complex financial securities that come in many variations. This is due to the surrender options which are embedded in life insurance policies and they are also path dependent. Contract with guaranteed rate of returns every year before maturity are common throughout Ghana and other developing countries. The contracts in Ghana are sometimes equipped to terminate the contract prior to maturity.

Existing models do not take into account the likelihood that someone surrendering is multi morbid. This paper seeks to address the issue concerning the fact that policyholders can terminate the contract prior to maturity due to the co-existence of two or more diseases in their lives which calls for some forms of financial burden, and since there is an option for them to surrender prior to maturity then surrendering becomes very paramount. Consequently, what will happen to the value of the insurance portfolio, also the surrender value that is due a policy holder when such an action is taken by him or her.

Analytical solution to the valuation problems cannot be found because of the complexities and the presence of path-dependence derivatives in the life insurance contract. Hence the need to resort to numerical methods in the valuation of insurance contract in Ghana.

1.3 Objectives of the study

The objectives of the study are:

1. To implement a Black-Scholes model that accommodates the survival rate of a non-monotonic hazard function of the policyholder .
2. To determine which of the finite difference methods is appropriate for the valuation of the American style life insurance contact using the Black-Scholes equation embedded in the survival rate.

3. To compare the Crank Nicolson numerical approach, the Implicit numerical approach and the Explicit numerical approach in the numerical valuation of life insurance contracts in Ghana which contains surrender option.

1.4 Methodology

Since path dependence demands derivation of closed-form solution formulas, the problem can be reduced to allow for the development and implementation of a finite difference approximation algorithm for fast and accurate numerical valuation of life insurance contracts. Agents are assumed to operate in a continuous time frictionless economy with a perfect financial market, so that tax effects, transaction costs, divisibility, liquidity, and short sales constraints and other imperfections can be ignored. As regards the specific contracts, we ignore the effects of expense charges and mortality (Dufie (1999)). In the case of this study the effect of mortality is not ignored since the survival rate of the insured influences the value of the insurance portfolio.

1.5 Justification of the Study

The study analyses a participating contract embedding a surrender option which is also known as the so called participating (or with profits) policy in the Black-Scholes framework. The study takes a finite difference algorithm approach to the market valuation of the equity and liabilities in life insurance companies.

1.6 Thesis Organization

This thesis is organized into five main chapters. Chapter 1 presents the introduction of the thesis. This consists of the background of the study, the research problem statement, objectives of the study, methodology, thesis justification and organization. Chapter 2 is the literature review, which looks at briefly work done by other researchers on the topic. Chapter 3 is the formulation of the mathematical model. Chapter 4 contains the data collection and analysis, formulation of model instances, algorithm, computational procedure, results and discussion. Chapter 5 looks at summary, conclusions, and recommendation of the results.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

In this section there is a review of the work of several authors regarding definition, concept of surrender options and various studies done to discover how to value life insurance contracts.

2.2 Background on Life Insurance Contracts

Unit-linked contracts are directly connected to the investment portfolio of the contract. Embedded in these types of contracts are maturity guarantees, which could be explained as option given a right to a fixed minimum amount when the contract is matured. Valuation of these contracts are highlighted, for example in Siu (2005).

Policyholder is given portion of the profit attained by the portfolio linked to the contract. The bonus policy dictates the profit division between the policyholder, reserves among others. Policyholders are also given contracts which may offer them the chance to select between several bonus mechanisms at a point in time of the contract term. Switching from a bonus to other bonuses is an embedded option,which is an opportunity for the policyholder. It is always necessary to enquire about the value of such opportunity. This provision is called a *switch option*. (Tanskanena and Lukkarinenb (2003)). Insurance contracts are grouped into two, European and American which is the same in regular literature. Contracts containing surrender option are contracts in which the policyholder can terminate the contract before maturity. American options are contracts that are with surrender option whereas European options are contracts that are without surrender options..

2.3 Insurance Accounting

Insurance accounting simply shift the traditional way of valuing insurance debt to the fair value of the liabilities. This has being in existence partly because in the balance sheet while the assets are evaluated, the liabilities are not. Jarrow and Turnbull (1996). Interest rate guarantee is a problem for insurance companies and insurance contracts due to low interest rates. Insurance firms and financial institutions like Nissan Mutual Life, have got into problems partly because they have always underestimated the embedded option in their written insurance contracts.Hull (2003). This interest rate issue can be addressed by the upcoming IAS standard for insurance contacts. IAS adopted a radical valuation: the valuation of liabilities are to be marketed in large quantities among independent investors in a liquid market. The process referred to as *fair valuation* and it is defined as ”the value at which an asset is exchanged or the settlement of a liability in an arm’s length transaction between knowledgeable willing people” IASB(2001). With the draft IAS standard, sophisticated future cash flow estimates which are backdated with risk free interest, should be used in the valuation.

International Accounting Standard Board(IASB) is shifting from the use of exact fair valuation. Comprehending fair valuation is relevant, because the actual importance of the evaluation approach, indirect obligation and reliable valuation of embedded options are still in place.

To Comprehend the feasibility of fair valuation, the construction of the model approach for valuing the participation life insurance contracts need to be done by improving the work of Grosen and Jorgensen (2000), and coming out with an analytical method for valuing the contract. The model gives analytical outcome for certain simple bonus mechanism. Iterated integral whose results rely on numerics are obtained from most bonus policies. Designing of the model gives chance for including most types of bonus mechanisms, thereby resulting in a simple and practical method to the comprehending of valuing participating contracts. Briefly we look at how to include a known term riskless interest structure to the model.

2.4 Options

An option is a contract or a financial derivative and trading these contracts is a cost-effective alternative to trading directly on the stock market. In financial engineering, option pricing is a major topic because options are widely used for hedging and speculations. Currently, on the market are different option contracts which includes European option,path-dependence and American option. In brief, all stock options are based on companies stock price movement. Investors can decide whether or not to exercise the contracts any time before the maturity or at the pre specified price. European option holders can exercise the contract at maturity. A path-dependence option is fairly different because its price is dependent on the initial price movement of the trading asset,example Asian options. A good asymmetry exists between obligation and rights of trading parties; the buyer of an option has the right but no obligation to exercise the contract, on the other hand his counter party bears the responsibility but not the right to complete the transaction. Therefore an appropriate price for the option is necessary for the need of both parties (buyer and seller) to be satisfied. .

European options valuation are simple and easy to compute because its exercise limit price is known on the date of maturity. Analytical methods for European options without dividend is derived from the Black and Scholes (1973) and Merton (1973). The American option has a larger executing time, there is an inherent possibly for traders to get higher payoff, that is why it is at the moment the main option contract traded on the most large stock exchange across the globe. Therefore on the market, it is very needed for the seller of the option to demand for a fair option premium so as to make it profitable for the buyer. An investor must be very circumspect in the calculation before exercising the option when the payoff is high. The premium, also known as the option price, is what needs to be carefully modelled for real-world trading, this originally motivated this research.

Further studies is needed to price American option correctly because of its early exercise factor. Under the non-dividend-paying assumption, the American call option can be regarded as a European-style option and can therefore be valued by the standard Black-Scholes formula. The idea is simple; when the option price is greater than the exercise price, the investor gets a premium greater than the intrinsic value by selling the call option instead of getting only the intrinsic value by exercising the option. Therefore, there is no economic benefit to exercise the American call early. Thus, the focus of the research work is narrowed to the pricing of American put options. For these options, early-exercising may lead to a higher payoff, which complicates the pricing formula. The Black-Scholes method must be adapted since the expected payoff depends on the underlying assets price movement. American put option has no fixed exercise boundary thus has no closed form solutions, an appropriate approximation to the option price is essential.

Technology allows for the use of numerical methods in approximating the price. Example finite element method, finite difference method, binomial method and Monte Carlo simulation.

2.5 Factors Affecting Option Pricing

Option prices are influence by various factors. The difference between the underlying stocks current price and the strike price is essential as it determines the final payoff; market volatility has a significant impact on option prices as it reflects the potential risk of quick price changes in the market; the interest rate is the return on riskfree investments. Investors would regard it as the benchmark to decide the level of premiums they are willing to pay for the protection against market risk. In order to find optimal price for options for these risk factors, much studies is needed to come out with a compact formula for the option. Ali (2013). Stock prices are assumed to follow Brownian motion in the last century with a drift term of zero (Bachelier 1900). Bachelier’s ideas however have two major drawbacks ; first prices could be negative based on the assumption that the drift term is zero, secondly the assumption that all investors are risk neutral. Number of academic boards have come out with modification of the model proposed by Bachelier by stating that stock returns rather follows a Brownian motion and not the stock price itself (Sprenkle ,1961). This assertion was also supported later by Bones (1964) and Samuelson (1965). Tolerance level of risk the returns of expected rate were incorporated in the formula. The model did not define the parameters used, hence making the practical use of the model irrelevant. This research work gave the basics for the Black-Scholes model, which was published in 1973 (Black and Scholes). In the study, an explicit formula was derived by Black and Scholes which can be used in the pricing of an European option with no dividend, this is done by constructing a risk free portfolio. The assumptions of the model have been criticised because it was built on a set of assumptions, its significance is not undermined in any scientific literature. The Black-Scholes model the original (PDE) was changed into a standard heat equation given the fixed exercise boundaries and the prescribed assumptions. In this way, an exact analytical solution to the approximation to the option price was derived. The model suggests individual's risk tolerance level has no effects on the option price. Hence the model does not include expected return of the stock . Since we can get the estimated values of the volatility and interest rate from historical data, the model could be simply fitted to an empirical data . Black-Scholes model has a number of disadvantages based on one of its weak assumptions; example, transaction cost and non-negligible tax exist in the market, which makes the market frictionless. Again continuous trading is not allowed at the opening market. Properties of dividend paying stock is accounted for by improving upon the Black-Scholes formula to fit the idea behind pricing theory. Research has shown that asset returns follow a distribution having heavier tails than expected, hence their model assumption is challenged. Furthermore, the Black-Scholes model is based on the assumption that the behaviour of stock price follows a Markov process. It is argued that the formula will not be valid should this assumption be flawed (Chance (1995)). Considering this, risk neutrality is an alternative to derive the Black-Scholes formula since it does not necessarily require the Markov property assumption to hold.

In Cox and Ross paper (1976) and Harrison and Pliska's contribution (1981), probabilities were assigned to the intrinsic option prices from the next time step which is then used to compute the expected option price. In a risk neutral world risk neutral rate was used to discount stock prices and the volatility of the underlying assets was obtained by using the probability to reflect on them. The martingale pricing comes with sophisticated approach than no-arbitrage pricing method by Black-Scholes. To check option volatility, (Kerman (2002)). introduced a double-exponential jump diffusion model by adding a probability measure of return jumps into the formula. One major disadvantage of the extended model is that the introduction of the return jumps prevents the formulation of riskless hedging. An equality is replaced by an inequality in the equation of the original Black-Scholes formula when valuing American options.

(Wilmott et al., 1995). The idea is American option is valued bigger than the European option in a riskless portfolio because of its early exercise option price. The exercise boundary makes it impossible to get analytical solution to the pricing of the option. Black-Scholes formula can be solved now by using numerical methods. The finite difference method was introduced in 1977 to approximate the value of American option(Brennan and Schwartz). This method works by discretising the continuous stock and time into fine mesh grid. The key idea is to approximate the Black-Scholes PDE. Approximating the Black-Scholes PDE can be realized using Taylor's series expansions. Through this conversion, at each mesh point, the PDE is written in linear form differences. (Zhilin (2007)). Option prices at each grid point is solved using systems of linear equations. To solve the explicit scheme, the direct LU-factorization method could be applied easily. To solve the implicit scheme, both LU-factorization and Gauss-Seidel iteration method are possible candidates whereas the latter is more efficient. In particular, when the large system is sparse, the iterative successive over-relaxation (SOR) can speed up the convergence of the Gauss-Seidel method (Kreyszig, 2006). Finite difference method is used to obtain option price on each grid point. Some techniques for solving the exercise boundary are needed for pricing American options. According to Mitchell and Griffiths (1980), this method has a high level of convergence and is accurate as long as it is stable. However, If the mesh is not properly established it gives rise to instability and time consuming computations; the two major draw backs of this approach.(Young and Mohlenkamp (2014)).

In 1979, as an alternative, Cox et al. (1979) came out with a model for option pricing and it was called the binomial model. In effect, the explicit finite difference method is equivalent to its trinomial variant. The binomial model is mostly because of it easy computations and concept. The binomial model when used to value pathdependent options, there is an exponential growth of the parameters as time steps increases. Error term also increases with increasing time steps.

In terms of analytical approximations to the price of American options, Geske and Johnson (1984) proposed a quadratic approximation formula. One major merit of this analytical approximation method lies in the relative accuracy as it is free from the truncation error arising from numerical discretisation. American option pricing can be divided into several price units in their model. Each unit is treated singularly as an European option of which exercise is done only at maturity. Extrapolation method of using prices of two neighbouring units is used to value an American option on any exercise date. Despite the merits, this method is still an approximation to the price of an American put. The exercise boundary cannot be determined therefore making extrapolation being impaired by the accuracy of the model..

American put option's price can be valued by using the method proposed by Carr et al. (1992). The European-style counterpart and its premium are equated to an American-style option. Since the premium is related to the exercise limit or boundary, it can be expressed analytically in integral form. The exercise limit or boundary and the associated option price were solved by adopting numerical integration methods. Other kinds of new methods and approaches were discussed by (Broadie and Detemple (1996)). In 2001, Longstaff and Schwartz (2001) leastsquare Monte Carlo simulation method was initiated to solve for the value of an American option. It is easy to implement this method when solving for option with multiple factors of risk. Nonetheless, the accuracy of the ultimate estimator is not ensured since is time consuming when simulating large number of paths.; As an alternative, a penalty approach was proposed by Manon(2012) to accompany the finite element method.

2.6 Valuation of Life Insurance Liabilities

Insurance regulatory and Development Authority (IRDA) in India define Life Insurance as a financial cover for contingency linked with human life, like death, disability, accident. retirement etc. Human life is subject to risk of death and disability due to natural and accidental causes. When human life is lost or a person is disabled permanently or temporarily, there is loss of income to the household.

Anders and Peter (2002) presented a model which explicitly takes into account the fact that the holders life insurance contracts (LICs) have the first claim on the company's asset whereas equity holders have limited liability, that is, interest rate guarantees are common elements of LICs, and that LICs according to the socalled contribution principle are entitled to receive a fair share of any investment surplus. He further built a regulatory mechanism in the form of an investment surplus. He further built a regulatory mechanism in the form of an intervention rule into the model. The mechanism was shown to significantly reduce the insolvency risk of the issued contracts and it implies that the various claims on the company's asset become more exotic and obtained barrier option properties. He derived closed valuation formulas. Numericals were also used to illustrate how the model can be used to establish the set of initially fair contracts and to determine the market values of the contracts after inception.

Daniel et al (2010) introduced a model used for valuing life insurance contracts with embedded options. Numerical methods for the valuation are described within their generic set up. Contracts containing early surrendering features pose numerical challenging problems so they particularly focus on such contracts. Using the participating contract as an example, different methods were applied to it to check and compare the efficiency of a simple Black-Scholes model. The effect of early exercise characteristics of their example were studied of which it was analysed to check how the impact of the model risk after introducing L,evy model. In their study, it was observed that the Monte Carlo method gives fast results for European contracts, that is, contracts without any early exercise features, but it was inefficient for the valuation of long-term non-European contracts. In this case, accurate results may not be obtained because the number of necessary simulation steps may be extremely high. Secondly, they presented a discretization approach based on the consecutive solution of certain partial (integro) differential equations (PDE approach). Christopher(2009). Based on the the consistent value of the market was able to explain the rational behind the new valuation approach ; New regime was formed to tackle the issues faced by life insurers; and came out how to address these issues. This was done by analysing 38 life insurance companies valuation report who used the new method and, specifically, the information they used in modelling. He found that the basis of market consistency gives a number of merits over the local regime way of which liabilities are valued in the United Kingdom. He also realized that there have got some challenges ahead.

(Eric (1995)) addressed the problem of convexity and duration of the liabilities and equity of an insurance company, because the insurance landscape was affected. The assessment of risk measure is crucial because they consist of primary components of any good asset-liability management method. To overcome all the pitfalls that are normally encountered in an insurance firm much effort is required to look accurately the picture of the risk . Fabio et al(2006) analysed the interest structure and the role of mortality in evaluating a fair value of a life insurance company. They discussed a fair value accounting impact on reserve evaluations and compare it to the traditional deterministic model based on local rules for an Italian balance sheet calculation and a stochastic one based on a diffusion process for both mortality and financial risks. They separated the embedded derivatives from their host contracts so the fair value of a traditional life insurance contract would be expressed as the value of four component: the basic contract, the participation option, the option to annuities and the surrender option.

2.7 Model Framework

There are financial and actuarial approaches to assess financial guarantees within life insurance contracts. Some of the financial approach is concerned with the riskneutral valuation and has been researched by various authors.

The rational insurance premium on an equity -linked insurance contract was obtained through the application of the theory of contingent claims pricing. The premium was determined in an economy with the equity following a geometric

Brownian motion, whereas the interest rate was assumed to be constant. Nielsen and Sandmann(1995) realized that, further consideration with deterministic interest rate allow for interest rate by assuming an Ornstein-Uhlenbeck process implying a closed form solution of the single premium endowment policy. They presented a model for the multi premium case in the contest of a stochastic interest rate process. It was shown that the insurance contract includes an Asian-like option contract. No closed form solution will be obtained. They discuss different numerical approaches and apply Monte Carlo simulations with a variance reduction technique.

Neilsen and Sandmann(1995) concluded that, in an economy with stochastic development of the term structure of interest rates a model for the determination of the fair valuation premium on an equity linked life insurance contract has been established. An essential part of the premium of the equation consist of a contingent claim with a character as an Asian option. However it was shown that the stochastic interest rate and the long time maturity of the insurance contract prohibited the application of the ”usual” solution methods: Edgeworth expansion or fast Fourier transform. The approximation formula developed by Vorst(1992) cited in Neilsen and Sandmann(1995 exhibited a better performance than the two just mentioned for medium term contracts. Neilsen and Sandmann (1995) applied and advocated for Monte Carlo simulations to overcome the difficulties. The result obtained was compared to the Edgeworth and Vorst approximation and found to be perfected to these. They realized that, although the Monte Carlo simulations are more time consuming than the other methods they did not take it as a serious critical point against simulations as far as the fair premium only has to be calculated once when the contract is entered.

The use of advanced data mining techniques to improve decision making has already taken root in property and casualty insurance as well as in many other industries. however since in their opinion, the application of such techniques for more objectives, consistent and optimal decision making in the life insurance industry is still in a nascent stage, they described the ways data mining and multivariate analytic techniques can be used to improve decision making processes in such functions as life insurance underwriting and marketing, resulting in more profitable and efficient operations. They implemented predictive modelling in life insurance underwriting and marketing and demonstrated the segmentation power of predictive modelling and resulting business benefits.

The liability structure of the insurance company is implied by participating life insurance contracts and based on a model suggested . In other for policyholders to initiate contracts, they must pay a single premium P_0 and if the company's initial capital is E_0, then the sum of the initial contribution $A_0 = E_0 + P_0$. This sum of initial contribution A_0 is invested in the reference portfolio. Hence for $0 < k \leq 1$, it holds that $P_0 = k.\, A_0$ and $E_0 = (1 - k).A_0$ where k represents the leverage of the company. if P denote the policyholders' account, that is, the book value of the policy reserves. the policy reserve P is a year-to-year, or cliquet-style, guarantee, which means it annually earns the maximum guaranteed interest rate or a fraction α of the annual surplus generated by the insurer's investment portfolio. Hence for $t = 1,2,\ldots T$, the development of the policy reserve is given by

$$P(t) = P(t-1).(1 + max[g, \alpha(\frac{A(t)}{A(t-1)} - 1)])$$

The value of liabilities L(T) are summarized as

$$L(T) = P(T) + \Delta[k.A(T)]^{+} = P(T) + \Delta.B(T) - D(T)$$

Where D(T) denotes the default put option, E(T), the residual claims of the equity holds and is determined as the difference between the market value of the reference portfolio A(T) and the policyholder's claim L(T). i.e.

$$E(T) = A(T) - L(T) = max(A(T) - P(T,0) - \Delta.B(T) \geq 0$$

In the standard Black Scholes frame work, the total market value of asset A evolves according to a geometric Brownian motion as stated earlier. In Black Scholes model for asset prices, the standard Brownian motion $(W^p(t),0 \leq (t) \leq T$ on a probability space (Ω,F,P) and $(F_t,0 \leq (t) \leq T)$, be the filtration generated by the Brownian motion. The total market value of the asset A in standard Brownian motion evolves according to a geometric motion under the objective measure P is given by

$$dA(t) = mA(t)dt + \sigma A(t)dW^p(t),$$

with constant asset drift m, volatility σ and P-Brownian motion $W^p(t)$, assuming a complete perfect, and frictionless market. The solution of the stochastic differential equation is

$$A(t) = A(0).e^{((m-\frac{\sigma^2}{2})t+\sigma.W^p(t))} = A(t-1).e^{((m-\frac{\sigma^2}{2}+\sigma.(W^p(t)-W^p(t-1))}$$

According to Nadine and Alexander(unpublished), fair pricing of embedded options in life insurance contracts is usually conducted by using the appropriate concept of risk-neutral valuation. This concept assumes a perfect hedging strategy, which insurance companies can hardly pursue i practice. They extended the risk-neutral valuation concept with a risk measurement approach and accomplish this by first calibrating contract parameters that lead to the same market value risk-neutral valuation.They then measure the resulting risk assuming that insurers do not follow perfect hedging strategies. They use lower partial moments as the relevant risk measure, comparing shortfall probability, expected shortfall, and shortfall variance. This research showed that even when contracts have the same market value, the insurance company's risk can vary widely, a finding that allows us to identify key risk drivers for participating life insurance contracts.

2.8 Numerical Valuations Approach

Gerstner et al (unpublished) propose a discrete time asset-liability management(ALM) model which was to simulate life insurance product balance sheet. Incorporating in the model is the life insurance product features or characteristics, the surrender option, a reserve-dependent bonus declaration, a dynamic assets allocation and a two-factor stochastic capital market. All terms in the model could be computed recursively which gives a simple implementation and efficient evaluation of the model equations. The modular design of the model permits straightforward modifications and extensions to handle specific requirements. In practise, the simulation of stochastic ALM models is usually performed by Monte Carlo methods which suffer from relatively low convergence rates and often very long run times, though. As alternatives to Monte Carlo simulation, they proposed deterministic integration schemes, such as quasi-Monte Carlo and sparse grid methods for the numerical simulation of such models. Their efficiency is demonstrated by numerical examples which show that the deterministic methods often perform much better than Monte Carlo simulation as well as by theoretical considerations which show that ALM problems are often of low effective dimension.

Russel and Collins (1992) described the application of the Monte Carlo technique to a practical situation in a company to solve the problem of rate-making with real problem in the transfer of coverage from one carrier to another by a policyholder who finds himself or herself in a large deficit position with the original carrier in the field of insurance. This situation can be avoided if the policyholder is willing to pay an additional charge for a guarantee of an upper limit on the amount of deficit carried forward from one year to the following years. In order to determine such a charge, it is necessary to know the probability of, the expected value of, and the variation of claims in excess of a given amount. The basic problem to be solved, of course, is that of determining the frequency distribution of the annual claim cost of a given group of lives for a given year. It was desired that the following properties of the group be allowed to vary over rather wide ranges:

- The size of the group
- The age distribution of the group
- The sex distribution of the group
- The total amount of insurance, and
- The distribution of the insurance on individual lives.

Since the analytical solution of such a problem was complex, they use the Monte Carlo technique, which is admirably suited to a problem of this nature.

Bjarke et al (2001), came out with a model for valuing traditional participating life insurance policy. The difference approaches used include the implicit finite difference, the explicit finite difference and the Crank Nicolson. Among them, it was realiized was more accurate than the implicit and the explicit finite difference because the error associated with the final solution with Crank Nicolson is smaller than the other two methods.

CHAPTER 3

METHODOLOGY

3.1 Introduction

This chapter seeks to explain the methodology used in the numerical approach used in valuation of the participation surrender option of life insurance contracts, embedded in the model is the survival rate which is the rate at which an insured becomes multi-morbid . There are three formulas of finite difference method, the explicit method(the forward difference scheme), the implicit method(the backward difference scheme) and the Crank Nicolson method(the central difference scheme).

3.2 Preliminaries

In this study the algorithm was derived from the simple idea of approximating partial derivative of a given partial differential equation by finite difference which is the fundamental sole for finite difference methods. As a tool for solving (PDEs), this process transforms analytical differential equation into a set of algebraic equations. As in many numerical algorithms, the starting point is a finite series approximation. Again, in this study the algorithm was derived from the application of the finite difference scheme to solve boundary-value partial differential equations proposed by Bjarke et al (2001). This is because in this scheme the European-style and the American-style contract, path-dependence variable is appropriately treated as a parameter.

Definition 3.1: Differential Equation

A differential equation is an equation involving the unknown function $y = f(t)$, together with derivatives y^{0},y^{00},...,y^{n}.

Mathematically, a differential equation may be express implicitly as

$$F(t,y,y',y'',\ldots,y^{n}) = 0 \tag{3.1}$$

Explicitly, the general form of a differential equation can be written as

$$y^{n} = f(t,y',y'',\ldots,y^{(n-1)}) \tag{3.2}$$

Definition 3.2: Ordinary Differential Equations

An ordinary differential equation(ODE) is an equation involving an unknown function of a single variable together with one or more of it derivatives.

Definition 3.3: Order of Differential Equations

A first order differential equation is of the form

$$y' = f(t,y) \tag{3.3}$$

and the equation is said to be in normal form.

A differential equation of of order n is of the form

$$f(t,y,y',y'',\ldots,y^{n}) = 0 \tag{3.4}$$

and is said to be normal form.

A typical n^{th} order linear differential equation is given by

$$y^{(n)} + a_1(t)y^{(n-1)} + a_2(t)y^{(n-2)} + \ldots + a_{(n-1)}(t)y' + a_n(t)y = f(t) \tag{3.5}$$

Definition 3.4: Partial Differential Equation (PDE)

A *partial differential equation*(PDE) is an equation that involves an unknown function (the dependent variable) and some of its partial derivatives, with respect to two or more independent variables. Mathematically, PDE is of the form

$$F(t_1, ..., t_n, u, \frac{\partial u}{\partial t_n}, ..., \frac{\partial^2 u}{\partial t_1 \partial t_1}, ..., \frac{\partial^2 u}{\partial t_1 \partial t_n}, ...) = 0 \tag{3.6}$$

If F ia a linear function of u and its derivatives, then the PDE is called linear. An n^{th} - order PDE has the highest order derivative of order n. A simple PDE is

$$\frac{\partial u}{\partial t}(t, y) = 0 \tag{3.7}$$

This relation implies that the function $u(t,y)$ is independent of t. However the equation gives no information on the function's dependence on the variable y. Hence the general solution of this equation is

$$u(t,y) = f(y) \tag{3.8}$$

where f is an arbitrary function of y. General linear second order PDE is of the form

$$a(t,y)u_{tt} + 2b(t,y)u_{ty} + c(t,y)u_{yy} + d(t,y)u_t + e(t,y)u_y + g(t,y)u = f(t,y) \tag{3.9}$$

where (t,y) ∈ Ω is a domain in t-y coordinates.

Definition 3.5: Stochastic Differential Equation (SDE)

A stochastic differential equation (SDE) is a differential equation in which one or more of the terms is stochastic process, resulting in a solution which is itself a stochastic process(Davis, 2005). In probability theory, a stochastic process or sometimes random process (widely used) is a collection of random variables; this is often used to present the evolution of some random variable, or system, over time. This is the probabilistic counterpart to a deterministic process (or deterministic system). In valuing insurance liabilities, the randomness of the underlying asset is modelled by SDE's. For example, asset price with respect to time is given by

$$ds_t = \alpha(A_t,t)dt + \sigma(A_t,t)dW_t, for t \in [0,\infty). \tag{3.10}$$

3.3 The General Approach Use In Solving Partial Differential Equations Numerically

1. First we attempt to replace the partial derivative with truncated Taylor series approximation.
2. Then we create a linear system from the new expression with the aid of the boundary conditions.
3. Then we use the initial conditions to begin iteration through the system.
4. When we using forward Euler we use straight forward iteration.
5. When using backward Euler we iterate using Jacobi iteration.

3.4 Finite Difference Approximation

A finite difference method typically involves the following steps:

1. Generate a grid, for example $(x_i,t^{(k)})$, where we want to find an approximate solution.
2. Substitute the derivatives in an ODE/PDE system of equation with finite difference schemes. The ODE/PDE then become a linear or non-linear system of algebraic equations.

3. Solve the system of algebraic equations.

4. Implement and debug the computer code.

5. Do the error analysis, both analytically and numerically.

3.4.1 Types of Finite Difference Methods

Depending on how we approximate the partial derivative with respect to time, we have three different finite difference schemes:

1. Explicit finite difference scheme, when forward difference formula is used

2. Implicit finite difference scheme, when backward difference formula is used

3. Crank-Nicolson finite difference scheme, when the central difference formula is used.

3.4.2 Finite Difference Formulas of Ordinary Differential Equations (ODE)

There are three commonly used finite difference formulas to approximate first order derivative of a function f(x). They are forward finite difference, backward finite difference and central finite difference. Let's consider Tylor's series expansion of a function f(x) in the neighbourhood of $x = x_i$:

$$f_{i+1} = f_i + \Delta x f_i' + \frac{(\Delta x)^2}{2!} f_i'' + \frac{(\Delta x)^3}{3!} f_i''' + \frac{(\Delta x)^4}{4!} f_i'''' + \dots \tag{3.11}$$

where $\Delta x = x_{i+1} - x_i$. Solving equation 3.11 for f_i, we have

$$f_i' = \frac{f_{i+1} - f_i}{\Delta x} - \frac{(\Delta x)}{2!} f_i'' - \frac{(\Delta x)}{3!} f_i''' - \dots \tag{3.12}$$

Using the mean-value theorem, equation 3.12 becomes

$$f_i' = \frac{f_{i+1} - f_i}{\Delta x} - \frac{(\Delta x)}{2} f''(\xi); x_i < \xi < x_{i+1} \quad (3.13)$$

Where $0(\Delta x) = -\frac{(\Delta x)}{2} f''(\xi)$, the order of Δx, indicates the error is proportional to the length (Δx) and also a second derivative of f. Hence

$$f_i' \approx \frac{f_{i+1} - f_i}{\Delta x} \quad (3.14)$$

Equation 3.14 is called the Forward Difference Formula. Similarly, the Backward Difference Formula from the Taylor series

$$f_{i-1} = f_i - \Delta x f_i' + \frac{(\Delta x)^2}{2!} f_i'' - \frac{(\Delta x)^3}{3!} f_i''' + \ldots \quad (3.15)$$

is given by

$$f_i' \approx \frac{f_i - f_{i-1}}{\Delta x} \quad (3.16)$$

Where the error $0(\Delta x) = -\frac{(\Delta x)}{2} f''(\xi)$. Finally, subtracting equation 3.15 from 3.11 we have the central difference formula

$$f_i' \approx \frac{f_{i+1} - f_{i-1}}{2\Delta x} \quad (3.17)$$

with the error $0(\Delta x) = -\frac{(\Delta x)^2}{2} f'''(\xi)$.

3.4.3 Finite Difference Approximation for Partial Differential Equations (PDE)

With regards to many financial engineering problems, the function f depends on two or more independent variables, hence the need for finite difference approximation of partial derivatives. Since the partial derivatives denotes the local variation of a function with respect to a particular independent variable while all other independent variables are held constant, finite difference approximation of ordinary derivatives can be adapted for the partial derivatives. If there are two independent variables, we use the notation (i,j) to designate the pivot point, and if there are three independent variables, (i,j,k) are used where, i, j, and k are the counters in x, y and z directions.

Figure 3.1 below is a two dimensional finite difference grid. If we consider the function $f(x,y)$, then the finite difference approximation for the partial derivative $\frac{\partial(x,y)}{\partial(x)}$ at $x = x_i, y = y_i$ can be found by fixing the value of y at y_i and treating $f(x,y_i)$ as a one-variable function. The forward, backward and the central difference of $\frac{\partial f}{\partial x}$ can be expressed as

$$\frac{\partial f}{\partial x}|_{i,j} \approx \frac{f(x_i + \Delta x, y_j) - f(x_i, y_j)}{\Delta x} \tag{3.18}$$

$$\frac{\partial f}{\partial x}|_{i,j} \approx \frac{f(x_i, y_j) - f(x_i - \Delta x, y_j)}{\Delta x} \tag{3.19}$$

$$\frac{\partial f}{\partial x}|_{i,j} \approx \frac{f(x_i + \Delta x, y_j) - f(x_i - \Delta x, y_j)}{2\Delta x} \tag{3.20}$$

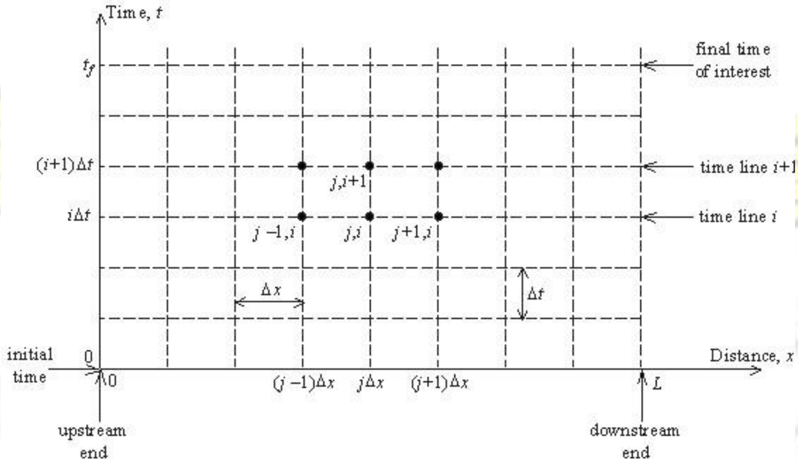

Figure 3.1: Two dimensional grid

Second Order Partial Derivative Central Difference Approximation

The central difference approximation of second partial derivatives at (x_i,y_j) can be derived as

$$\frac{\partial^2 f}{\partial x^2}|_{i,j} \approx \frac{f(x_i + \Delta x, y_j) - 2f(x_i, y_j) + f(x_i - \Delta x, y_j)}{\Delta x^2} \tag{3.21}$$

$$\frac{\partial^2 f}{\partial y^2}|_{i,j} \approx \frac{f(x_i, y_j + \Delta y) - 2f(x_i, y_j) + f(x_i, y_j - \Delta y)}{\Delta y^2} \quad (3.22)$$

Error of Finite Difference Approximation of Partial Derivatives

To fine the error associated with finite difference approximation of partial derivatives, we use the Taylor series expansion of $f(x,y)$ around the point(x_i,y_j). That is,

$$f_{i\pm1,j} = f_{i,j} \pm \Delta x \frac{\partial f}{\partial x}|_{i,j} + \frac{(\Delta x)^2)}{2!}\frac{\partial^2 f}{\partial x^2}|_{i,j} \pm \frac{(\Delta x)^3)}{3!}\frac{\partial^3 f}{\partial x^3}|_{i,j} + \ldots \quad (3.23)$$

$$f_{i\pm1,j} = f_{i,j} \pm \Delta y \frac{\partial f}{\partial y}|_{i,j} + \frac{(\Delta y)^2)}{2!}\frac{\partial^2 f}{\partial y^2}|_{i,j} \pm \frac{(\Delta y)^3)}{3!}\frac{\partial^3 f}{\partial y^3}|_{i,j} + \ldots \quad (3.24)$$

Truncating equation 3.23 after the n^{th} order, we have the error

$$R_{x,n} \simeq (-1)^{n+1}\frac{(\Delta y)^{n+1}}{(n+1)!}\frac{\partial^{n+1} f(x,y)}{\partial y^{n+1}}|_{i,j} \quad (3.25)$$

and truncating equation 3.24 after the n^{th} order gives the error

$$R_{y,n} \simeq (-1)^{n+1}\frac{(\Delta y)^{n+1}}{(n+1)!}\frac{\partial^{n+1} f(x,y)}{\partial y^{n+1}}|_{i,j} \quad (3.26)$$

3.4.4 Finite Difference Approximation for One Dimensional Partial Differential Equations

Let consider a one dimensional partial differential equation

$$\frac{\partial \phi}{\partial t} = \alpha \frac{\partial^2 \phi}{\partial x^2} \qquad 0 \le x \le L, \qquad t \ge 0 \quad (3.27)$$

Where $\varphi = \varphi(x,t)$ is the dependent variable, and α is a constant coefficient. In a practical computation, the solution is obtained only for a finite time say t_{max}. solution to equation

3.27 requires specification of boundary conditions at $x = 0$ and $x = L$, and the initial conditions as $t = 0$. Simple boundary and initial conditions

are

$$\varphi(0,t) = \varphi_0, \qquad \varphi(L,t) = \varphi_L \qquad \varphi(x,0) = f_0(x). \tag{3.28}$$

The finite difference method involves using discrete approximation like

$$\frac{\partial \phi}{\partial x} \approx \frac{\phi_{i+1} - \phi_i}{\Delta x} \tag{3.29}$$

Figure 3.2 below is a mesh on a semi-infinite strip used for solution to the one dimensional equation above. The solid squares indicate the location of the (known) initial values.The open squares indicate the location of (known) boundary values or conditions. The open circles indicate the position of the interior points when the finite difference approximation is computed. Where the quantities on the right hand side are defined on the finite difference mesh. Approximations to the governing differential equation are obtained by replacing all continuous derivative by discrete formulas such as those in equation 3.29. The relationship between the continuous(exact) solution and the discrete approximation is shown in above.

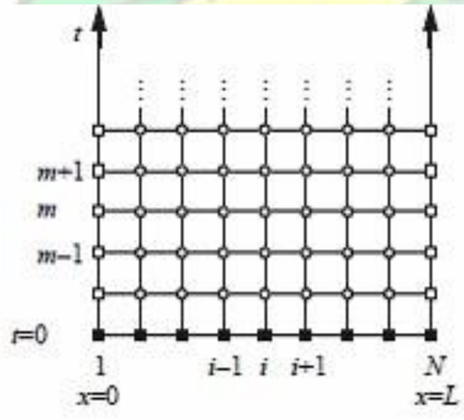

Figure 3.2: Mesh of a semi finite strip

Note that φ_{im} from the finite difference model is a distinct step from translating the continuous problem to discrete problem.

3.4.5 Finite Difference Approximation for Two Dimensional PDEs

Let consider a two dimensional PDE

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = g(x, y) \tag{3.30}$$

such that $a \le x \le b$ and $c \le y \le d$. if $U(a,y) = U_a$, $U(b,y) = U_b$, $U(x,c) = U_c$ and $U(x,d) = U_d$, where U_a, U_b, U_c and U_d are boundary conditions at y and x respectively. Note that, Δx is not necessarily equal to Δy, but for this case we let $\Delta x = \Delta y =$ h At the generic points where x is in the direction of i and y is in the direction of j, the pde can be written as

$$\frac{\partial^2 U}{\partial x^2}\Big|_{i,j} + \frac{\partial^2 U}{\partial y^2}\Big|_{i,j} = g_{i,j}$$

Using the central difference approximation scheme we have

$$\frac{\partial^2 U_{i,j}}{\partial x^2} \approx \frac{U_{i-1,j} - 2U_{i,j} + U_{i+1,j}}{h^2} \tag{3.31}$$

and

$$\frac{\partial^2 U_{i,j}}{\partial y^2} \approx \frac{U_{i,j-1} - 2U_{i,j} + U_{i,j+1}}{h^2} \tag{3.32}$$

adding equation 3.31 and 3.32, we have

$$\frac{\partial^2 U_{i,j}}{\partial x^2} + \frac{\partial^2 U_{i,j}}{\partial y^2} \approx \frac{U_{i-1,j} + U_{i+1,j} - 2U_{i,j} + U_{i,j-1} + U_{i,j+1}}{h^2} = g_{i,j} \tag{3.33}$$

$$\Rightarrow U_{i-1,j} + U_{i+1,j} - 4U_{i,j} + U_{i,j-1} + U_{i,j+1} = h2g_{i,j} \tag{3.34}$$

At the first node of the grid, say $P_1 : i = 1, j = 1$

$$\Rightarrow U_{0,1} + U_{2,1} - 4U_{1,1} + U_{1,0} + U_{1,2} = h2g_{1,1}$$

let $U_{0,1} = U_a$, and $U_{1,0} = U_c$

$$\Rightarrow -4U_{1,1} + U_{2,1} + U_{1,2} = h2g_{1,1} - U_a - U_c$$

$$\Rightarrow -4P_1 + P_2 + P_4 = h^2 g_{1,1} - U_a - U_c \qquad (3.35)$$

At the second node, say $P_2 : i = 2, j = 1$

$$P_1 - 4P_2 + P_3 + P_5 = h^2 g_{2,1} - U_c \qquad (3.36)$$

summarizing the generic points, we have

2 2 2

$= h^2 g_{1,1} \quad -U_a \quad -U_c$

$= h^2 g_{2,1} \quad -U_c$

$= h^2 g_{3,1} \quad -U_b$

$= h^2 g_{1,2} - U_a =$

$h^2 g_{2,2}$

$+P_9 \quad = h^2 g_{3,2} \quad -U_b$

$= h^2 g_{1,3} \quad -U_a \quad -U_d$

$+P_9 \quad = h^2 g_{2,3} \quad -U_d$

$-4P_9 \quad = h^2 g_{3,3} \quad -U_b \quad -U_d$

b^1

b_2

b_3

b_4

P_1 : $-4P_1$ $+P_2$ P_4

P_2 : P_1 $-2P_2$ $+P_3$ P

P_3 : P_2 $-4P_3$

P_4 : P_1 $-P_4$ $+$

P_5 : P_2 $+P_4$ $-$

P_6 : P_3 $+$

P_7 : P_4

P_8 : P

P_9 :

writing the above system to matrix

-4

1

0

1

0

0 $\quad$ $1 \quad 0 \quad 1 \quad 0$

$-4 \quad 1 \quad 0 \quad 1$

0 $\quad$ $1 \quad -4 \quad 0 \quad 0$

$0 \quad 0 \quad -4 \quad 1$

0 $\quad$ $1 \quad 0 \quad 1 \quad -4$

$0 \quad 1 \quad 0 \quad 1$

0 $\quad$ $0 \quad 0 \quad 1 \quad 0$

$0 \quad 0 \quad 0 \quad 1$

0 $\quad$ $0 \quad 0 \quad 0 \quad 0$

$= b_5$

b_6

b_7

b_8

b_9

where $b_1 = h\,g_{1,1} - U_a - U_c$, $b_2 = h\,g_{2,1} - U_c$, $b_3 = h\,g_{3,1} - U_b$, $b_4 = h^2g_{1,2} - U_a$, $b_5 = h^2g_{2,2}$, $b_6 = h^2g_{3,2} - U_b$, $b_7 = h^2g_{1,3} - U_a - U_d$, $b_8 = h^2g_{2,3} - U_d$ and $b_9 = h^2g_{3,3} - U_b - U_d$ if We let

$$A = \begin{pmatrix} -4 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 4 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Also let $B_1 = b_1 : b_3$, $B_2 = b_4 : b_6$, $B_3 = b_7 : b_9$, $X_1 = P_1 : P_3$, $X_2 = P_4 : P_6$ and $X_3 = P_7 : P_9$, we obtain the matrix below

$$\begin{pmatrix} A & I & O \\ I & A & I \\ O & I & A \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} \tag{3.37}$$

Which is further simplified in the form

$$HX = B \tag{3.38}$$

Systems expressed in the form $HX = B$ have several solution techniques in solving it.

Solution Techniques to Systems of Linear Algebraic Equations

A system of linear equation algebraic equations is nothing but a system of 'n' algebraic equations satisfied by a set of n unknown quantities. The aim is to find these n unknown quantities satisfying the n equations. It is a very common practice to write the system of n equations in matrix form as in equation 3.37

$$HX = B$$

Where H is an $n \times n$, non singular matrix and X and B are $n \times 1$ matrices out of which B is known. For a small n the elementary methods like crammer's rule, matrix inversion are very convenient to get the unknown vector X from the system $HX = B$. However for large n these methods will become computationally very expensive because of the evaluation of matrix determinants involved in these methods. Hence to make the solution methods less expensive one has to find alternate means which does not require the evalution of any determinants to find X from $HX = B$ There are two types of methods exists to solve these linear systems:

- Direct Method
- Iterative Method

Direct Method

These are the methods which can find the solution of the system in a finite number of steps known apriori. Some of the important direct methods are

- Gauss Elimination Methods

There are two basic steps in this type of elimination method they are:

1. Forward Elimination

2. Backward Elimination

In forward elimination the augmented matrix(the elements of the vector B has joined with the coefficient matrix H as $(n + 1)th$ column) and is denoted by $H|B$ is converted into upper diagonal form by making use of matrix row transformations(one can also convert into lower triangular form in which case the process is called backward elimination). Then by starting with the last row of the upper triangular matrix(first row for lower triangular matrix) the unknown quantity is obtained by back (forward) substitution

- Decomposition Methods

In this method we have two types:

1. LU Decomposition Method
2. QR Decomposition Method

In these methods the coeffficient matrix H of the given system of equation $HX = B$ is written as a product of a lower triangular matrix L and an upper triangular matrix U, such that $H = LU$, where the elements of $L = (l_{ij} = 0, i < j)$ and the element of $U = (u_{ij} = 0, i > j)$

Iterative methods

Iterative techniques for solving linear syetems of algebraic equation are Jacobi, Guass-Seidel and SOR method. The basic idea is to solve the i^{th} equation in the system for i^{th} variable(Laurene,2008). Let consider the 4×4 system below:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = b_1 \quad (3.39)$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 = b_2 \quad (3.40)$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 = b_3 \quad (3.41)$$

$$a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 = b_4 \tag{3.42}$$

Solving for x_1, x_2, x_3, x_4 in equation 3.39 t0 3.42, we have

$$x_1 = -\frac{a_{12}}{a_{11}}x_2 - \frac{a_{13}}{a_{11}}x_3 - \frac{a_{14}}{a_{11}}x_4 + \frac{b_1}{a_{11}} \tag{3.43}$$

$$x_2 = -\frac{a_{21}}{a_{22}}x_1 - \frac{a_{23}}{a_{22}}x_3 - \frac{a_{24}}{a_{22}}x_4 + \frac{b_2}{a_{22}} \tag{3.44}$$

$$x_3 = -\frac{a_{31}}{a_{33}}x_1 - \frac{a_{32}}{a_{33}}x_2 - \frac{a_{34}}{a_{33}}x_4 + \frac{b_3}{a_{33}} \tag{3.45}$$

$$x_4 = -\frac{a_{41}}{a_{44}}x_1 - \frac{a_{42}}{a_{44}}x_2 - \frac{a_{43}}{a_{44}}x_3 + \frac{b_4}{a_{44}} \tag{3.46}$$

Iterative methods are stooped at certain conditions. Below are two possibilities:

1. Iteration are stopped when the norm of the change in the solution vector x from iteration to the next is sufficiently small or
2. Whwn norm of residual vector,k$Ax - b$k, is below a specific tolerance.

3.4.6 Jacobi Iteration

In the jacobi method, the system $Ax = b$ is transformed into the system $X = Hx+d$, where H has the zeros on the diagonal and X is a vector which is updated from previous vector x.

The systems of equations in 3.43 to 3.46 in a matrix form is

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{a_{12}}{a_{11}} & -\frac{a_{13}}{a_{11}} & -\frac{a_{14}}{a_{11}} \\ -\frac{a_{21}}{a_{22}} & 0 & -\frac{a_{23}}{a_{22}} & -\frac{a_{24}}{a_{22}} \\ -\frac{a_{31}}{a_{33}} & -\frac{a_{32}}{a_{33}} & 0 & -\frac{a_{34}}{a_{33}} \\ -\frac{a_{41}}{a_{44}} & -\frac{a_{42}}{a_{44}} & -\frac{a_{43}}{a_{44}} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} \frac{b_1}{a_{11}} \\ \frac{b_2}{a_{22}} \\ \frac{b_3}{a_{33}} \\ \frac{b_4}{a_{44}} \end{bmatrix} \tag{3.47}$$

Which is in the form $X = Hx + d$. Iteratively

$$X_{i+1} = Hx_i + d \tag{3.48}$$

Equation 3.48 is the Jacobi iterative technique. In this iteration, MATLAB will be used to run the iteration.

3.4.7 Convergence, Consistency and Stability

Convergence

One step finite difference scheme approximating a partial differential equation is a convergent scheme if for any solution to the partial differential equation, $u(t,x)$, and solutions to the finite difference scheme u^n_i such that v_i^0 converges to $u_0(x)$ as $i\Delta x$ converges to x, then v_i^n converges to $u(t,x)$ as $(n\Delta t, i\Delta x)$ converges to (t,x) as Δt, Δx converge to 0 (Singiresu, 2002).

Def: A numerical method is convergent if its global error computed up to a given x satisfies

$$\lim_{h\to 0} \|\varepsilon\| \equiv \lim_{h\to 0} \|y - Y\| = \lim_{h\to 0} max|y_n - Y_n| = 0 \quad (3.49)$$

This implies that, the numerical solution Y_n is computed with no round-off error.

Consistency

Def: A numerical method is said to be consistent if for a partial differential equation $P_U = S$ then $P_{\Delta t,\Delta x U} = f$ we say the finite difference scheme is consistent with the partial differential equation if for any smooth function $\varphi(x,t)$

$$P\varphi - P_{\Delta t,\Delta x}\varphi \to 0 \qquad as, \Delta t, \Delta x \to 0 \quad (3.50)$$

Stability

A one step finite difference Scheme with constant coefficients is stable in a stability region Λ if and only if there is a constant K (independent of θ, Δt, and Δx) such that,

$$|g(\theta,\Delta t,\Delta x)| \leq 1 + K\Delta t \quad (3.51)$$

with $(\Delta t, \Delta x)\epsilon\Lambda$. If $g(\theta,\Delta t,\Delta x)$, is independent of of Δt and Δx the stability condition may be replaced with restricted stability condition $|g(\theta)| \leq 1$

3.5 Life Insurance Portfolio

3.5.1 Life Insurance Model

Considering the accounting equation of an insurance company:

$$A_t = L_t + B_t$$

The above accounting equation is a simplified form of the asset and liability situation given in relation to a given contract. A_t denotes the market value of the insurer's asset portfolio, L_t denotes the policyholder's account balance and $B_t = A_t - L_t$ is the bonus reserve at time t. If charges are not taken into consideration, the policyholder's account balance at time zero L_0 equals the single up-front Premium P, that is, $L_0 = P$. Note that the policyholder may surrender his or her contract during the term of the contract with regard to the fact that he or she becomes multi-morbid while the contract is still in place. If the contract is lapsed at time

$v_0\epsilon(1, ..., T)$, the policyholder receives the current account balance L_{v0}. It is assumed that shareholders are paid dividends during the anniversaries as compensations for the adopted risk.

3.5.2 Stock Price Model

The price of a stock can be modelled by a continuous stochastic process which is the sum of a predicted and an unpredicted part. However, this type of model does not take into account market crashes. If those are to be taken into consideration, the stock price needs to contain a third component which models the unexpected jumps.

Constant Drift and Volatility Model

Let S_t denote the price of a stock at time t. If F_t denotes the information set at time t, then S_t is a continuous process that is F_t-adapted. The return on the stock during the time interval Δt measures the percentage increase in the stock price between instances t and $t + \Delta t$ is given by $\frac{S_{t+\Delta t}-S_t}{St}$. When Δt is infinitesimally small, we obtain the instantaneous return, $\frac{dS_t}{S_t}$. This is suppose to be the sum of two components:

- The predicted part αdt
- The noisy part due to unexpected news σdz_t

Adding these two components yields

$$\frac{dS_t}{S_t} = \alpha dt + \sigma dz_t \tag{3.52}$$

Which leads to the stochastic equation:

$$dS_t = \alpha S_t dt + \sigma S_t dz_t$$

The goal is to come up with $Y_t = F(S_t)$ so that Y_t will not contain any reference to S_t

$$Y_t = log(S_t) \tag{3.53}$$

Applying Ito's lemma

$$dG = (\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2)dt + \frac{\partial G}{\partial x}\sigma dz$$

From equation 3.53 $\frac{\partial Y_t}{\partial S_t} = \frac{1}{S_t},\ \frac{\partial^2 Y_t}{S_t^2} = \frac{-1}{S_t^2},\ \frac{\partial Y_t}{\partial t} = 0,\ a = \alpha$ and b=σ

$$dG = (\frac{\partial Y_t}{\partial S_t}\alpha S_t + \frac{\partial Y_t}{\partial t} + \frac{1}{2}\frac{\partial^2 Y_t}{\partial {S_t}^2}\sigma^2 S_t^2)dt + \frac{\partial Y_t}{\partial S_t}\sigma S_t dz$$

$$dY_t = (\alpha - \frac{1}{2}\sigma^2)dt + \sigma dz \tag{3.54}$$

$$dlogS_t = (\alpha - \frac{1}{2}\sigma^2)dt + \sigma dz \tag{3.55}$$

applying integration to both sides we obtain

$$\int_0^t dlogS_t = \int_0^t (\alpha - \tfrac{1}{2}\sigma^2)dt + \int_0^t \sigma dz$$

$$logS_t = logS_0 + (\alpha - \tfrac{1}{2}\sigma^2)t + \sigma z_t$$

$$S_t = S_0 e^{(\alpha - \frac{1}{2}\sigma^2)dt + \sigma z_t} \quad (3.56)$$

$$S_t = S_0 e^{(\alpha - \frac{1}{2}\sigma^2)dt + \sigma z_t} \quad (3.57)$$

The process S satisfies the stochastic differential equation which is a geometric Brownian motion with parameter $\mu = \sigma - \frac{1}{2}\sigma^2$. Hence the stock price becomes:

$$S_t = S_0 e^{(\mu t + \sigma z_t)} \quad (3.58)$$

3.6 Numerical Approach

With regards to European contracts there exist a closed form solutions to price such contract, for American and Asian contracts there exist no closed form solutions so numerical approach or method is adopted for a market participant to price such contracts. Among these numerical approach or method are Monte Carlo Simulations, Finite Difference Approximation and Risk Neutral Valuation. This paper incorporates a survival rate(rate at which an insured becomes multi-morbid) into the Black-Scholes model and use that to value the surrender option embedded in the life insurance contract. The model is solved using the numerical approach of the finite difference approximation, of which the solution that will be obtained from the implicit method, explicit method and Crank Nicolson method will be compared and analysed.

3.7 Black-Scholes Analysis

Black-Scholes analysis is established based on two concepts:

- The concept of hedging
- The concept of arbitrage

The Concept of Hedging

Is an investment position intended to offset potential losses or gains that may be incurred by a company on investment. In simple language a hedge is used to reduce any substantial losses or gains suffered by an individual or an organisation. A hedge can be constructed from many types of financial instruments, including stock, exchange traded funds, insurance, forward contracts, swaps options many types of over-the-counter and derivative products and futures contracts. Hedging is the practice of taking a position in one market to offset and balance against the risk adopted by assuming a position in a contract or opposing market or investment. Delta Hedging: In finance, it is the process of setting or keeping the delta of the portfolio as close to zero as possible. In practice, maintaining a zero delta is very complex because there are risk associated with re-hedging on large movement in the underlying stocks prices and research indicates portfolios tend to have lower cash flows if re-hedged too frequently. Mathematically delta hedging is given as:

$$\Delta = \frac{\partial V}{\partial A}$$

The Concept of Arbitrage

Is the practice of taking advantage of a price difference between two or more markets. Striking a combination of matching deals that capitalize upon the imbalance, the profit being the difference between the market prices. An arbitrage can also be defined as a transaction that involves no negative cash flow at any probabilistic or temporal state and a positive cash flow in at least one state. In simple terms it is the possibility of risk-free profit after transaction cost, for instance an arbitrage is present when there is the opportunity to instantaneously buy low and sell high. In principle and in academic use, an arbitrage is risk-free; in common use, as in statistical arbitrage, it may refer to expected profit though losses may occur and in practice, there are always risk in arbitrage, some minor(such as fluctuations of prices decreasing profit margin) some major (such as devaluation of currency or derivatives).

3.8 Derivation of Black-Scholes Model

Black-Scholes assumed that one risky asset like stock, and one riskless asset like cash or bond are both traded in the market .

1. There is a constant interest rate on the riskless asset.
2. The instantaneous log returns of the stock price is an infinitesimal random walk with drift.
3. The stock is a non-dividend paying stock
4. There are no arbitrage opportunities(that is, there is no way to make a riskless profit).
5. Borrowing and lending of stock and cash at a riskless rate and at a fractional amount are allowed
6. Short selling is possible.
7. The market is frictionless

Bases on the model assumption above the stock price follows a geometric Brownian motion:

$$\frac{dA}{A} = \mu dt + \sigma dW \tag{3.59}$$

Where W is a stochastic variable(Brownian motion). Note that W, and consequently its infinitesimal increment dW, represents the only source of uncertainty in the price history of the stock. Intuitively, W(t) is a process that wiggles up and down in such a random way that its expected change over any time interval is zero.(In addition its variance over time T is equal to 1). A good discrete analogue for W is a simple random walk, thus equation 3.85 above states that the infinitesimal rate of return on the stock has an expected value of μdt and a variance of $\sigma^2 dt$.

The payoff of an option V(A,t) at maturity is known. To find its value at an earlier time we need to know how V evolves as a function of S and t by Ito's lemma, for two variables we have:

$$dV = (\frac{\partial V}{\partial A}\mu A + \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial A^2}\sigma^2 A^2)dt + \frac{\partial V}{\partial A}\sigma A dW \tag{3.60}$$

Now let take a portfolio; delta hedging portfolio comprising of a short position in one option and a long position of $\frac{\partial V}{\partial A}$ shares at time t. The value of these holdings is:

$$\Pi = -V + \frac{\partial V}{\partial A}A \tag{3.61}$$

During the period of time $[t,t+\Delta t]$ all the gains or loss from differences in the values of the portfolio is:

$$\Delta\Pi = -\Delta V + \frac{\partial V}{\partial A}\Delta A$$

(3.62) Now we discretize the equation for $\frac{dA}{A}$ and dV by replacing differentials with deltas:

$$\Delta A = \mu A\Delta t + \sigma A\Delta W \tag{3.63}$$

$$\Delta V = (\frac{\partial V}{\partial A}\mu A + \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial A^2}\sigma^2 A^2)\Delta t + \frac{\partial V}{\partial A}\sigma A\Delta W \tag{3.64}$$

And appropriately substitute equation 3.63 and 3.64 into 3.62 for $\Delta\pi$ we have:

$$\Delta\Pi = (-\frac{\partial V}{\partial t} - \frac{1}{2}\frac{\partial^2 V}{\partial A^2}\sigma^2 A^2)\Delta t \tag{3.65}$$

Notice that ΔW term has vanished. Thus uncertainty has been eliminated and the portfolio is effectively riskless.In order for the portfolio to maintain its riskless property, it must be rebalanced at every point in time as $\frac{\partial V}{\partial A}$ will not remain the same for different time values of t. Thus shares need to be bought and sold continuously in fractional amounts as was stated in the assumption. The returns on other instruments which are riskless must be the same as the returns on the portfolio; if not it creates room for arbitrage opportunities. Taking returns of the riskless interest rate r we should get during the time period $[t,t + \Delta t]$

$$r\Delta\Pi\Delta t = \Delta\Pi$$

when we equate the two equations that is, equation 3.61 and 3.65 for $\Delta\Pi$ we obtain

$$(-\frac{\partial V}{\partial t} - \frac{1}{2}\frac{\partial^2 V}{\partial A^2}\sigma^2 A^2)\Delta t = r(-V + \frac{\partial V}{\partial A})\Delta t$$

Simplifying, gives us the Black-Scholes partial derivative

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 A^2 \frac{\partial^2 V}{\partial A^2} + rA\frac{\partial V}{\partial A} - rV = 0 \tag{3.66}$$

V is the option price which is the function of the stock price S and time t V(S,t), the riskless rate is r, and the volatility is σ The option is hedge by constantly buying and selling the trading asset in order to eliminate risk. This act of hedge means that there is only one correct price for the option, as given by the Black-Scholes formula. The Black-Scholes is the fundamentals of derivatives and is commonly used by practitioners because of it solid interpretations.

The equation can be rewritten as:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 A^2 \frac{\partial^2 V}{\partial A^2} = rV - rA\frac{\partial V}{\partial A} \tag{3.67}$$

3.8.1 Dividend Paying Asset

The contract pays no dividend . We will therefore cater for these dividends which are paid during the life of the contract.

Let λ represent the known dividend. Meaning the policy holder receives a dividend $\lambda A\Delta t$ with interval of time Δt. The value of the share is reduced after paying dividend therefore making the returns r becomes $(r-\lambda)$. So the Geometric Brownian motion from equation 3.85 becomes:

$$dA = (r - \lambda)Adt + \sigma AdX \tag{3.68}$$

Hence the Black-Scholes equation becomes

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 A^2 \frac{\partial^2 V}{\partial A^2} + (r - \lambda) A \frac{\partial V}{\partial A} - rV = 0 \tag{3.69}$$

For a continuous dividend asset, replace r with $r - \lambda$ Hence we have:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 A^2 \frac{\partial^2 V}{\partial A^2} + rA \frac{\partial V}{\partial A} - rV = 0 \tag{3.70}$$

3.9 Survival rate and Stock price

The approach described here involves a Black-Scholes dividend paying asset price, the short rate and a survival rate from a hazard function of a non-monotonic hazard. The survival rate from a hazard function of a non-monotonic hazard actually determines the rate of an insured being multi-morbid. Since embedded in life insurance contracts is an American option, which is an American put option that gives the holder the right to sell the contract back to the insurer. Also being multi-morbid calls for some form of financial burden or obligation, consequently surrendering is done base on the policy holders rate of being multi-morbid.

3.10 Assumptions of the Rate of Multi-morbid

1. The rate of being multi-morbid s, is a probability that lies between 0 and 1.
2. The multi-morbid rate is the median rate of being multi-morbid for all rates as time changes or whiles the contract is still in place
3. The addition of the dividend rate λ and the multi-morbid rate s is less than the short rate r mathematically ($r > s$)

Hence the Black-Scholes model becomes:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 A^2 \frac{\partial^2 V}{\partial A^2} + (r - s) A \frac{\partial V}{\partial A} - rV = 0 \tag{3.71}$$

3.11 Finite Difference Approximation

In the grid form let $V(t,A)$ be $V(n,m)$, the expansions of $V(t,A+\Delta A)$ and $V(t,A-\Delta A)$ in Taylors series are:

$$V(t,A+\Delta A) = V(t,A) + \frac{\partial V}{\partial A}\Delta A + \frac{1}{2}\frac{\partial^2 V}{\partial A^2}\Delta A^2 + \frac{1}{6}\frac{\partial^3 V}{\partial A^3}\Delta A^3 + 0(\Delta A^4) \quad (3.72)$$

$$V(t,A-\Delta A) = V(t,A) - \frac{\partial V}{\partial A}\Delta A + \frac{1}{2}\frac{\partial^2 V}{\partial A^2}\Delta A^2 - \frac{1}{6}\frac{\partial^3 V}{\partial A^3}\Delta A^3 + 0(\Delta A^4) \quad (3.73)$$

Using equation 3.72, the forward difference is given by

$$\frac{\partial V}{\partial A}(t,A) = \frac{V(t,A+\Delta A)-V(t,A)}{\Delta A} + 0(\Delta A)$$

$$\approx \frac{V_{n,m+1} - V_{n,m}}{\Delta A} \quad (3.74)$$

Similarly equation 3.73 gives the corresponding backward difference as

$$\frac{\partial V}{\partial A}(t,A) = \frac{V(t,A)-V(t,A-\Delta A)}{\Delta A} + 0(\Delta A)$$

$$\approx \frac{V_{n,m} - V_{n,m-1}}{\Delta A} \quad (3.75)$$

Subtracting equation 3.73 from 3.72 we get

$$\frac{\partial V}{\partial A}(t,A) = \frac{V(t,A+\Delta A)-V(t,A-\Delta A)}{2\Delta A} + 0(\Delta A^2)$$

$$\approx \frac{V_{n,m+1} - V_{n,m-1}}{2\Delta A} \quad (3.76)$$

Using central difference method to get the second order partial derivative. We sum equation 3.72 and 3.73 and it becomes

$$\frac{\partial^2 V}{\partial A^2}(t,A) = \frac{V(t,A+\Delta A)-2V(t,A)+V(t,A-\Delta A)}{2\Delta A} + 0(\Delta A^2)$$

$$\approx \frac{V_{n,m+1} - 2V_{n,m} + V_{n,m-1}}{\Delta A^2} \quad (3.77)$$

Expanding $V(t+\Delta t,A)$ in Taylor's series

$$V(t+\Delta t,A) = V(t,A) - \frac{\partial V}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 V}{\partial t^2}\Delta t^2 - \frac{1}{6}\frac{\partial^3 V}{\partial t^3}\Delta t^3 + 0(\Delta t^4) \quad (3.78)$$

Time is approximated using the forward difference

$$\frac{\partial V}{\partial t}(t,A) = \frac{V(t+\Delta t,A)-V(t,A)}{\Delta t} + 0(\Delta t)$$

$$\approx \frac{V_{n+1,m} - V_{n,m}}{\Delta t} \tag{3.79}$$

Boundary and Initial Conditions

Boundary conditions as well as initial conditions are to be considered when solving a PDE problem, if not the solution to the PDE will be infinite. These conditions need to be specified. Example the payoff of the European put option is given by $\max(K - S_T, 0)$ The put option is equal to the strike price when the underlying asset is out-of- -the-money. that is,

$$V_{n,0} = K \qquad for\ n = 0,1,\ldots,N \tag{3.80}$$

The put option value is zero when the price of the asset increases. Therefore, we choose $A_{max} = A_M$ and get

$$V_{n,M} = 0 \qquad for\ n = 0,1,\ldots,N. \tag{3.81}$$

Initial condition can be imposed and the price of the put option is known in advance

$$V_{N,m} = max(K - m\Delta A, 0) \qquad for\ m = 0,1,\ldots,M. \tag{3.82}$$

This approach can be used to solve the price of the American call. Hence equation 3.82 becomes

$$V_{N,m} = max(m\Delta A - K, 0) \qquad for\ m = 0,1,\ldots,M \tag{3.83}$$

3.11.1 Approaches of Finite Difference Scheme

Explicit Method

An expression can be formed to give the next value $V_{n,m}$ explicitly in the form $V_{m-1,n+1}, V_{m,n+1}$ and $V_{m+1,n+1}$.if we know the option price at maturity. By knowing the maturity the equation, is dicretized to get the expression $V_{m,n}$ explicitly in terms

of the given values $V_{m-1,n+1}$, $V_{m,n+1}$ and $V_{m+1,n+1}$.

Equation 3.72 is discretizing the time by using forward approximation and the stock price by using central difference approximation. This gives us

$$\frac{V_{n+1,m}-V_{n,m}}{\Delta t} + \frac{(r-s)m\Delta A}{2\Delta A}[V_{n+1,m+1} - V_{n+1,m-1}]$$
$$+\frac{\sigma^2 m^2 \Delta A^2}{2\Delta A^2}[V_{n+1,m-1} - 2V_{n+1,m} + V_{n+1,m+1}] = rV_{n,m} \tag{3.84}$$

This gives

$$V_{n,m} = \frac{1}{1+r\Delta t}[\beta_{1m}V_{n+1,m-1} + \beta_{2m}V_{n+1,m} + \beta_{3m}V_{n+1,m+1}] \tag{3.85}$$

for n = $0,1,\ldots,N-1$ and $m = 1,2,\ldots,M-1$

The accuracy of the forward approximation for the time is $0(\Delta t)$ and that of the central approximation for the stock is $0(\Delta S^2)$. The accuracy of the is $0(\Delta t \Delta S^2)$. The weights in equation 3.85 are given by

$$\begin{aligned}
\beta_{1m} &= \frac{1}{2}\sigma^2 m^2 \Delta t - \frac{1}{2}(r-s)m\Delta t,\\
\beta_{2m} &= 1 - \sigma^2 m^2 \Delta t,\\
\beta_{3m} &= \frac{1}{2}(r-s)m\Delta t + \frac{1}{2}\sigma^2 m^2 \Delta t.
\end{aligned} \tag{3.86}$$

The weights add up to one. These are probabilities of the risk of the three prices $A - \Delta A, A$, $A + \Delta A$ at $t + \Delta t$. Is assumed that the returns of the asset is the same in the risk neutral world. The probabilities of the risk neutral need to be positive for the explicit method of the finite difference approximation to work well. Negative probabilities of the risk neutral in the explicit methods prevent it from working well. Therefore producing results which does not converge to the solution of the partial differential equation. The condition to have non-negative probabilities is that $\sigma^2 m^2 \Delta t < 1$ and $(r - s) < \sigma^2 m$ Equation 3.120 gives a system of tridiagonal matrix which is $Au + \varepsilon = b$. The boundary condition gives rise to vector ε when $m = 0$ and M for all $n > 0$. This is given us

$$\begin{pmatrix} \beta_{20} & \beta_{30} & 0 & 0 & 0 & \dots & 0 \\ \beta_{11} & \beta_{21} & \beta_{31} & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \beta_{1M-1} & \beta_{2M-1} & \beta_{3M-1} \\ 0 & 0 & 0 & \dots & 0 & \beta_{1M} & \beta_{2M} \end{pmatrix} \begin{pmatrix} V_{n+1,0} \\ V_{n+1,1} \\ \dots \\ V_{n+1,M-1} \\ V_{n+1,M} \end{pmatrix} = \begin{pmatrix} V_{n,0} \\ V_{n,1} \\ \dots \\ V_{n,M-1} \\ V_{n,M} \end{pmatrix} \quad (3.87)$$

The system of equation is given us $AV_{n+1,m} = V_{n,m}, for m=0,1,...,$M and error term is neglected.

At maturity $V_{n+1,m}$ is fixed based on the initial condition. Solving backward for $V_{n,m} (m = 0,1,...,M)$. Matrix A is used because of the probabilities, $\beta_{km} (k = 1,2,3)$ are fixed. The option price is obtained by the backward iteration.

Iterating to find the solution errors are generated when solving the difference equation to get numerical solution. The system become stable if the errors are not magnifying while we iterate. Accuracy and stability are the two problems encountered when using numerical approach to solve for the solution to a PDE. The idea is we want to get accurate solution with little computations. This makes stability and accuracy an important factors to be considered when looking for numerical solution to a solving a PDE.

3.11.2 Stability Analysis

The errors generated in the approximation are the fundamental source of errors which are,error of truncation in the discretization of both the stock price and time. The meaning of the error of truncation basically means the numerical method solve a problem which is different from what we are solving. The factors of the approach are:

- Consistency
- Stability
- Convergence

Lax Equivalence Theorem linked together these factors. It states that with a linear value problem and an appropriate finite difference approach, convergence of the approach will depend on the stability of the scheme.

Stability Conditions

Let $V_{n+1} = AV_n$ be a system of equations. Matrix A and the column vectors V_{n+1} and V_n are as represented in equation 3.86 we have

$$\begin{aligned} V_n &= AV_{n-1} \\ &= A^2V_{n-2} \\ &\cdots \\ &= A^2V_0 \qquad \textit{for } n = 1,2,\ldots,N \end{aligned} \tag{3.88}$$

Where V_0 is the vector of initial values. We are concerned with the stability and we investigate the propagation of a perturbation. Perturb the vector of initial values V_0 to V_0^*. The exact solution at the n^{th} time-row then be

$$V_n^* = A^n V_0^* \tag{3.89}$$

ε, which is the error vector or perturbation is given by

$$\varepsilon = V^{*} - V$$

and the use of the perturbation vector in equation 3.87 and 3.88 gives

$$\begin{aligned}\varepsilon_n &= V_n^{*} - V_n \\ &= A^n (V_0^{*} - V_0) \qquad (3.90) \\ &= A^n \varepsilon_0 \qquad \textit{for } n = 1,2,...,N.\end{aligned}$$

Hence, for compatible matrix and vector norms

$$\|\varepsilon_n\| \le \|A^n\|\|\varepsilon_0\|.$$

The Eigenvalues of a Common Tridiagonal Matrix

The eigenvalue of the $N \times N$ matrix

$$\begin{pmatrix} y & z & & & \\ x & y & z & & \\ & \dots & \dots & \dots & \\ & & x & y & z \\ & & & x & y \end{pmatrix} \qquad (3.91)$$

are $\lambda_n = y + 2[\sqrt{xz}]cos\frac{n\pi}{N+1}, \quad for \quad n = 1,2,...,N$, where x,y and z may be real or complex

The Stability Issue of Explicit Method

Matrix A in equation 3.120 is used to analyse the explicit finite difference method's stability, where the β_{km}, for $k = 1,2,3$ are given by equation 3.121. Matrix A is symmetric and real. If λ_n is the n^{th} eigenvalue of matrix A, then we get

$$\|A\|_2 = \rho(A) = max|\lambda_n|$$

The eigenvalues λ_n gives

$$\lambda_n = \beta_{2m} + 2[\beta_{1m}\beta_{3m}]^{\frac{1}{2}} cos\frac{nx}{N}, \qquad for \quad n = 1, 2, ..., N-1, \tag{3.92}$$

The values of $\beta^0 s$, are substituted to obtain

$$\lambda_n = 1 - \sigma^2 m^2 \Delta t + \sigma^2 m^2 \Delta t [1 - \frac{r^2}{\sigma^4 m^2}]^{\frac{1}{2}} [1 - 2sin^2 \frac{n\pi}{2N}] \tag{3.93}$$

for $n = 1,2,...,N-1$. Further, applying the binomial theorem and ignoring some terms to obtain

$$\lambda_n \approx 1 - 2\sigma^2 m^2 \Delta t sin^2 \tfrac{n\pi}{2N}.$$

Hence the equations become stable when

$$\|A\|_2 = max|1 - 2\sigma^2 m^2 \Delta t sin^2 \tfrac{n\pi}{2N}| \leq 1.$$

That is,

$$-1 \leq 1 - 2\sigma^2 m^2 \Delta t sin^2 \frac{n\pi}{2N} \leq 1. \qquad for \quad n = 1, 2, ..., N-1 \tag{3.94}$$

as $\Delta t \to 0, N \to \infty$ and $sin^2 \frac{(N-1)\pi}{2N} \to 1$. Hence

$$0 \leq \sigma^2 m^2 \Delta t \leq 1 \tag{3.95}$$

Alternatively, when $1 - \sigma^2 m^2 \Delta t \geq 0$, then $\sigma^2 m^2 \Delta t \leq 1$, and

$$\|A\|_\infty = \beta_{1m} + \beta_{2m} + \beta_{3m} = 1.$$

When $1 - \sigma^2 m^2 \Delta t < 0, \sigma^2 m^2 \Delta t > 1$, then $|\sigma^2 m^2 \Delta t| = \sigma^2 m^2 \Delta t - 1$, and

$$\|A\|_\infty = 2\sigma^2 m^2 \Delta t - 1 > 1$$

Therefore by Lax's Equivalence theorem, the scheme is stable, convergent and consistent for $0 \le \sigma^2 m^2 \Delta t \le 1$.

In equation 3.120, the other condition is that $r < \sigma^2 m$. These conditions are necessary for the weights $\beta_{nm}(k = 1,2,3)$ to be positive, otherwise, they will be

negative.

Implicit Method

We express $V_{n+1,m}$ implicit in-terms of the unknown $V_{n,m-1}, V_{n,m}$and $V_{n,m+1}$. We discretize the Black-Scholes PDE in equation 3.105 using the forward difference for time and central difference for the stock price to have

$$\frac{V_{n+1,m} - V_{n,m}}{\Delta t} + (r - s)m\Delta A \frac{[V_{n,m+1} - V_{n,m-1}]}{2\Delta A}$$
$$+ \frac{1}{2}\sigma^2 m^2 \Delta A^2 \frac{[V_{n,m+1} - 2V_{n,m} + V_{n,m-1}]}{2\Delta A^2} = rV_{n+1,m} \quad (3.96)$$

and re-arranging we have

$$V_{n+1,m} = \frac{1}{1 - r\Delta t}[\alpha_{1m} V_{n,m-1} + \alpha_{2m} V_{n,m} + \alpha_{3m} V_{n,m+1}] \quad (3.97)$$

for n = $0,1,...,N - 1$ and $m = 1,2,...,M - 1$

Similar to the explicit method, the implicit method is accurate to $0(\Delta t \Delta S^2)$. The parameters$\alpha'_{km} s$ for k=1,2,3 are given as

$$\alpha_{1m} = \frac{1}{2}(r - s)m\Delta t - \frac{1}{2}\sigma^2 m^2 \Delta t,$$
$$\alpha_{2m} = 1 + \sigma^2 m^2 \Delta t,$$
$$\alpha_{3m} = \frac{1}{2}(r - s)m\Delta t - \frac{1}{2}\sigma^2 m^2 \Delta t. \quad (3.98)$$

The system of equations can be expressed as a tridiagonal system

$$\begin{pmatrix} V_{n+1,0} \\ V_{n+1,1} \\ \vdots \\ V_{n+1,M-1} \\ V_{n+1,M} \end{pmatrix} = \begin{pmatrix} \alpha_{20} & \alpha_{30} & 0 & \cdots & 0 & 0 & 0 \\ \alpha_{11} & \alpha_{21} & \alpha_{31} & \cdots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \cdots & \alpha_{1M-1} & \alpha_{2M-1} & \alpha_{3M-1} \\ 0 & 0 & 0 & \cdots & 0 & \alpha_{1M} & \alpha_{2M} \end{pmatrix} \begin{pmatrix} V_{n,0} \\ V_{n,1} \\ \vdots \\ V_{n,M-1} \\ V_{n,M} \end{pmatrix} \quad (3.99)$$

which can be is rewritten as $AV_{n,m} = V_{n+1,m}$. for $m = 0,1,\ldots,M$. Let $V_{n,m} = V_n$, and solve for V_n given matrix A and column vector V_{n+1} and this gives $V_n = A^{-1}V_{n+1}$.

The matrix A has $\alpha_{2m} = 1 + \sigma^2 m^2 \Delta t$ in the diagonal which is positive. When the diagonal elements are multiplied it gives a non zero matrix. Hence the matrix is non singular. The system is solved by finding the inverse matrix A^{-1}. When the boundary conditions are applied together with equation 3.133, this gives rise to some changes in the elements of matrix A with $\alpha_{20},\alpha_{2M} = 1$ and $\alpha_{30},\alpha_{1M} = 0$.

Stability Issue of Implicit Method

The same principle and idea is used to the test the stability of the implicit finite difference approximation.

The eigenvalues λ_n are given by

$$\lambda_n = \alpha_{2m} + 2[\alpha_{1m}\alpha_{3m}]^{\frac{1}{2}} cos\frac{nx}{N}, \qquad for \quad n = 1, 2, ..., N-1, \tag{3.100}$$

The values of $\alpha^0 s$, is substituted and we have

$$\lambda_n = 1 + \sigma^2 m^2 \Delta t + \sigma^2 m^2 \Delta t[1 - \frac{r^2}{\sigma^4 m^2}]^{\frac{1}{2}}[1 - 2sin^2\frac{n\pi}{2N}] \tag{3.101}$$

for n = 1,2,...,N − 1. Furthermore, applying binomial theorem on the square root and ignoring some terms to obtain

$$\lambda_n \approx 1 + 2\sigma^2 m^2 \Delta t - 2\sigma^2 m^2 \Delta t sin^2 \tfrac{n\pi}{2N}.$$

$$\|A\|_2 = max|1 + 2\sigma^2 m^2 \Delta t - 2\sigma^2 m^2 \Delta t sin^2 \tfrac{n\pi}{2N}| \leq 1.$$

that is,

$$-1 \leq 1 - 2\sigma^2 m^2 \Delta t - 2\sigma^2 m^2 \Delta t sin^2 \frac{n\pi}{2N} \leq 1. \qquad for \quad n = 1, 2, ..., N-1 \tag{3.102}$$

as $\Delta t \to 0, N \to \infty$ and $sin^2 \frac{(N-1)\pi}{2N} \to 1$. Hence equation 3.138 reduces to |1| ≤ 1

$1 + \sigma^2 m^2 \Delta t \geq 0$ and kAk = 1.

Therefore by Lax's equivalence theorem the scheme is consistent convergent and unconditionally stable.

Crank Nicolson Method

This method is the average of the implicit and the explicit methods. Equation 3.119 is the explicit scheme and equation 3.133 is the implicit scheme. When the average of the two is taken we get

$$\tfrac{V_{n+1,m} - V_{n,m}}{\Delta t} + \tfrac{(r-s)m\Delta A}{4\Delta A}[V_{n+1,m+1} - V_{n+1,m-1} + V_{n,m+1} - V_{n+1,m+1}]$$

$$+ \tfrac{\sigma^2 m^2 \Delta A^2}{4\Delta A^2}[V_{n,m-1} - 2V_{n,m} + V_{n,m+1} + V_{n+1,m-1} - 2V_{n+1,m} + V_{n+1,m+1}]$$

$$= \frac{1}{2}[rV_{n,m} + rVn + 1, m] \tag{3.103}$$

re-arranging we get

$$[\frac{1}{4}(r-s)m\Delta t - \frac{1}{4}\sigma^2 m^2 \Delta t]V_{n,m-1} + [1 + \frac{1}{2}r\Delta t + \frac{1}{2}\sigma^2 m^2 \Delta t]V_{n,m} + [-\frac{1}{4}\sigma^2 m^2 \Delta t - \frac{1}{4}(r-s)m\Delta t]V_{n,m+1}$$

$$= [1 - \frac{1}{2}r\Delta t - \frac{1}{2}\sigma^2 m^2 \Delta t]V_{n+1,m} + [\frac{1}{4}(r-s)m\Delta t + \frac{1}{4}\sigma^2 m^2 \Delta t]V_{n+1,m+1} + [\frac{1}{4}\sigma^2 m^2 \Delta t - \frac{1}{4}(r-s)m\Delta t]V_{n+1,m-1} \quad (3.104)$$

and we simplify to get

$$\rho_{1m}V_{n,m-1} + \rho_{2m}V_{n,m} + \rho_{3m}V_{n,m+1} = X_{1m}V_{n+1,m-1} + X_{2m}V_{n+1,m} + X_{3m}V_{n+1,m+1} \quad (3.105)$$

for $n = 0,1,...,N-1$ and $m = 1,2,...,M-1$. Then, the parameters ρ_{km} and X_{km} for $k = 1,2,3$ are given as

$$\begin{aligned}
\rho_{1m} &= \frac{1}{4}(r-s)m\Delta t - \frac{1}{4}\sigma^2 m^2 \Delta t\\
\rho_{2m} &= 1 + \frac{1}{2}r\Delta t + \frac{1}{2}\sigma^2 m^2 \Delta t\\
\rho_{3m} &= -\frac{1}{4}\sigma^2 m^2 \Delta t - \frac{1}{4} \quad \Delta t\\
X_{1m} &= \frac{1}{4}\sigma^2 m^2 \Delta t - \frac{1}{4}(r-s)m\Delta t\\
X_{2m} &= 1 - \frac{1}{2}r\Delta t - \frac{1}{2}\sigma^2 m^2 \Delta t\\
X_{3m} &= \frac{1}{4}(r-s)m\Delta t + \frac{1}{4}\sigma^2 m^2 \Delta t
\end{aligned} \quad (3.106)$$

We express the system of equation in equation 3.141 as $CV_n = DV_{n+1}$. This gives a tridiagonal matrix giving by

$$\begin{pmatrix} \rho_{20} & \rho_{30} & 0 & \cdots & 0 & 0 & 0 \\ \rho_{11} & \rho_{21} & \rho_{31} & \cdots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix} \begin{pmatrix} V_{n,0} \\ V_{n,1} \\ \ldots \end{pmatrix}$$

$$
\begin{pmatrix}
\vdots & & & & & & \\
0 & 0 & 0 & \cdots & \rho_{1M-1} & \rho_{2M-1} & \rho_{3M-1} \\
0 & 0 & 0 & \cdots & 0 & \rho_{1M} & \rho_{2M}
\end{pmatrix}
\begin{pmatrix}
\vdots \\
V_{n,M-1} \\
V_{n,M}
\end{pmatrix}
\tag{3.107}
$$

$$
=\begin{pmatrix}
X_{20} & X_{30} & 0 & \cdots & 0 & 0 & 0 \\
X_{11} & X_{21} & X_{31} & \cdots & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & X_{1M-1} & X_{2M-1} & X_{3M-1} \\
0 & 0 & 0 & \cdots & 0 & X_{1M} & X_{2M}
\end{pmatrix}
\begin{pmatrix}
V_{n+1,0} \\
V_{n+1,1} \\
\cdots \\
V_{n+1,M-1} \\
V_{n+1,M}
\end{pmatrix}
$$

The diagonal entries of matrix C is $\rho_{2m} = 1 + r\Delta/2 + \sigma^2 m^2 \Delta t/2$ and this is always positive which makes the diagonal elements non zero. Since the entries are non zero ia obvious that the matrix is non singular.

Checking Accuracy of the Crank-Nicolson method

The approximation from the Taylor's theorem gives an error of truncation which affect the scheme's accuracy. The Crank-Nicolson method of approximation has an accuracy up to $0(\Delta t^2, \Delta S^2)$. We show this accuracy by equating the central difference and th symmetric central difference at $V_{n+\frac{1}{2},m} = V(t+\Delta t/2, A)$. We expand $V_{n+\frac{1}{2}m}$ to yield

$$V_{n+1,m} = V_{n+\frac{1}{2},m} + \frac{1}{2}\frac{\partial V}{\partial t}\Delta t + 0(\Delta t^2) \tag{3.108}$$

and expanding $V_{n,m} = V_{n+\frac{1}{2},m}$ gives

$$V_{n,m} = V_{n+\frac{1}{2},m} - \frac{1}{2}\frac{\partial V}{\partial t}\Delta t + 0(\Delta t^2) \tag{3.109}$$

The two equations are added together and it yields

$$\tfrac{1}{2}[V_{n,m} + V_{n+1,m}] = V_{n+\frac{1}{2},m} + 0(\Delta t^2)$$

$$V_{n+\frac{1}{2},m-1} - 2V_{n+\frac{1}{2},m} + V_{n+\frac{1}{2},m+1}$$

$$= \frac{1}{2}[V_{n,m-1} - 2V_{n,m} + V_{n,m+1}] + \frac{1}{2}[V_{n+1,m-1} - 2V_{n+1,m} + V_{n+1,m+1}] + 0(\Delta t^2). \quad (3.110)$$

Dividing by (ΔA^2)

the equality will be

$$\frac{\partial^2 V(t+\frac{1}{2}\Delta t, A)}{\partial A^2} = \frac{1}{2}[\frac{\partial^2 V(t,A)}{\partial A^2} + \frac{\partial^2 V(t+\Delta t, A)}{\partial A^2}] + 0(\Delta t^2, \Delta A^2) \tag{3.111}$$

The m which is the subscript is an arbitrary and the central difference approximation is as follows.

$$V_{n+\frac{1}{2},m+1} - V_{n+\frac{1}{2},m-1}$$

$$= \frac{1}{2}[V_{n,m+1} - V_{n,m-1}] + \frac{1}{2}[V_{n+1,m+1} - V_{n+1,m-1}] + 0(\Delta t^2) \tag{3.112}$$

Dividing by $(2\Delta A)$ we obtain the equality

$$\frac{\partial V(t+\frac{1}{2}\Delta t, A)}{\partial A} = \frac{1}{2}[\frac{\partial V(t,A)}{\partial A} + \frac{\partial V(t+\Delta t, A)}{\partial A}] + 0(\Delta t^2, \Delta A^2) \tag{3.113}$$

Now we subtracting equation 3.113 from equation 3.112 to obtain the approximation of $\partial V/\partial t$ centered at $(t+\frac{1}{2}\Delta t, A)$

$$\frac{\partial V(t+\frac{1}{2}\Delta t, A)}{\partial t} = \frac{V_{n+1,m} - V_{n,m}}{\Delta t} + 0(\Delta t^2) \tag{3.114}$$

Therefore the Black-Scholes equation at $(t+\frac{1}{2}\Delta t, A)$ is approximated as

$$\begin{aligned} &\frac{V_{n+1,m} - V_{n,m}}{\Delta t} + \frac{rm\Delta A}{4\Delta A}[V_{n,m+1} - V_{n,m-1} + V_{n+1,m+1} - V_{n+1,m-1}] \\ &+ \frac{\sigma^2 m^2 \Delta A^2}{4\Delta A^2}[V_{n,m-1} - 2V_{n,m} + V_{n+1,m-1} - 2V_{n+1,m} + V_{n+1,m+1}] \\ &= rV_{n,m} \end{aligned} \tag{3.115}$$

and re-arranging, we get an equation of the form of equation 3.115, the leading error of the scheme which is given as $0(\Delta t^2, \Delta A^2)$.

CHAPTER 4

ANALYSIS

4.1 Introduction

In this chapter, the application of the Black-Scholes partial differential equation is looked at. The Black-Scholes PDE is solved using the finite difference method. This is achieved by approximating the PDE to obtain a systems of algebraic equations. This is done in order to obtain numerical solutions to PDE's. The method is very powerful and comes along with simple techniques that is able to generate accurate solutions to PDE's arising in financial and other physical sciences

In solving a Black-Scholes PDE the implicit, explicit and Crank-Nicolson methods are used, and these methods are different in terms of stability, consistency and accuracy. In this chapter we compare and contrast the convergence of the original Black-Scholes PDE and the Black-Scholes PDE with the survival rate incorporated in the valuation of surrender option.

4.2 Analysis Using Matlab

The tridiagonal matrix obtained is very large. Large matrices requires high memory and its execution also takes a lot of computer time. Matlab is designed in such a way to accommodate all these flaw backs. The implicit finite difference method can be expressed as $f_n = A^{-1} f_{n+1}$.

Matlab can solve for the inverse of the matrix.
H

Table 4.1: Eigenvalues of the Implicit Finite Difference Approximation as N→ ∞

N=2000		N=4000		N=5000		N=6000		N=7000	
j	λ	j	λ	j	λ	j	λ	j	λ

1991	1.0197	3991	1.0099	4991	1.0079	5991	1.0066	6991	1.0057
1992	1.0156	3922	1.0078	4992	1.0063	5992	1.0052	6992	1.0045
1993	1.0120	3993	1.0060	4993	1.0048	5993	1.0040	6993	1.0034
1994	1.0088	3994	1.0044	4994	1.0035	5994	1.0029	6994	1.0025
1995	1.0061	3995	1.0031	4995	1.0025	5995	1.0021	6995	1.0018
1996	1.0039	3996	1.0020	4996	1.0016	5996	1.0013	6996	1.0011
1997	1.002	3997	1.0011	4997	1.0009	5997	1.0007	6997	1.0006
1998	1.0010	3998	1.0005	4998	1.0004	5998	1.0003	6998	1.0003
1999	1.0003	3999	1.0001	4999	1.0001	5999	1.0001	6999	1.0001

4.3 Stability Analysis of Implicit Finite Difference Approximation

The eigenvalues of the the resulting matrix are given in Table 4.1 for volatility of 0.2231 and surrender time of 2 years for maturity of 30 years.

N=10000		N=11000	
j	λ	j	λ
9991	1.0040	10091	1.0036
9992	1.0031	10092	1.0029
9993	1.0024	10093	1.0022
9994	1.0018	10094	1.0016
9995	1.0012	10095	1.0011
9996	1.0008	10096	1.0007
9997	1.0004	10097	1.0004

9998	1.0002	10098	1.0002
9999	1.0001	10099	1.0000

It could be verified from table 4.1 that, as the number of time steps N increases, the eigenvalue approaches i and this shows that the implicit finite difference scheme is unconditionally stable.

4.4 Stability of Crank-Nicholson Approximation

The table below shows the eigenvalues of the matrix of the scheme as $N \to \infty$

Table 4.2 indicates that as $N \to \infty$, the eigenvalues approaches one showing the stability of Crank-Nicolson's approximation. Also the Crank-Nicolson approximation is with the accuracy of $0(\Delta t^2, \Delta A^2)$ and that also indicates that how accurate the results is to the actual value.

Table 4.2: The eigenvalues of the Crank-Nicholson method as $N \to \infty$

N=100		N=500		N=1000		N=2000		N=4000	
j	λ_j	j	λ_j	j	λ_j	j	λ_j	j	λ_j
93	1.0572	493	1.0119	993	1.0060	1993	1.0030	3993	1.0015
94	1.0398	494	1.0088	994	1.0044	1994	1.0022	3994	1.0011
95	1.0284	495	1.0062	995	1.0031	1995	1.0016	3995	1.0008
96	1.0189	496	1.0040	996	1.0020	1996	1.0010	3996	1.0005
97	1.0111	497	1.0023	997	1.0012	1997	1.0006	3997	1.0003
98	1.0055	498	1.0011	998	1.0006	1998	1.0003	3998	1.0001
99	1.0020	499	1.0004	999	1.0002	1999	1.0001	3999	1.0000

4.5 Comparing the Convergence of the Implicit and Crank-Nicolson's Method for Valuation of Surrender Option with no Dividend

The convergence of the fully implicit and the Crank-Nicolson with relation to the Black-Scholes value of surrender options were considered earlier in chapter 3. Table 4.3 shows the value of life insurance contract containing surrender option with the data from Life Insurance Company A and Table 4.4 shows that of the data from
Life Insurance Company B. The data from company A are as follows: Asset price,A = 50, Strike price, K = 52, Risk-free interest rate, r = 0.05, Surrender period, t = 2 years, Maturity period, T = 30 years, Volatility, σ = 0.2331 and the Dividend payment rate, φ = 0.03. The surrender value of the life insurance contract is 5.4650 with the value at maturity being 8.22 for non-dividend paying asset.

The data from Insurance Company B are as follows:
Asset price, A = 250, Strike price, K = 260, Risk-free interest rate, r = 0.06, Surrender period, t = 7 years, Maturity period, T = 30 years, Volatility, σ = 0.24. The surrender value of the life insurance contract is 36.04 with the value at maturity being 40.15 for non-dividend paying asset.

Table 4.3: The comparison of the two methods in the valuation of surrender options with no-dividend payment for company A. Surrender value at t=2 years, Expected value = 5.4650.

Number of steps	Fully Implicit	Crank- Nicolson
30	5.3770(0.0880)	5.4204(0.04204)
150	5.4413(0.0237)	5.4503(0.0147)
210	5.4462(0.0188)	5.4531(0.1191)
330	5.4501(0.0149)	5.4546(0.0104)
390	5.4511(0.0139)	5.4550(0.0100)
450	5.4518(0.0132)	5.4552(0.0098)
570	5.4529(0.0121)	5.4556(0.0094)

630	5.4533(0.0177)	5.4557(0.0093)
690	5.4536(0.0114)	5.4558(0.0092)
720	5.4537(0.0113)	5.4559(0.0091)
780	5.4554(0.0112)	5.4559(0.0091)
810	5.4541(0.0109)	5.4560(0.0090)
840	5.4542(0.0108)	5.4560(0.0090)
870	5.4543(0.0107)	5.4560(0.0090)
1000	5.4546(0.0104)	5.4561(0.0089)

Table 4.4: The valuation of surrender option with no-dividend payment at maturity(T=30 years) for company A. Expected Value=8.22

Number of steps	Fully Implicit	Crank- Nicolson
100	7.4244(0.7956)	7.5107(0.7093)
250	7.5275(0.6925)	7.5622(0.6578)
400	5.7.5535(0.6665)	7.5757(0.6443)
550	7.5656(0.6544)	7.5817(0.6383)
650	7.5706(0.6494)	7.5842(0.6358)
700	7.5725(0.6475)	7.5851(0.6349)
800	7.5757(0.6443)	7.5867(0.6333)
1000	7.5801(0.6399)	7.5890(0.6310)
1150	7.5824(0.6376)	7.5901(0.6299)
1300	7.5842(0.6358)	7.5910(0.6290)
1500	7.5860(0.6340)	7.5919(0.6281)
1700	7.5874(0.6326)	7.5926(0.6274)
2000	7.5890(0.6310)	7.5934(0.6266)
3000	7.5919(0.6281)	7.5949(0.6251)

Table 4.5: The comparison of the two methods in the valuation of surrender option with no-dividend payment for company B. Surrender time at t=7 years. Expected value =36.04

Number of steps	Fully Implicit	Crank- Nicolson
30	34.4780(1.5620)	35.0667(0.9733)
90	35.2300(0.8100)	35.4022(0.6378)
150	35.3591(0.6809)	35.4687(0.5713)
200	35.4089(0.6311)	35.4923(0.5477)
250	35.4388(0.6012)	35.5052(0.5348)
300	35.4586(0.5814)	35.5146(0.5254)
350	35.4730(0.5670)	35.5209(0.5191)
400	35.4836(0.5564)	35.5256(0.5144)
500	35.4986(0.5414)	35.5323(0.5077)
600	35.5086(0.5314)	35.5367(0.5033)
700	35.5157(0.5243)	35.5399(0.5001)
800	35.5210(0.5190)	35.5423(0.4977)
900	35.5252(0.5148)	35.5441(0.4959)
970	35.5276(0.5124)	35.5452(0.4948)

Note: *In the tables above, the figures in the bracket are the differences between the actual values obtained from the various numerical methods.*

4.6 Comparing the Convergence of the Implicit and Crank-Nicolson's Method for Valuation of Surrender Option with survival rate

4.6.1 Numerical Methods for Surrender Option Valuation with Rate of Multimobidity for Company A

Simulation of Survival Data

Survival data is obtained through simulations, the simulation is done for 10000 times of which the confidence intervals of the last survival rate of each simulation is of importance as shown in the R code of the appendix c, The maximum of both the lower confidence and upper confidence intervals are then obtained of which any random number is picked from the interval and fix into the Black Scholes setup as the rate of an insured being multi-morbid and use in the valuation of surrender options as the first scenario. The averages of both the lower confidence and upper confidence intervals are then obtained of which any random number is selected from the interval and fix into the Black-Scholes setup as the rate of an insured being multi-morbid and use in the valuation of surrender options as the second scenario. Also the median of both the lower and upper confidence intervals are also obtained of which any random number is selected from the interval and fix into the Black-

Scholes setup and use in the valuation of surrender options as the third scenario. Lastly the minimum of the lower confidence and upper confidence intervals are then obtained of which any random number is selected from the interval and fix into the Black-Scholes setup as the rate of an insured being multi-morbid and use in the valuation of surrender options. The disparities among these valuations are discussed in chapter five.

Table 4.6: The comparison of the two methods in the valuation of surrender options with no-dividend payment for company A. Surrender value at t=2 years,with maximum rate of being multi-morbid between 0.436 and 0.691(ie, s=0.5) Expected value = 5.4650.

Number of steps	Fully Implicit	Crank- Nicolson
100	28.5885	77.9099
150	28.6136	77.9098
210	28.6279	77.9098
330	28.6409	77.9097
390	28.6444	77.9097
450	28.6470	77.9097
570	28.6505	77.9097
630	28.6518	77.9097
690	28.6521	77.9097
720	28.6330	77.9097
780	28.6541	77.9097

810	28.6545	77.9097
840	28.6548	77.9097
870	28.6551	77.9097
1000	28.6562	77.9097

Table 4.7: The comparison of the two methods in the valuation of surrender options with no-dividend payment for company A. Surrender value at t=2 years,with mean rate of being multi-morbid between 0.214 and 0.453(ie, s=0.3) Expected value = 5.4650.

Number of steps	Fully Implicit	Crank- Nicolson
100	19.8259	36.0963
150	19.8385	36.0981
210	19.8453	36.0990
330	19.8512	36.0994
390	19.8528	36.0994
450	19.8539	36.0994
570	19.8555	36.0995
630	19.8561	36.0995
690	19.8565	36.0995
720	19.8567	36.0995
780	19.8571	36.0995
810	19.8572	36.0995
840	19.8574	36.0995
870	19.8575	36.0995
1000	19.8558	36.0995

Table 4.8: The comparison of the two methods in the valuation of surrender options with no-dividend payment for company A. Surrender value at t=2 years,with median rate of being multi-morbid between 0.216 and 0.458(ie, s=0.25) Expected value = 5.4650.

Number of steps	Fully Implicit	Crank- Nicolson
100	17.4888	28.2978
150	17.2322	28.2999
210	17.2373	28.3009

330	17.2417	28.3014
390	17.2428	28.3014
450	17.2436	28.3014
570	17.2448	28.3015
630	17.2452	28.3015
690	17.2455	28.3015
720	17.2457	28.3015
780	17.2459	28.3016
810	17.2461	28.3016
840	17.2462	28.3016
870	17.2463	28.3016
1000	17.2466	28.3016

Table 4.9: The comparison of the two methods in the valuation of surrender options with no-dividend payment for company A. Surrender value at t=2 years,with minimum rate of being multi-morbid between 0.03 and 0.214 (ie, s=0.035) Expected value = 5.4650.

Number of steps	Fully Implicit	Crank- Nicolson
100	6.6783	6.7042
150	6.6850	6.7068
210	6.6888	6.7083
330	6.6918	6.7090
390	6.6693	6.7090
450	6.6931	6.7091
570	6.6939	6.7093
630	6.6942	6.7094
690	6.6944	6.7094
720	6.6946	6.7094
780	6.6947	6.7094
810	6.6948	6.7094
840	6.6949	6.7094
870	6.6950	6.7094
1000	6.6952	6.7094

Tables 4.3, 4.4 and 4.5 shows that the Crank-Nicolson finite scheme converges faster than the fully implicit finite scheme as N−→ ∞, The Crank-Nicolson finite difference approximation is closer to the value of the surrender value for large values of N than the Fully implicit finite difference approximation. (see figure 4.1, 4.2 and 4.3)

Chart Describing the Valuation of Surrender Option With no Dividend

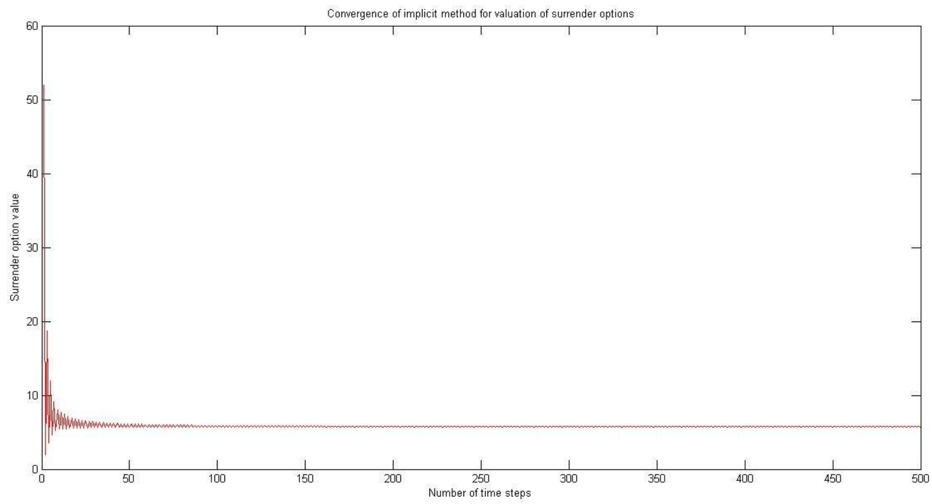

Figure 4.1: Chart on Fully Implicit method for the valuation of surrender option with no dividend

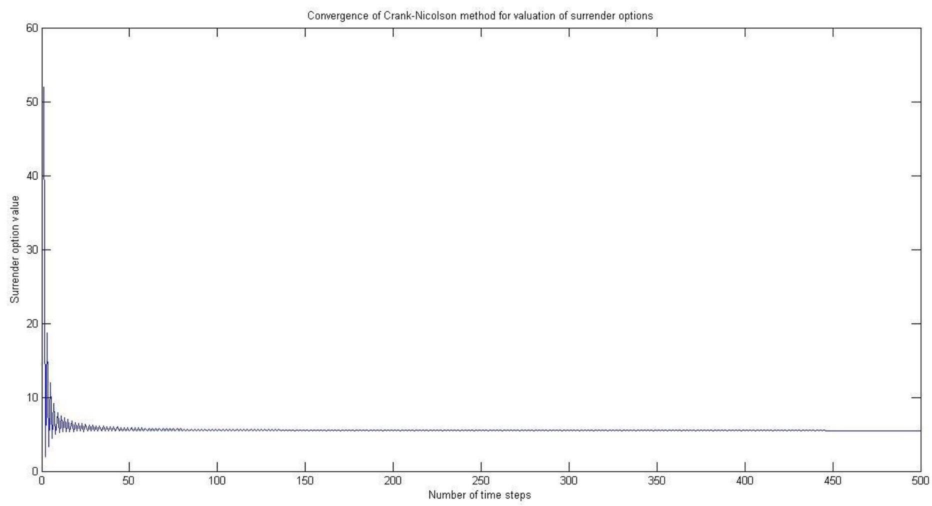

Figure 4.2: Chart on Crank-Nicolson method for the valuation of surrender option with no dividend

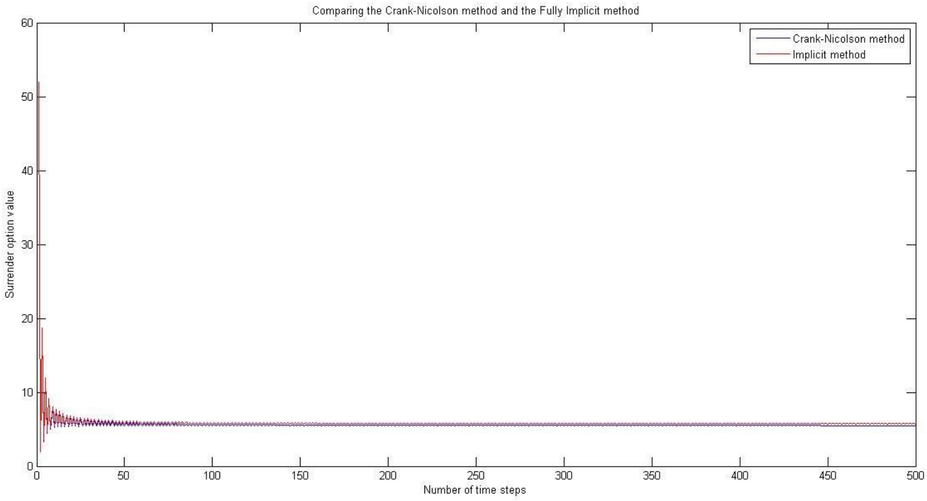

Figure 4.3: Chart Comparing the Crank-Nicolson method and the Fully Implicit method.

Tables 4.6, 4.7 and 4.8 and 4.9 shows that the implicit finite scheme converges faster than the Crank Nicolson finite scheme as N−→ ∞, The implicit finite difference approximation is closer to the value of the surrender value for large values of N than the Crank-Nicolson finite difference approximation. (see Figure 4.4, 4.5, 4.6, 4.7, 4.8)

Chart Describing the Valuation of Surrender Option with Rate of Being Multi-morbid Incorporated

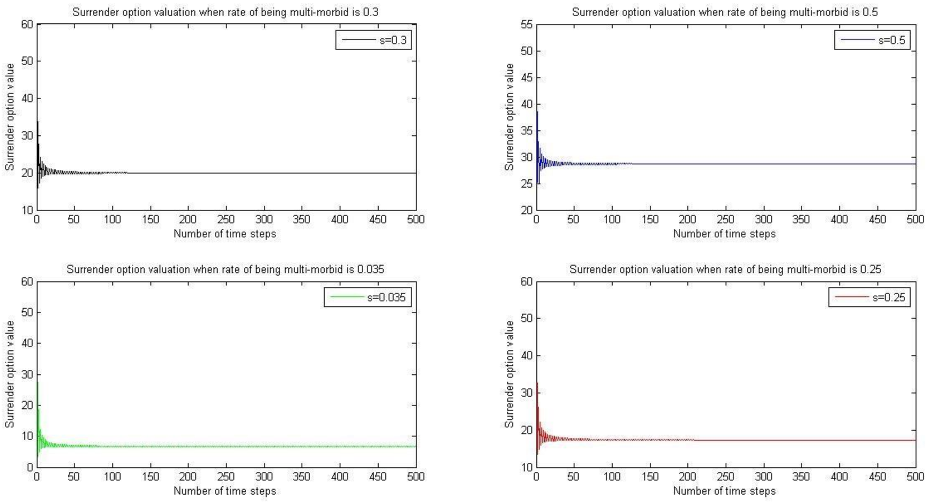

Figure 4.4: Chart comparing the valuation of surrender option with different rate of being multi-morbid using implicit finite difference scheme

Chart Describing the Valuation of Surrender Option with Rate of Being Multi-morbidity Incorporated

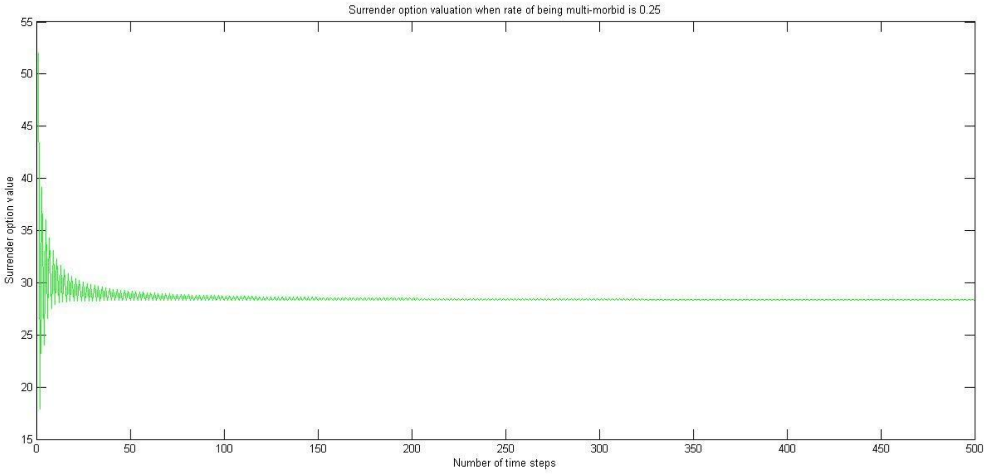

Figure 4.5: Chart for the valuation of surrender option with median rate of being multi-morbid using Crank Nicolson finite difference scheme

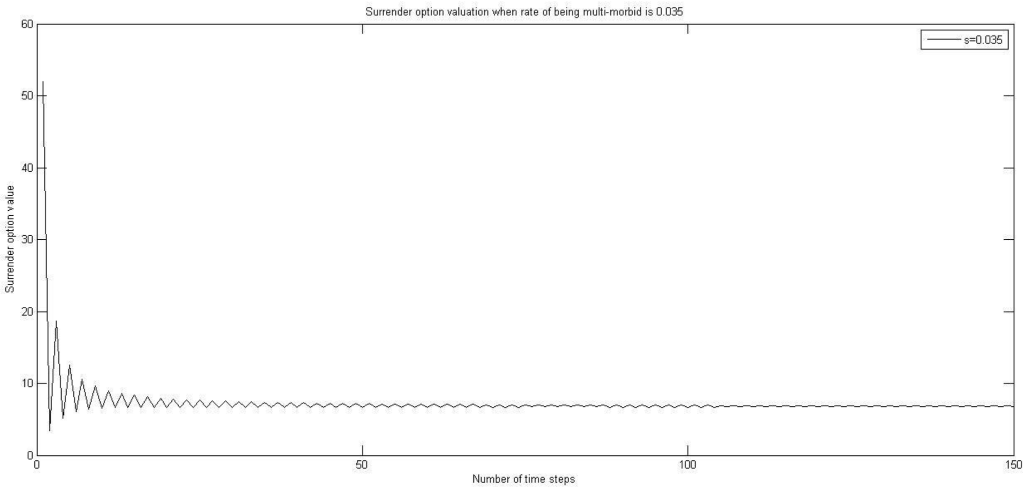

Figure 4.6: Chart for the valuation of surrender option with a minimum rate of being multi-morbid using Crank-Nicolson finite difference scheme

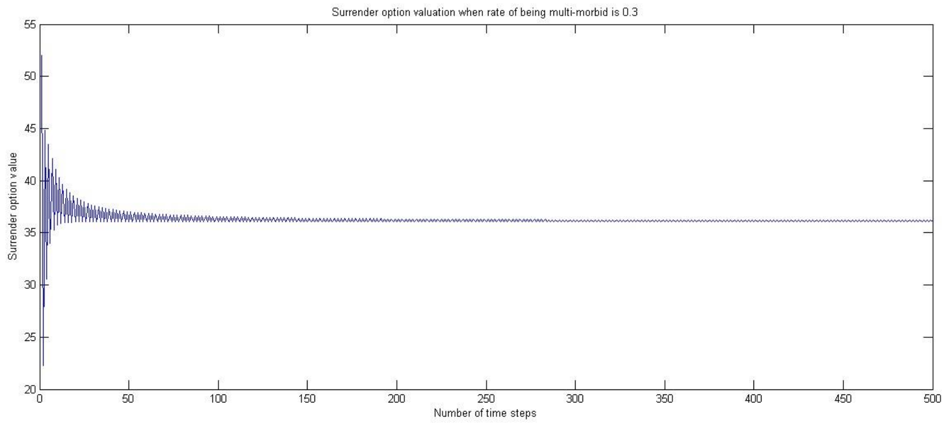

Figure 4.7: Chart for the valuation of surrender option with a mean rate of being multi-morbid using Crank-Nicolson finite difference scheme

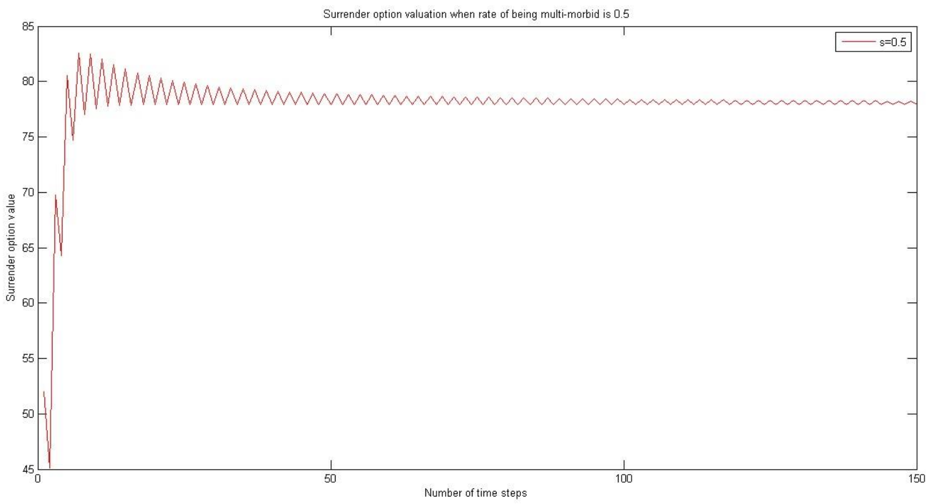

Figure 4.8: Chart for the valuation of surrender option with a maximum rate of being multi-morbid using Crank-Nicolson finite difference scheme.

CHAPTER 5

CONCLUSIONS AND RECOMMENDATIONS

5.1 Introduction

In this chapter the conclusions were made based on the study findings and the recommendations were also made based on the conclusions drawn.

5.2 Summary of Results

Findings from the analysis show that asset price discretization and time discretization contain two fundamental source of errors of which Lax Equivalence theorem indicated that the three fundamental factors that characterized a numerical scheme are consistency, stability and convergence.

The study used the eigenvalue to test the three finite difference methods. The results showed that, the explicit finite difference scheme is conditionally stable but the implicit finite difference and the Crank-Nicolson methods were unconditionally stable.

It was also observed that the Crank-Nicolson Finite Difference Approximation converges faster than the Implicit Finite Difference Approximation, which implies that the Crank-Nicolson Finite Difference Approximation gives more accurate results than the Implicit Finite Difference Approximation.

5.3 Conclusions

Rate of being multi-morbid increases as age increases, Most insurers would not like to insure someone who is multi-morbid, or someone with a higher rate of being multimorbid. This is because the company may encounter losses when the person comes in to surrender after few years the contact is put in place. Numerical valuation of surrender option with no rate of being multi-morbid incorporated in the valuation has a pay off which is a bit lower than the actual pay off, of which the Crank-Nicolson method converges faster than the implicit method. This means that the Crank-Nicolson approach gives more accurate results than the implicit approach.

On the other hand numerical valuation of surrender option with rate of being multimorbid incorporated into the Black-Scholes model has a pay off that is dependent on how high or low the policy holder's rate of being multi-morbid is when he or she comes in to surrender at any point in time. It is observed that higher rate of being multi-morbid when one comes in to surrender will result in higher pay off of which the insurance company would be losing. Also it is also observed that lower rate of being multi-morbid that is a rate that is less or equal to the drift of the Black Scholes model will results in lower pay off when someone comes to surrender of which the insurance company would be gaining and the insured or the policyholder would be losing.

With regards to the numerical analysis the Crank-Nicolson method converges faster than the Implicit method when valuation is done without incorporating the rate of being multi-morbid in the Black-Scholes model, consequently Crank-Nicolson given more accurate results than the implicit method. On the other hand it is observed in the computational analysis that when the valuation is done with the rate of being multi-morbid incorporated into the Black-Scholes model the implicit finite difference method converges faster than the Crank-Nicolson finite difference method, as a results the implicit method giving more accurate results than the Crank-Nicolson method.

5.4 Recommendations

In finding the value of American style life insurance contract where by the rate of being multi-morbid is ignored in the valuation using Black-Scholes model, the CrankNicolson method converges faster and gives more accurate results than the implicit finite

difference scheme. Again in finding the value of American style insurance contract where by the rate of being multi-morbid is incorporated in the Black Scholes model the implicit finite scheme converges faster than the Crank-Nicolson finite scheme. But the explicit finite scheme may not give accurate results because of its
conditional stability.

5.5 Further Studies

The scope of this research is centred around the valuiation of surrender options, which is a single premium American life insurance contract on a dividend paying asset. The researcher wish to look at the valuation of surrender option which involves continuous-instalment premiums or multiple premiums

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APPENDIX

Appendix I- Matlab Codefor Crank-Nicolson Scheme

```
function[P]=CrankNicolsonFDBS(S,K,r,sigma,T,N,M,s,)
```

```
% S is the asset price
% K is the strike price
% T is the maturity period
% N is the number of iterations in the time direction
% M is the number of iterations in the asset direction
% s is the rate of being multimorbid % sigma is
 the volatility dt=T/N ; ds=2*S/M ;
 A=zeros(M+1,M+1) ; f=max(K-(0:M)*ds,0)' ;
 for m=1:M-1
     A(m+1,m)=((r-lambda)*m*dt-sigma.^2*m.^2*dt)/4 ;
     A(m+1,m+1)=1+0.5*(r-s)*dt+0.5*sigma.^2*m.^2*dt ;
     A(m+1,m+2)=(-(r-s)*m*dt-sigma.^2*m.^2*dt)/4 ; end
 A(1,1)=1 ;
 A(M+1,M+1)=1 ;
 A ;
 for m=1:M-1
     B(m+1,m)=(-(r-s)*m*dt+sigma.^2*m.^2*dt)/4 ;
     B(m+1,m+1)=1-0.5*(r-s)*dt-0.5*sigma.^2*m.^2*dt ; B(m+1,m+2)=((r-
     s)*m*dt+sigma.^2*m.^2*dt)/4 ; end
 B(1,1)=1 ;
 B(M+1,M+1)=1 ;
 B ;
 for i=N:-1:1 f=A^(-1)*(B*f) ;
f=max(f,(K-(0:M)*ds)') ; end
 f ;
 P=f(round((M+1)/2)) ;
```

Appendix II Matlab Code for Implicit Scheme

```
function[Rec,V]=ImplicitFDBS_nodividend(A,K,r,volatility,T,N,M,s)
%If no dividend payment was made, enter zero for the dividend_yield
% A is the asset price
% K is the strike price
% T is the maturity period
% N is the number of iterations in the time directions % M is the
number of iterations in the asset direction
% s is the rate of being multimorbid
sigma=volatility;
y=length(N);
Table=zeros(y,3); for
j=1:y dt=T/N(j);
dA=2*A/M(j);
    B=zeros(M(j)+1,M(j)+1); f=max(K-(0:M(j))*dA,0)'; f(1)=f(1)-
    (0.5*r*1*dt-0.5*sigma^2*1*dt); x=1/(1-r*dt); for
    m=1:M(j)-1
        B(m+1,m)=x*(0.5*(r-s)*m*dt-0.5*sigma.^2*m.^2*dt);
        B(m+1,m+1)=x*(1+sigma.^2*m.^2*dt);
        B(m+1,m+2)=x*(-(r-s)*m*dt-sigma.^2*m.^2*dt)/2; end
    B(1,1)=1; B(M(j)+1,M(j)+1)=1; for i=N(j):-1:1 f=B\f; f=max(f,(K-
    (0:M(j))*dA)'); end f;
    V=f(round((M(j)+1)/2));
    Table(j,1)=j;
    Table(j,2)=N(j);
    Table(j,3)=V;
end Rec=Table;
```

Appendix III- R Codes for Simulating Survival Data

```
y<-function(n){
        V<-matrix(0,nrow=n, ncol=2) for(i in
    1:n) {
      lifetimes<-rexp(60,rate=1/15) censtimes<-15+5*runif(60) ztimes<-
      pmin(lifetimes,censtimes) status<-as.numeric(censtimes>lifetimes)
      kwasi<-summary(survfit(Surv(lifetimes,status)~1)) t<-
      length(kwasi$lower) g<-kwasi$lower[t] h<-kwasi$upper[t] d<-
      cbind(g,h)
V[i,]<-d
    }
V
}
```