KWAME NKRUMAH UNIVERSITY OF SCIENCE AND TECHNOLOGY

COMPUTING POWERS OF POSITIVE INTEGERS USING THE MODIFIED DETACHED COEFFICIENTS METHOD AND THE STAIRCASE METHOD.

BY

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DECLARATION

I hereby declare that this submission is my own work towards the Master of Science (Msc) and that, to the best of my knowledge, it contains no material previously published by another person nor material which has been accepted for the award of any other degree of the university, except where due acknowledgement has been made in the text.

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ABSTRACT

The Modified Detached Coefficients Method is used to find power numbers ,say Q = $(d_1d_2\ldots d_m)^n$ to any positive nth power. The power number is first converted to a multinomial, Say

$$Q= (d_1d_2d_3)^3$$

$Q = (d_1p^{m-1} + d_2p^{m-2} + d_mp^0)^n$…(1),where m is the number of digits and n is a positive power of the expansion.

Then equation (1) is expanded using the multinomial expansion.

The coefficients (d's) of the p's are extracted to become the result of the power number if and only if all the coefficients are single digits otherwise convert the coefficients which are more than 1 to p's by replacing 10's by p. For example, supposing 54 is a coefficient it is replaced by 5p+4,i.e. 5*10+4 =5*p+4. Where 10=p.

The Staircase Method is also used to find power numbers. The procedure is the same as the Modified Detached Coefficients Method but after the expansion of the converted form of the power number. The coefficients (d's) of the p's are arranged in the staircase form.

The result of the power number is then gotten by adding the staircase numbers arrangement column wise as done in multiplication of two numbers.

Results of both methods were done manually.

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LIST OF ABBREVIATIONS

IDL ----------Institute of Distance Learning

KNUST-----Kwame Nkrumah University of Science and Technology

DEDICATION

I dedicate this thesis to my parents Mr. and Mrs. Donatus Avoka and to my siblings Avoka Juliana, Avoka Andrews, Avoka Jarvis, Avoka Victor and Avoka Emmanuel.

Also, this piece of work is dedicated to the special individuals who have been a blessing to me: Ayine Ruth, Nabwomya John Millim, Wilfred Bormeh and Anarho Richard.

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I am most grateful to God for his mercies and Guidance through graduate school and for His Grace I can still hope.

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CHAPTER 1

INTRODUCTION

This chapter gives a brief account of the background of the study, the problem statement, objectives of the study, methodology, justification and the thesis organization.

1.1 BACKGROUND OF THE STUDY

Contributions to the theory of numbers have interested many researchers, mathematicians of the highest class of old that have contributed original work to the number theory that cannot be forgotten. Notably among them are Blaise Pascal (1623-1662), Leonhard Euler (1707-1783), Gabriel Lame (1795-1870), Godfred Harold Hardy (1877-1947), Srinivara Ramanuyan (1887-1920), Paul Erdo's (1913-1996) and many others.

However, since the middle of the century, when the number theory began through Pierre-de Fermat (1601-1665). Fermat influence was limited by the lack of interest in publishing the discoveries which are known mainly from letters to friends and marginal notes in the books he read. In 1629, Fermat invented analytic geometry, but most of the credit went to Descartes; who hurried into print with his own ideas came directly from Fermat. In a series of letters written in 1654, Fermat and Pascal developed the fundamental concepts of the theory of probability.

China discovered a magic square as far back as 200BC, the magic square were subsequently introduced into India, Japan and later to Europe.

Also, in 1770 Euler published his Anleitung Zur Algebra in two volumes. A French translation, with numerous and valuable additions by Langrange, was brought out in 1794, and a treatise on arithmetic by Euler was appended to it. The first volume treats of determinate algebra. This contains one of the earliest attempts to place the fundamental processes on a scientific basis: the same had attracted D'Alembert's attention. This work also includes the proof of the binomial theorem for unrestricted real index which is still known by Euler's name, the proof is founded on the principle of the permanence of equivalent forms, but Euler made no attempt to investigate the convergence of the series: that he should have omitted this essential step is the more curious as he had himself recognized the necessity of considering the convergence of infinite series: Vandermonde's proof given in 1764 suffers from the same defect.

The second volume of the algebra treats of indeterminate or Diophantine algebra. This contains the solutions of some of the problems proposed by Fermat, and which hitherto remained unsolved.

Moreover, Pascal employed his arithmetical triangle in 1653, but no account of this method was printed till 1665. The constructed triangle with each horizontal line being formed from the one above it by making every number in it equal to the sum of those above it.

The numbers in each line are now called figurate numbers. Those in the first line are called numbers of the first order, those in the second line , natural numbers or numbers of the second order ,those in the third line, numbers of the third, and so on.

Pascal's arithmetical triangle, to any required order, is got by drawing a diagonal downwards from right to the left of the triangle. The numbers in any diagonal give the coefficients of the expansion of a binomial.

1.2 PROBLEM STATEMENT

Ward (2007) proposed the method of Detached Coefficients to find the power of positive integers using binomial expansion. Selected coefficients from the terms of the expansion were put together to become the result of the power number. Ward(2007) only considered power of numbers that produce binomial and multinomial coefficients.

However, his method did not work when the coefficient of a term is more than 1 digit. I, therefore want to improve the method of Ward (2007).

1.3 OBJECTIVES OF THE STUDY

The objectives are:

(i) Compute squares and cubes of integers up to five digits using the modified Detached Coefficients method and the Staircase method.

(ii) Determine which of the methods is faster in terms of computation.

1.4 METHODOLOGY

The problem of the method of Detached Coefficients by Ward (2007) is that when the coefficient(s) of the terms of a binomial /multinomial expansion is more than 1 digit it does not give the result of a power number I in tend to research on. The problem would be modeled using the multinomial expansion. The Modified

Detached Coefficients Method and the Staircase Method would be used as methods of solution to resolve the problem of Ward (2007).

The methods of solution would be done manually to solve problems related to power numbers while information would be gotten from the Internet, Encyclopedia and extracts from the mathematics book in the KNUST Library.

1.5 JUSTIFICATION

(i) It will benefit students and researchers who are interested in researching into the sum of positive integers and series.

(ii) It is easier to implement when finding powers of large digit integers.

1.6 THESIS ORGANISATION

The project is made up of five chapters. The first chapter is the introduction; the second chapter is the literature review of the study. The third chapter is the derivation of the method whiles chapter four is the results from illustrative examples and finally chapter five is conclusion and recommendation.

CHAPTER 2

LITERATURE REVIEW

According to Beery and Jacqueline (2006), Aryabhata (499) wrote: ‘The sixth part of the product of three quantities consisting of the number of terms, the number of terms plus one, and twice the number of terms plus one is the sum of the squares. The square of the sum of (original) series is the sum of the cubes’.

According to Beery and Jacqueline (2006), Gerson (1344), gave a similar proof that $(1+2+3+\ldots.+n)^2 = 1^3+2^3+3^3+..n^{3,}$ which he stated as follows:’ The square of the sum of the natural numbers form 1 up to a given number is equal to the sum of the cubes of the numbers from 1 up to the given number’.

According to Beery and Jacqueline (2006), Briggs and Hutton (1624) noted that difference of squares, cube, and fourth and higher powers are eventually constant , but Briggs did not extend his observation to formulas for sums of these powers.

According to Beery and Jacqueline (2006) Fermat (1636) in his letter to mersenne and Roberval stated the following results about the generalized triangular numbers:’ The last side multiplied by the next greater makes twice the triangle. The last side multiplied by the triangle of the next greater side makes three times the pyramid. The last side multiplied by the pyramid of the next greater side makes four times the triangle. And so in by the same progression in infinitum.’

According to Beery and Jacqueline (2006), Pascal (1665),in his famous Traite du Triangle Arithmetique or Treatise on the Arithmetical Triangle, Pascal described in words a general formula for the sum of powers of the first n terms of an arithmetic progression of which the sum of powers of the first n positive integers is a special case. If our positive integer power is m, then we are to find a formula for the sum $1_m+2_m+3_m+\ldots+n_m$ or $\Sigma k=1nk_{m.}$

Form a binomial having as its first term a literal quantity A and for second term the difference of the given progression. Raise this binomial to a power of which the exponent is one more than the power proposed, noting the coefficients of the successive powers of the A in the resulting development.

According to Beery and Jacqueline (2006), Bernoulli (1713), in his Bernoulli’s book on probability, Ars conjectandi, derived symbolic formulas for the sums of positive integers powers using the method conjecture for fermat, then noted a pattern that would make computation of these formulas much simpler.

Bernoulli first showed how to find the sum of the first n positive integers, obtaining $(\int n = 12nn + 12n$. (Note that he used an integral sign, to represent the ‘’sum’’ sign. He then derived the sums of the first n squares and cubes, obtaining

$\int nn=13n_3+12nn+16n$ and $\int n_3 =14n_4+12n_3+14nn.$

According to Beery and Jacqueline (2006) Fermat (1636) derived a simple recursive procedure for finding the sum of powers of positive integers.

Pascal (1654) found an improved method for finding the sum of powers of positive integers using a binomial expansion formula.

According to Beery and Jacqueline (2006), Faulhaber(1631) in his book ‘Academiea Algebrae’ ,derived a general formula for finding the sum of powers of positive integers.

Langrange (1775), in Langrange's four-square theorem, Langrange proved that every positive integer can be written as the sum of at most four squares, although four may be reduced to three except for numbers of the form $4^n(8k+7)$. Diophantus first studied a problem equivalent to finding three squares whose sum is 3a+1, and stated that for this problem, a must not be of the form 8n+2, which is however an insufficient condition, where a is whole number and n is a positive integer.

Bachet (1621), subsequently excluded 8n+2 and 32n+9. Finally, Fermat (1636) remarked that Bachet's condition failed to exclude a = 37, 149, etc., and gave the correct sufficient condition that a must not be of the form $((24k+7)4^n-1))/3$, so 3a+1 not of the form $(24k+7)4^n$, or equivalently $(8m+7)4^n$.

Fermat (1636) stated no integer of the 8k+7 is the sum of three rational squares, and in 1638, Descartes proved this for integer squares. Fermat (1658) subsequently asserted (but did not prove) that 2p where p is any prime of the form 8n-1(i.e., any prime of the form 8n+7) is the sum of three squares.

Langrange (1775) made some progress on Fermat's assertion , but could not completely prove it.

Legendre (1785) remarked that Fermat's assertion is true for all odd numbers (not just primes), and then gave an incomplete proof that either every number or its double is a sum of three squares.

Beguelin (1774) had concluded that every integer congruent to 1,2,3,5 or 6 (mod8) is a sum of three squares, but without adequate proof(Dickson 2005, p.15).

Legendre (1798) in his Théorie des nombres, Legendre proved that every positive not of the form 8n+7 or 4n is a sum of three squares having no common factor.

The number of representations of n by k squares, allowing zeros and distinguishing signs and order, is denoted $r_k(n)$. The function $r_2(n)$ is often written simply as r(n), and is intimately connected with the Leibniz series and with Gauss's circle problem(Hilbert and cohn-vossen 1999).

Jacobi (1829) gave analytic expressions for $r_k(n)$ for cases k = 2, 4, 6 and 8(Hardy and Wright 1979, Hardy 1999).The cases k = 2, 4, and 6 were found by equating coefficients of the Jacobi theta functions $\varnothing_3(x)$, $\varnothing_3^2(x)$ and $\varnothing_3^4(x)$. The solutions for k = 10 and 12 were found by Liouville (1864).

Hardy and Wright (1979) and Glaisher (1907) gives a table of r2s(n) for up to 2s = 18. However, the formulas for 2s = 14 and 2s = 16 contained functions defined only as the coefficients of modular functions, but not arithmetically (Hardy and wright1979,p.316).

Ramanujan (2000) extended Glaisher's table up to k =24. Bouyguine (1915) found a general for $r_{2s}(n)$ in which every function has arithmetic definition (Hardy and Wright 1979, Dickson 2005). $r_3(n)$ was found as a finite sum involving quadratic reciprocity symbols by

Dirichlet. $r_5(n)$ and $r_7(n)$ were found by Eisenstein , Smith ,and minkowski. Mordell, Hardy , and Ramanujan have developed a method applicable to representations by an odd numbers of squares (Hardy 1920; Mordell 1920,1923; Estermann 1937 ; Hardy 1999).

According to Beery and Jacqueline (2006) Faulhaber (1631), in his Academia Algebrae, faulhaber presented formulas for sums of powers of the first n positive integers from the 13th to the 17th powers.

Faulhaber (1617) had presented formulas for sums of powers up to the 12th power in his Continuatio seiner neuen wunderkunste. Furthermore, D.E. Knuth has argued that, in Academia Algebrae, Faulhaber actually encoded sums of powers of positive integers up to the 23rd power.

According to Beery and Jacqueline (2006) Harriot (1557), wrote formulas for sum of squares, cubes and fourth power on a manuscript sheet headed" Ad aggregate Z. C. ZZ. Etc" ("for sum of squares, cubes, square-squares, etc.") (Harriot, folio 240). In this heading, we have substituted z, c, and zz for symbols for squares, cube, and fourth powers (Square-squares) that Harriot probably borrowed from Robert Record's famous algebra book, whetstone of witte (1557).

According to Beery and Jacqueline (2006), Harriot (1575) cited page 25 of Francisco maurolico's Arithmeticorum Libri Duo, copying out the following theorem:

'The square of every triangular number is equal to the sum of cube from unity (one) through the cube of the side of the triangle'.

Polignac (1849) conjectured that odd number larger than can be written as the sum of odd prime and a power of 2. He found a counterexample 959 soon.

Ro-Manoff (1934) proved that the set of positive odd integers which can be expressed in the 2n + p has positive lower asymptotic density, where n is a nonnegative integer and p a prime. Corput (1950) proved that counterexample to de Polignac's conjecture form a set of positive lower density.

Erdo's (1950) proved that there is an infinite arithmetic progression of positive odd integers each of which has no representation of the form 2n + p. {ai (mod mi)}k is called a covering system if integer is congruent to ai (mod mi) for at least one value of i.

Refining the argument of Erdo's ,s R. Cocker proved that there are infinitely many positive odd integers not representable as the sum of a prime power and two positive power of two positive powers of two. Before this Z.W.Sun and M.H. handled the integers of the form c(2a + 2b) + p for many values of constant c.

Taeisinger (1915) proved that the harmonic number **$H_n := 1 + 1/2 + \cdot \cdot \cdot + 1/n$** is never an integer except for **H_1.** . The more general result that the sum of reciprocals of consecutive terms, not necessary starting with 1, is never an integer was proved by Kurschak (1918).

Erdos (1932) proved that the sum of reciprocals of any integer in arithmetical progression is never reciprocal and then an integer.

According to Beery and Jacqueline (2006) Fibonacci (1202), in his book ‘’Liber Abaci’’ Sum of odd integers, starting 1, gives the square of integers. Interpretation: “the average number of the Nth row is N.

Therefore, the sum of the row, which is the sum of odd integers, is NXN $=N^2$.

According to Beery and Jacqueline (2006) Fibonacci (1202) arranged the same odd numbers in a different pattern and came up with a very elegant proof regarding the sum of integers cubed.

Interpretation: “the average number of the Nth row is N^2. So the N th row has N terms and their average is N^2.

Therefore the sum of the row is $N^2xN=N^3$. From the left hand side, the last term of the old integers on the N th row is a triangular number TN, because TN =1+2+…+N. And it is known that sum of the first p old integers is p^2.

Therefore Fibonacci concluded $1^3+2^3+3^3+\ldots+N^3= (1+2+3+\ldots+N)^2= (TN)^2 =\{ (1/2) N (N+1)\}^2$

According to Beery and Jacqueline (2006), Al- Haytham (2006). For his volume computations, al-Haytham needed formulas for the sums of the first n integer cubes and the first n fourth power. He may have used a diagram like that in figure 6 (Baron, p.70) to describe the relationship. $(4+1)\Sigma i=14i=\Sigma i=14i2=\Sigma p=14\Sigma i=1pi$.

According to Beery and Jacqueline (2006), Pythagoras (2006). The Pythagoreans experimented with number properties by arranging pebbles on a flat surface. As a result, they saw that we would describe as a sum of successive positive integers as a triangle or triangular number.

Their pebbles experiments led them to see that two copies of the of the triangular number could be fitted together to form an oblong number, hence for example twice the triangular number 15=1+2+3+4+5 could be view as the oblong number 5x6=30 Twice a triangular number is an oblong number, or in modern notation, $2(1+2+3+\ldots+n) = n(n+1)$. In general, $1+2+3+\ldots..+n=n(n+1)2$ for any positive integer n.

According to Beery and Jacqueline (2006) Archimedes (2006) knew the Pythagorean “formula” for the sum of the first n positive integers. He almost certainly knew also a “formula” for the sum of the squares of the first n positive integers. His Lemma to proposition 2 in on Conoids and Spheroids and his proposition 10 in on Spirals translated to our symbols, say that $(n+1)n_2+(1+2+3+\ldots.+n)=3(1_2+2_2+3_2+\ldots+n_2)$.

Lagrange (1770) in his paper *Réflexions sur la résolution algébrique des équations (Thoughts on Solving Algebraic Equations)* devoted to solutions of algebraic equations, in which he introduced Lagrange resolvents. Lagrange's goal was to understand why equations of third and fourth degree admit formulae for solutions, and he identified as key objects permutations of the roots. An important novel step taken by Lagrange in this paper was the abstract view of the roots, i.e. as symbols and not as numbers. However, he did not consider composition of permutations.

Waring (1782) in his paper *Meditationes Algebraicae* (*Meditations on Algebra*) came up with the expanded version of Lagrange's resolvents. He proved that the main theorem on symmetric functions, and specially considered the relation between the roots of a quartic equation and its resolvent cubic.

Vandermonde (1771), developed the theory of symmetric functions from a slightly different angle, but like Lagrange, with the goal of understanding solvability of algebraic equations.

Al-Tusi (1262) determined the coefficients of the expansion of a binomial to any power

giving the binomial formula and the Pascal triangle relations between binomial coefficients.

Edwards (2002) has postulated that the work of al-Karaji in expanding the Binomial Triangle might have borrowed Brahmegupta's work, given that it was available and al- Karaji definitely had read other Hindu texts available in Baghdad which was the great cultural and scientific center of Muslims. The binomial coefficients have been studied in cultures around the world, both in the text of binomial expansions and in the question of how many ways to choose k items out of a collection of n things.

Hsien and Yanghui (1261) and Shih-chieh(1303) came up with the idea of taking ''six tastes one at time, two at a time, three at a time, etc'' was written down correctly in India 300 years before the birth of Christ in a book called the ‘’ Bhagabati Sutra’’.

According to Beery (2006) Al- Khayyam (2002) wrote a letter claiming to have been able to expand binomials to Sixth power and higher, but the actual work does not survive.

Tartaglia (1523) first published the generation of the figurate numbers. Some 30 years he published the triangle in a table form in his paper known as General Treatise. He did publish a general formula for solving cubic equations.

Cardano (1539) the Italian algebraist, correctly determines that the number of ways to take 2 or more things from a set of n things is $2^n - n - 1$.

Stifel (1544) published the extended Figurate Triangle. He gives credit to Cardano’s work published five years earlier.

Viéte (591), gives names to the first few columns of the Triangle in Latin; "numeri trianguli", "pyramidales", "triangulo-trianguli", "triangulo-pyramidales"

Oughtred (1631) published his Clavis Mathematicae, which influences his student John Wallis and is later owned in a 3rd edition printing by Isaac Newton; both Wallis and Newton are instrumental in the work that connects the binomial coefficients to the new field of calculus.

Vall´ee (2003) published a paper entitled ‘Dynamical analysis of a class of Euclidean algorithms’ and developed a general technique for analyzing the average-case behavior of the Euclidean-type algorithms. ‘This is a deep and important paper which merits careful study, and will likely have a significant impact on future directions in algorithm analysis.’ Jeffrey O. Shallit reviewed, ‘The method involves viewing these algorithms as a dynamical system, where each step is a linear fractional transformation of the previous one. ... Then a generating function (Dirichlet series) is used to describe the cost of the algorithms, and Tauberian theorems are used to extract the coefficients.’

Vall´ee (2006), further proposed a detailed and precise dynamical and probabilistic analysis of the more natural variants of Euclid’s algorithm. The paper ‘presented a clear, clever, and unified overview of the methodology (dynamical analysis) and of the tools.’

One of the strengths of this approach comes from the fact that it combines sophisticated tools taken from, on the one hand, analytic combinatorics and functional analysis (moment generating functions, Dirichlet series, quasi-power theorems and Tauberian theorems), and on the other hand, from dynamical systems and ergodic theory (including Markovian dynamical systems, induction, transfer operators and related concepts).

Buchberger (1965) gave an algorithm for finding a basis $g_1, \ldots, g_k$ of the ideal $I =< f_1, \ldots, f_m >$ such that the leading term of any polynomial in I is divisible by the leading term of some polynomial in $G = \{g_1, \ldots, g_k\}$. Such a basis is called Grobner bases by

Buchberger. An analogous concept was developed independently by Hironaka (1964) and he then named it as standard bases.

Grimm (1969) made an important conjecture that if m + 1, ...,m + n are consecutive composite numbers, then there exist n distinct prime numbers p1, ..., pn such that m+1 is divisible by pi for 1 _< i >_ n. This implies that for all sufficiently large integer n, there is a prime between n_2 and $(n+1)_2$. It is nice that for $m \leq 1$, that there exists a prime in the interval $(m_2, (m + 1)_2)$ is equivalent with that $m_2 + 1, ..., m_2 + 2m$ is a W sequence.

Heilbronn (1968), studied the average length of a class of finite continued fractions. This is an important result on Euclids algorithm.

Tonkov et al (1975), improved Heilbronn's estimate respectively using an idea of Heilbronn, Andrew C. Yao and Donald E. Knuth studied the sum of the partial quotients qi in Euclids algorithm. They proved a well-known result which states that the sum S of all the partial quotients of all the regular continued fractions.
Zheng (1994) improved the result of Andrew C. Yao and Donald E. Knuth. As an application, Conrey et al (1996) studied the mean values of Dedekind Sums.

Magne (1545) discovered an algebraic formula for the solution to both the cubic and Quadratic equations.

Hui (263AD) wrote a commentary on a book known as mathematical at a hand book of practical problems that was compiled in the first two centuries BC. He did some analysis of a mathematical statement called Gou-Gou theorem. The theorem in the West is known as the Pythagorean theorem that describes a special relationship that exists between the sides of a Right triangle.

Mercator (1668) used a geometric series to develop a power series for representation for ln(1+x).

Taylor (1715) published Methodus Incramentorum Directa et Inversa which contained the result known as Taylor's theorem.

Maclaurin (1742) text ''Treatise on fluxions'' used the approach "order of contact" to Develop power series. Maclaurin concentrated on series centered at c=0 and therefore the series is known as maclaurin series.

Burton (2007) the order of contact method matches derivatives of the function to Derivatives of power series in order to find the power series coefficients.

Rabin and Shallit (1986) showed how to represent any positive integer n as a sum of four Squares in random polynomial by formulating algorithms for representing these integers as Sums of squares.

Bumby (1990) gave a deterministic, polynomial-time algorithm in case n is prime. Determining the number of such representations for composite n is random polynomial-time equivalent to factoring n.

Gauss (1801) considered the representations by three squares in 'Disquisitiones Arithmeticae. V' and found the number of representations in terms of two other functions of n.

Uspensky (1929) considered representations by more general ternary quadratic forms.

Landau (1908) proved that the number of integers up to N which are sums of two squares of two squares is roughly K*N/sqrt(log(N). The best estimate of K is 0.76422365…Shiu(1986).

Osbaldestin and Shiu (1989) showed that the number of integers up N which are sums of three squares is roughly 5N/6 .

Kronecker (1870) gave a definition of an abelian group in the context of ideal groups of a number field, generalizing Gauss's work; but did not tie his definition with previous work on groups, particularly permutations groups.

Euler (1770) published his Anleitung Zur Algebra in two volumes. A French translation, with numerous and valuable additions by Langrange, was brought out in 1794, and a treatise on arithmetic by Euler was appended to it. The first volume treats of determinate algebra. This contains one of the earliest attempts to place the fundamental processes on a scientific basis: the same had attracted D'Alembert's attention. This work also includes the proof of the binomial theorem for unrestricted real index which is still known by Euler's name, the proof is founded on the principle of the permanence of equivalent forms, but Euler made no attempt to investigate the convergence of the series: that he should have omitted this essential step is the more curious as he had himself recognized the necessity of considering the convergence of infinite series: Vandermonde's (1764) proof given suffers from the same defect.

The second volume of the algebra treats of indeterminate or Diophantine algebra. This contains the solutions of some of the problems proposed by Fermat, and which hitherto remained unsolved.

Fermats (1640) achievements in arithmetic include:Fermats little theorem , stating that, if a is not divisible by a prime p, then If a and b are coprime, then $a^2 + b^2$ is not divisible by any prime congruent to −1 modulo 4; and Every prime congruent to 1 modulo 4 can be written in the form $a^2 + b^2$.

Fermat (1659) stated to Huygens that he had proven the latter statement by the method of descent. Fermat and Frenicle also did some work (some of it erroneous or non- rigorous) on other quadratic forms. Fermat posed the problem of solving as a challenge to English mathematicians (1657). The problem was solved in a few months by Wallis and Brouncker.

Fermat considered their solution valid, but pointed out they had provided an algorithm without a proof (as had Jayadeva and Bhaskara, though Fermat would never know this.) He states that a proof can be found by descent. Fermat developed methods for (doing what in our terms amounts to) finding points on curves of genus 0 and 1.

Hardy and Wright (1938) proposed that the least numbers of those which have the same ratio with them measure those which have the same ratio the same number of times, the greater the greater and the lesser the lesser.

Hardy and Wright (1938) proposed that if two numbers by multiplying one another make some number, and any prime number measures the product, it will also measure one of the original numbers.

CHAPTER 3

METHODOLOGY

3.0 INTRODUCTION

This chapter gives definitions, derivations and methods of examination in accordance with the objective of the study.

3.1.1 Series

A series is a sum of the terms in a sequence. If there are n terms in the sequence , the sum is often write as Sn for the result, so that Sn = u1 + u2 + u3 + . . . + un .

3.1.2 Power series

A **power series** is a series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n.$$

The Taylor series at a point *c* of a function is a power series that, in many cases, converges to the function in a neighborhood of *c*. For example, the series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

is the Taylor series of e^x at the origin and converges to it for every *x*.

Unless it converges only at *x*=*c*, such a series converges on a certain open disc of convergence centered at the point *c* in the complex plane, and may also converge at some of the points of the boundary of the disc.

3.1.3 Taylor series.

Definition:

The Taylor series of a real or complex-valued function $f(x)$ that is infinitely differentiable at a real or complex number *a* is the power series

$$f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f^{(3)}(a)}{3!}(x-a)^3+\cdots.$$

which can be written in the more compact sigma notation as

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

where $n!$ denotes the factorial of n and $f^{(n)}(a)$ denotes the nth derivative of f evaluated at the point a. The derivative of order zero f is defined to be f itself and $(x-a)^0$ and 0! are both defined to be 1. In the case that $a = 0$, the series is also called a Maclaurin series.

3.1.4 Maclaurin series.

Definition:

It is a particular case of Taylor series, in the region near a = 0 such a polynomial is called

Maclaurin series.

The infinite series expansion for f(a) about a = 0 becomes

$$\mathbf{f(a) = f(0) + f'(0) + } f''\frac{(0)x^2}{2!} + \mathbf{f'''}\frac{(0)x^3}{3!} + \ldots$$

f'(0) is the first derivative evaluated at **a = 0,** $f''(0)$ is the second derivative evaluated at a = 0 and so on.

3.1.5 Convergence of the power series

Consider a power series, say

$$f(x) = 1 + x^2 + x^3 + x^4 + \dots$$

The power series converges depending on the value of x. If x is too large , then the series will

diverge:

$f(10) = 1 + 10 + 100 + 1000 + \dots = \infty$

However, if x is small enough , then the series will converge:

$f(0.1) = 1 + 0.1 + 0.01 + 0.001 + \dots = 1.111111$

Therefore, the series converges whenever $|x|<1$, and diverges whenever $|x|>1$.

In general, a power series converges whenever x is close to 0, and may diverge if x is far away from 0. The maximum allowed distance from 0 is called the radius of convergence.

3.1.6 Arithmetic series.

An arithmetic progression, or AP, is a sequence where each new term after the first is obtained by adding a constant d, called the common difference, to the preceding term. If the first term of the sequence is a then the arithmetic progression is a, a + d, a + 2d, a + 3d, . . .

where the n-th term is $a + (n - 1)d$.

The sum of the terms of an arithmetic progression gives an arithmetic series. If the starting value is a and the common difference is d then the sum of the first n terms is

$S_n = \frac{1}{2}n(2a + (n - 1)d)$.

If the value of the last term ℓ is known instead of the common difference d then the sum is written as Sn = $\frac{1}{2}$n(a + ℓ) .

3.1.7 Geometric series.

A geometric progression, or GP, is a sequence where each new term after the first is obtained by multiplying the preceding term by a constant r, called the common ratio. If the first term of the sequence is a then the geometric progression is

a, ar, ar^2, ar^3, . . .

where the n-th term is ar^{n-1}.

The sum of the terms of a geometric progression gives a geometric series. If the starting value is a and the common ratio is r then the sum of the first n terms is Sn $=\frac{a(1-r^n)}{1-r}$ provided that $r \neq 1$.

3.1.8 Convergence of goemetric series

Consider the sum, Sn $= \frac{a(1-r^n)}{1-r}$,

The formula contains the term , r^n , assuming where $r = \frac{1}{2}$ and as $-1 < r < 1$, then this term gets closer and closer to zero as n gets larger and larger So, if $-1 < r < 1$, then the ''sum to infinity" of a geometric series is;

$$Sn = \frac{a}{1-r}$$

$$n \rightarrow \infty$$

This is the limit of the sum , Sn as n 'tends to infinity'.

3.2 THE BINOMIAL THEOREM

A binomial is any expression of the form x + y. In other words, an expression involving two terms is a binomial expression. For example $a^2 - b^2$, 2a +3b, $z^3 - 2xy$ and (-2/x) + $(3/2)y^2$ are binomial expressions.

The expansion of $(x + y)^n$ is given by

$(x + y)^n = x^n + \binom{n}{n-1}x^{n-1}y + \binom{n}{n-2}x^{n-2}y^2 + \binom{n}{k}x^k y^{n-k} + \ldots + \binom{n}{1}xy^{n-1} + y^n$, where (for the purposes of this research) x, y and n are nonnegative integer.

3.2.1 Example of the Binomial Theorem

Consider the expansion of $(x + y)^4$. This can be written using the binomial theorem as

$$(x + y)^n = \binom{4}{4}x^4 + \binom{4}{3}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{1}xy^3 + \binom{4}{0}y^4$$

$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

Where $\binom{4}{4}$, $\binom{4}{3}$, $\binom{4}{2}$, $\binom{4}{1}$, and $\binom{4}{0}$ are the coefficients of the terms x^4 ,x^3y , x^2y^2 ,

xy^3 and y^4 respectively.

3.3 The Multinomial Theorem

The expansion o f $(x_1+x_2+\ldots+x_k)^n$ is found by adding all the of form

$$\binom{n}{n_1, n_2, \ldots, n_k} x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k}$$

Where $n_1 + n_2 + \ldots + n_k = n$.

3.3.1 Example of The Multinomial Theorem

Consider the expansion of $(x_1+2x_2+x_3)^3$.This can be written using the multinomial theorem as

$$(x_1+2x_2+x_3)^3 = \binom{3}{n_1,n_2,\ldots,n_k} x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k}$$

$$= \binom{3}{0,0,0} x_1^3 + \binom{3}{2,1,0} x_1^2 + \binom{3}{2,0,1} 3x_1^2x_3 + \binom{3}{1,2,0} x_1(2x_2)2x_3 +$$

$$\binom{3}{1,0,2} x_1x_3^2 + \binom{3}{1,1,1} x_1(2x_2)x_3 + \binom{3}{0,3,0} (2x_2)^3 + \binom{3}{0,2,1} (2x_2)^2x_3 +$$

$$\binom{3}{0,1,2} (2x_2)x_3^2 + \binom{3}{0,0,3} x_3^3$$

$$= x_1^3 + 6\,x_1^2 + 3x_1^2x_3 + 12x_1x_2x_3 + 3x_1x_3^2 + 12x_1x_2x_3 + 8x_2^3 + 12x_2^2x_3 +$$

$$6x_2x_3^2 + x_3^3$$

3.3.2 Expanding numbers of different digits as Multinomials

Example1:

$(d_1d_2d_3)^n = (d_1p^2 + d_2p^1 + d_3p^0)^n$

$= (x_1 + x_2 + x_3)^n$

Where $x_1 = d_1p^2$, $x_2 = d_2p^1$, $x_3 = d_3p^0$. Also d_1, d_2 , d_3 are positive integers and p=10

3.3.3 Multinomial Expansion of the form $(d_1d_2d_3)^n$,where values of variables are given

Example2:

Expanding $(123)^2$ using the multinomial expansion is given by;

$(123)^2 = (d_1p^2 + d_2p^1 + d_3p^0)^2$

$=(1p^2 + 2p^1 + 3p^0)^2$

3.3.4 Expanding $(d_1d_2d_3)^n$ using the Modified Detached Coefficients Method

Example3:

Expanding $(123)^2$ using the Modified Detached Coefficient Method is given by ;

$(123)^2 = (d_1p^2 + d_2p^1 + d_3p^0)^2$

$=(1p^2 + 2p^1 + 3p^0)^2$

$\Rightarrow(1p^2 + 2p^1 + 3p^0)^2 = p^4(1)+p^3(4)+p^2(10)+p^1(12)+p^0(9)$ when $10=p$.

$= p^4(1)+p^3(4)+p^2(10)+p^1(10+2)+p^0(9)$

$= p^4(1)+p^3(4)+p^2(p)+p^1(p+2)+p^0(9)$

$= 1p^4+4p^3+1p^3+1p^2+2p^1+9p^0$

$=\mathbf{1}p^4+\mathbf{5}p^3+\mathbf{1}p^2+\mathbf{2}p^1+\mathbf{9}p^0$

Therefore $(123)^2 = 15129$

3.3.5 Expanding $(d_1d_2d_3)^n$ using the Staircase Method.

Example4:

Expanding $(123)^2$ using the Staircase Method is given by;

$(123)^2 = (d_1p^2 + d_2p^1 + d_3p^0)^2$

$=(1p^2 + 2p^1 + 3p^0)^2$

$\Rightarrow(1p^2 + 2p^1 + 3p^0)^2 = p^4(1)+p^3(4)+p^2(10)+p^1(12)+p^0(9)$

$$
\begin{array}{lcr}
(123)^2 & = & 09 \\
 & & +12 \\
 & & +10 \\
 & & +04 \\
 & & +01 \\
\hline
(123)^2 & = & 15129
\end{array}
$$

3.4.1 Methodology of the modified detached coefficients method:

Given a power number $Q = (123)^2$ convert to equivalent multinomial expansion by inserting powers of $p = 10$ as coefficients of the digits in the number as shown below:

$Q = (123)^2 = (1*10^2+2*10^1+3*10^0)^2$

$=(1*p^2+2*p^1+3*p^0)^2$ where $p=10$

In general if x_1 , x_2 ,x_3 ,x_m are order digits of the number Q then $Q =(x_1x_2x_3\ldots x_m)^n$ is the required number, where n is a positive integer.

$\Rightarrow Q= (x_1p^{m-1}+x_2p^{m-2}+\ldots+x_mp^0)^n$

$=(y_1+y_2+\ldots+y_m)^n$,where $y_1 = x_1p^{m-1}$, $y_2 = x_2p^{m-2}$, $y_3 = x_mp^0$

Step 0 : Convert $(x_1x_2x_3\ldots x_m)^n$ to Multinomial form, i.e.,

$Q = (x_1p^{m-1}+x_2p^{m-2}+\ldots+x_mp^0)^n$

$=(y_1+y_2+\ldots+y_m)^n$, where $y_t = x_tp^{m-t}$

From the Multinomial Summation;

$Q = (y_1+y_2+\ldots+y_m)^n$ is expressed as

$$(y_1+y_2+\ldots+y_m)^n = \sum_{k_1+k_2+\cdots+k_m=n} \binom{n}{k_1,k_2,k_3} {y_1}^{k_1}{y_2}^{k_2} \ldots {y_m}^{k_m}$$

Step1: Expand the multinomial using the staircase multinomial expansion, i.e.

$$(\mathbf{p^{m-t}x_1+p^1x_2+\ldots+p^0x_m})^{\mathbf{n}} = \sum_{k_1+k_2+\cdots+k_m} \binom{n}{k_1,k_2,k_m} \prod_{t=1}^{m} {x_t}^{k_t}\, p^{(m-t)^{k_t}}$$

Where $1\le t\le m$,

P=terms determinant

$x_1,x_2,\ldots,x_m$ are positive integers.

$$\binom{\boldsymbol{n}}{\boldsymbol{k_1},\boldsymbol{k_2},\ldots,\boldsymbol{k_m}} = \frac{\boldsymbol{n}!}{\boldsymbol{k_1}!\boldsymbol{k_2}!\ldots\boldsymbol{k_m}!}$$

Step 2: Input the values of $x_1,x_2,\ldots,x_m$ into the expanded form.

Step 3: Substitute 10=p in the expanded form $\sum_{i=0}^{n} a_i p^i$, if all $a_i's$ are single digits then extract coefficients as results otherwise go to step 5 if not go to step 7. Where $a_1, a_2, \ldots, a_n$ are coefficients of the p's and p = 10.

Step 4: Write coefficients with digits more than 1 as placeholder in the expansion of p.

Step 5: Terms of the same powers are added together.

Step 6: Rearrange terms in descending powers of p. All coefficients in the current expansion will be of single digit.

Step7: Extract the coefficients of the p's as the result of the power number.

3.4.2 Methodology of the staircase method

Given a power number $Q = (123)^2$ convert to equivalent multinomial expansion by inserting powers of p = 10 as coefficients of the digits in the number as shown below:

$Q = (123)^2 = (1*10^2+2*10^1+3*10^0)^2$

$= (1*p^2+2*p^1+3*p^0)^2$ where p=10

In general if x_1 , x_2 ,x_3 ,x_m are order digits of the number Q then $Q =(x_1x_2x_3\ldots x_m)^n$ is the required number, where n is a positive integer.

$\Rightarrow Q= (x_1p^{m-1}+x_2p^{m-2}+\ldots+x_mp^0)^n$

$= (y_1+y_2+\ldots+y_m)^n$,where $y_1 = x_1p^{m-1}$, $y_2 = x_2p^{m-2}$, $y_3 = x_mp^0$

Step 0: Convert $(x_1x_2x_3\ldots x_m)^n$ to Multinomial form, i.e.,

$$Q = (x_1p^{m-1}+x_2p^{m-2}+\ldots+x_mp^0)^n$$

$$= (y_1+y_2+\ldots+y_m)^n \quad , \text{where } y_t = x_tp^{m-t}$$

From the Multinomial Summation;

$Q = (y_1+y_2+\ldots+y_m)^n$ is expressed as

$$(y_1+y_2+\ldots+y_m)^n \quad = \sum\nolimits_{k_1+k_2+\cdots+k_m\ =n} \binom{n}{k_1,k_2,k_3} y_1^{\,k_1} y_2^{\,k_2} \ldots y_m^{\,k_m}$$

Step1 :Expand the multinomial using the staircase multinomial expansion, i.e.

$$\mathbf{(p^{m-t}x_1+p^1x_2+\ldots+p^0x_m)^n} = \sum\nolimits_{k_1+k_2+\cdots+k_m} \binom{n}{k_1,k_2,k_m} \prod\nolimits_{t=1}^{m} x_t^{\,k_t}\, p^{(m-t)^{k_t}}$$

Where $1 \leq t \leq m$,

P=terms determinant

$x_1,x_2,\ldots,x_m$ are positive integers.

$$\binom{\boldsymbol{n}}{\boldsymbol{k_1},\boldsymbol{k_2},\ldots,\boldsymbol{k_m}} = \frac{\boldsymbol{n}!}{\boldsymbol{k_1}!\boldsymbol{k_2}!\ldots\boldsymbol{k_m}!}$$

Step 2: Input the values of $x_1,x_2,\ldots,x_m$ into the expanded form .

Step 3: The results of the terms are now arranged in the staircase form starting with the term that contains p^0 followed by $p^1,p^2,\ldots,p^m$.

Step 4: Add staircase digit wise, one digit to the left and placing them successively beneath the previous coefficient.

3.5.1 Computing the cubes of integers, using the coefficients method (3digits)

Methodology:

Problem: Evaluate $Q=(x_1x_2x_3)^3$ using the Modified Detached Coefficients method.

Step0**:** Convert Q to a multinomial, $Q=(p^2x_1+p^1x_2+p^0x_3)^3$

Step1: Use the staircase multinomial expansion to expand $(p^2x_1+p^1x_2+p^0x_3)^3$

$Q = (p^2x_1+p^1x_2+p^0x_3)^3 = p^6(x_1^3) + p^5(3x_1^2x_2) + p^4(3x_1x_2^2+3x_1^2x_3) + p^3(x_2^3+6x_1x_2x_3)$ $+ p^2(3x_2^2x_3+3x_1x_3^2) + p^1(3x_2x_3^2) + p^0\,(x_3^3)$--------(1)

Step 2: Input the values of x_1,x_2,x_3 into equation (1) in step (1)

Step 3: Write coefficients with digits more than 1 as placeholder in the expansion of p

Step 4: Terms of the same powers are added together.

Step5: Rearrange terms in descending powers of p. All coefficients in the current expansion will be of single digit.

Step 6: Extract the coefficients of the p's as the result of the power number.

Example 5:

Computing $Q = (123)^3$, Using the coefficients method.

Solution:

Step0: $Q = (123)^3 = (p^2x_1+px_2+p^0x_3)^3$

Step1: Use the staircase multinomial expansion to expand $(p^2x_1+p^1x_2+p^0x_3)^3$

$(p^2x_1+p^1x_2+p^0x_3)^3 =$

$p^6(x_1^3) + p^5(3x_1^2x_2)+p^4(3x_1x_2^2+3x_1^2x_3)+p^2(3x_2^2x_3+3x_1x_3^2)+p^1(3x_2x_3^2)+p^0(x_3^3)$------- (1)

Step 2: Input the values of $x_1=1, x_2 = 2, x_3 = 3$ in (1)

$(123)^3 =(1p^2+2p+3p^0)^3=p^6(1)+p^5(6)+p^4(21)+p^3(44)+p^2(63)+p^1(54)+p^0(27)$ when $10 = p$

$= p^6(1)+p^5(6)+p^4[2(10)+1]+p^3[4(10)+4]+p^2[6(10)+3]+p^1[5(10)+4]+p^0[2(10)+7]$

$= p^6(1)+p^5(6)+p^4[2(p)+1]+p^3[4(p)+4]+p^2[6(p)+3]+p^1[5(p)+4]+p^0[2(p)+7]$

$= p^6(1)+p^5(6)+2p^5+p^4+4p^4+4p^3+6p^3+3p^2+5p^2+4p^1+2p^1+7p^0$

$= 1p^6+8p^5+5p^4+10p^3+8p^2+6p^1+7p^0$

$=1p^6+8p^5+5p^4+(p)p^3+8p^2+6p^1+7p^0$

$=1p^6+8p^5+5p^4+1p^4+8p^2+6p^1+7p^0$

$=\mathbf{1}p^6+\mathbf{8}p^5+\mathbf{6}p^4+\mathbf{0}p^3+\mathbf{8}p^2+\mathbf{6}p^1+\mathbf{7}p^0$

Therefore $(123)^3 = 1860867$

3.5.2 Computing the cubes of integers, using the staircase method (3digits).

Methodology:

Problem: Evaluate $Q = (x_1x_2x_3)^3$ by Staircase Method.

Step 0: Convert Q to a multinomial form $Q= (p^2x_1+p^1x_2+p^0x_3)^3$,when p=10

Step 1: Expand $(p^2x_1+p^1x_2+p^0x_3)^3$ using the multinomial expansion.

$Q= (p^2x_1+p^1x_2+p^0x_3)^3 = p^6(x_1^3) + p^5(3x_1^2x_2) + p^4(3x_1x_2^2+3x_1^2x_3) + p^3(x_2^3+6x_1x_2x_3) + p^2(3x_2^2x_3+3x_1x_3^2) + p^1(3x_2x_3^2) + p^0(x_3^3)$

$Q=(x_1x_2x_3)^3 = P^6(x_1^3) + p^5(3x_1^2x_2) + p^4(3x_1x_2^2+3x_1^2x_3) + p^3(3x_2^2x_3+3x_1x_3^2) + p^2(3x_2x_3^2) + p^0(x_3^3)$-----(1) when p=10.

Step 2: Input the values of x_1,x_2,x_3 into equation(1) in step (1).

Step 3: Arrange the coefficients of p^{th} powers in the staircase form by starting coefficient of p^0 on the first line placing the coefficient of $p^1,p^2,\ldots,p^m$ with the last digit of p^i listed below the last but one digit of p^{i-1} .

Step4: Add staircase numbers arrangement column-wise as done in multiplication of two numbers.

Example 6:

Computing $Q=(123)^3$, using the staircase method

Step 0: $Q = (123)^3 =(p^2x_1+px_2+p^0x_3)^3$

Step 1: Use the staircase multinomial expansion to expand $(p^2x_1+p^1x_2+p^0x_3)^3$

$(p^2x_1+p^1x_2+p^0x_3)^3 = p^6(x_1^3) + p^5(3x_1^2x_2) + p^4(3x_1x_2^2+3x_1^2x_3) + p^2(3x_2^2x_3+3x_1x_3^2) + p^1(3x_2x_3^2) + p^0(x_3^3)$-------(1)

Step 2: Input the values of $x_1=1,x_2 = 2,x_3 = 3$ into equation(1) in step(1)

$(123)^3 = (1p^2+2p+3p^0)^3 = p^6(1)+p^5(6)+p^4(21)+p^3(44)+p^2(63)+p^1(54)+p^0(27)$ when $10 = p$

$$
\begin{array}{lcr}
(123)^3 & = & 27 \\
 & & +54 \\
 & & +63 \\
 & & +44 \\
 & & +21 \\
 & & +06 \\
 & & +01 \\
\hline
(123)^3 & = & 1860867
\end{array}
$$

CHAPTER 4

ANALYSIS AND RESULTS

4.0 INTRODUCTION

Ward (2007) Method of Detached Coefficients involves selecting the coefficients of polynomial terms and putting them together as the result of a power number.

Ward (2007) used the calculator to show power numbers that produces Binomial and Multinomial coefficients. Ward (2007) gave the examples below:

(i) $\mathbf{(1 + x)^2 = 1 + 2x + x^2}$ **……………………….(1)**

$\mathbf{11^2 = 121}$

(ii) $\mathbf{(1+x)^3 = 1 + 3x + 3x^2 + x^3}$ **……………………..(2)**

$\mathbf{11^3 = 1331}$

(iii) $\mathbf{(x_1 + x_2)^5 = x_1^5 + 5x_1^4 x_2 + 10x_1^3x_2^2 + 10x_1^2x_3^3 + 5x_1x_2^4 + x_2^5}$

…………..(3)

$\mathbf{11^5 \neq 15101051}$

$\mathbf{11^5 = 161051}$ **and not 15101051**

The equations above can be written as follows:

Equation (1) can be written as :

$(11x_1{}^0 + 1x_2{}^1)^2 = 1(x_1{}^0)^2 + 2(x_1{}^0)(x_2{}^1) + 1(x_2{}^1)^2$

$11^2 = 121$

Equation (2) can be written as:

$(1x_1{}^0 + 1x_2{}^1)^3 = 1(x_1{}^0)^3 + 3(x_1{}^0)^2(x_2{}^1) + 3(x_1{}^0)^1(x_2{}^1)^2 + 1(x_2{}^1)^3$

$11^3 = 1331$

Where $x_1{}^0 = 1$ and $x = x_2{}^1$

Also, equation (3) can be written as:

$$(p^1x_1 + p^0x_2)^5 = p^5 (x_1^5) + p^4(5x_1^4 x_2) + p^3(10x_1^3x_2^2) + p^2(10x_1^2x_3^3) + p^1(5x_1x_2^4) + p^0(x_2^5)$$

Using the method of Ward (2007) to find the result of a power number becomes a problem when the coefficients of a term is more than 1 digit. For example, the coefficients of $x_1^3x_2^2$ and $x_1^2x_3^3$ are 10 each which are two digits. Hence it is a problem to use the method of Ward (2007) to find 11^5.

Below are the methods of solution used to resolve the problem of Ward (2007).

4.1.1 Computing the fifth power of integers, using the modified detached coefficients method (2digits).

Problem: Evaluate $Q=(x_1x_2)^5$ using the Modified Detached Coefficients method.

Algorithm

Step 0: Convert Q to a binomial, $Q=(p^1x_1 + p^0x_2)^5$

Step 1: Use the multinomial expansion to expand $(p^1x_1 + p^0x_2)^5$

$$(p^1x_1 + p^0x_2)^5 = p^5 (x_1^5) + p^4(5x_1^4 x_2) + p^3(10x_1^3x_2^2) + p^2(10x_1^2x_3^3) + p^1(5x_1x_2^4) + p^0(x_2^5) \ldots\text{.(1)}$$

Step 2: Input the values of x_1, x_2 into equation (1) in step (1)

Step 3: Write coefficients with digits more than 1 as placeholder in the expansion of p.

Step 4: Terms of the same powers are added together.

Step 5: Rearrange terms in descending powers of p. All coefficients in the current expansion will be of single digit.

Step 6: Extract the coefficients of the p's as the result of the power number.

Illustration Ia:

Compute Q = $(11)^5$,Using the Modified Detached Coefficients Method.

Solution steps:

Step0: $Q = (11)^5=(p^1x_1+p^0x_2)^5$

Step1: Use the multinomial expansion to expand $(p^1x_1+p^0x_2)^5$

$$(p^1x_1 +p^0x_2)^5 = p^5 (x_1^5) + p^4(5x_1^4 x_2) + p^3(10x_1^3x_2^2) + p^2(10x_1^2x_3^3) + p^1(5x_1x_2^4) + p^0(x_2^5)\ldots.(1)$$

Step2: Input the values of $x_1=1,x_2=1$ in step (1)

$(11)^5 =(1p^1+1p^0)^5= p^5(1)+p^4(5)+p^3(10)+p^2(10)+p^1(5)+p^0(1)$ when 10 = p

$= p^5(1)+p^4(5)+p^3(p)+p^2(p)+p^1(5)+p^0(1)$

$= p^5(1)+p^4(5)+p^4+p^3+p^1(5)+p^0(1)$

$= 1p^5+ 6p^4+p^3+5p^1+1p^0$

$= \mathbf{1}p^5+ \mathbf{6}p^4+\mathbf{1}p^3+\mathbf{0}p^2+5p^1+\mathbf{1}p^0$

The coefficients are appended in their order of placement.

Therefore $(11)^5 = 161501$

NB: The coefficient of an absent p^m is zero, where m is the positive exponent of p.

4.1.2 Computing the squares of integers, using the modified detached coefficients method (4digits).

Problem: Evaluate $Q = (x_1x_2x_3x_4)^2$ using the modified detached coefficient method.

Algorithm.

Step 0 : Convert Q to a multinomial form $Q= (p^3x_1+p^2x_2+p^1x_3+p^0x_4)^3$

Step 1: Use the multinomial expansion to expand $(p^3x_1+p^2x_2+p^1x_3+p^0x_4)^2$

$(p^3x_1+p^2x_2+p^1x_3+p^0x_4)^2$

=

$p^6(x_1^2)+p^5(2x_1x_2)+p^4(x_2^2+2x_1x_3)+p^3(2x_1x_4+2x_2x_3)+p^2(2x_2x_4+x_3^2)+p^1(2x_3x_4)+p^0(x_4^2)$--

-(1) Step 2: Input the values of x_1, x_2, x_3, x_4 in (1)

Step 3: The coefficients of the p^{th} powers starting from the highest power to the lowest power becomes the result of $(x_1x_2x_3x_4)^2$ when 10=p.

Illustration Ib:

Computing Q=(1023)^2,Using the Modified Detached Coefficients Method.

Solution steps:

$(p^3x_1+p^2x_2+p^1x_3+p^0x_4)^2 =$

=

$p^6(x_1^2)+p^5(2x_1x_2)+p^4(x_2^2+2x_1x_3)+p^3(2x_1x_4+2x_2x_3)+p^2(2x_2x_4+x_3^2)+p^1(2x_3x_4)+p^0(x_4^2)$--

-(1)

Step 2: Input the values of $x_1=1$,$x_2=0$,$x_3=2$,$x_4=3$ in (1), we have;

$(1023)^2 \quad =(1p^3+0p^2+2p^1+3p^0)^2 = p^6(1)+p^5(0)+p^4(4)+p^3(6)+p^2(4)+p^1(12)+p^0(9)$

$=1p^6+0p^5+4p^4+6p^3+4p^2+12p^1+9p^0$

$=1p^6+0p^5+4p^4+6p^3+4p^2+(10+2)p^1+9p^0$

$=1p^6+0p^5+4p^4+6p^3+4p^2+(p+2)p^1+9p^0$

$=1p^6+0p^5+4p^4+6p^3+4p^2+p^2+2p^1+9p^0$

$=\mathbf{1}p^6+\mathbf{0}p^5+\mathbf{4}p^4+\mathbf{6}p^3+\mathbf{5}p^2+\mathbf{2}p^1+\mathbf{9}p^0$

The coefficients are appended in order of their placement

Therefore $(1023)^2=1046529$

4.1.3 Computing the squares of integers, using the modified detached coefficients method (5digits).

Problem: Evaluate Q = $(x_1x_2x_3x_4x_5)^2$ using the modified detached coefficient method.

Algorithm.

Step 0: Convert Q to a multinomial form Q= $(p^3x_1+p^2x_2+p^1x_3+p^0x_4)^2$

Step1: Use the multinomial expansion to expand $(p^4x_1+p^3x_2+p^2x_3+P^1x_4+p^0x_5)^2$

$(p^4x_1+p^3x_2+p^2x_3+p^1x_4+p^0x_5)^2 =$

$p^8(x_1^2)+p^7(2x_1x_2)+p^6(x_2^2+2x_1x_3)+p^5(2x_1x_4+2x_2x_3)+p^4(2x_1x_5+2x_2x_4+x_3^2)+p^3(2x_2x_5+2x_3x_4)+$

$p^2(2x_3x_5+x_4^2)+p^1(2x_4x_5)+p^0(x_5^2)$ ---(1)

Step 2: Input the values of x_1,x_2,x_3,x_4,x_5 in (1)

Step 3: The coefficients of the p^{th} powers starting from the highest power to the lowest power becomes the result of $(x_1x_2x_3x_4x_5)^2$ when 10=p

Illustration Ic:

Compute Q=$(10011)^2$,Using the Modified Detached Coefficients Method.

Solution steps:

Step 0: Convert Q to a multinomial form Q= $(p^4x_1+p^3x_2+p^2x_3+p^1x_4+p^0x_5)^2$

Step1: Use the staircase multinomial expansion to expand $(p^4x_1+p^3x_2+p^2x_3+P^1x_4+p^0x_5)^2$

$(p^4x_1+p^3x_2+p^2x_3+p^1x_4+p^0x_5)^2 =$

$p^8(x_1^2)+p^7(2x_1x_2)+p^6(x_2^2+2x_1x_3)+p^5(2x_1x_4+2x_2x_3)+p^4(2x_1x_5+2x_2x_4+x_3^2)+p^3(2x_2x_5+2x_3x_4)+$

$p^2(2x_3x_5+x_4^2)+p^1(2x_4x_5)+p^0(x_5^2)$ ---(1)

Step2: Input the values of $x_1=1, x_2=0, x_3=0, x_4=1, x_5=1$ in (1)

$(10011)^2=(p^3x_1+p^2x_2+p^1x_3+p^0x_4)^2$ =

$p^8(1)+p^7(0)+p^6(0)+p^5(2)+p^4(2)+p^3(0)+p^2(1)+p^1(2)+p^0(1)$

$=\mathbf{1}p^8+\mathbf{0}p^7+\mathbf{0}p^6+\mathbf{2}p^5+\mathbf{2}p^4+\mathbf{0}p^3+\mathbf{1}p^2+\mathbf{2}p^1+\mathbf{1}p^0$

The coefficients appended are in their order of placement.

Therefore $(10011)^2=100220121$

4.1.4 Computing the cubes of integers, using the modified detached coefficients method (4digits).

Problem: Evaluate $Q=(x_1x_2x_3x_4)^3$ using the modified detached coefficient method.

Algorithm.

Step 0 : Convert Q to a multinomial form $Q=(p^3x_1+p^2x_2+p^1x_3+p^0x_4)^3$

Step 1: Use the multinomial expansion to expand $(p^3x_1+p^2x_2+p^1x_3+p^0x_4)^3$

$(p^3x_1+p^2x_2+p^1x_3+p^0x_4)^3$ =

$p^9(x_1^3)+p^8(3x^2{}_1x_2)+p^7(3x_1x_2^2+3x_1^2x_3)+p^6(x_2^3+6x_1x_2x_3+3x_1^2x_4)+p^5(3x_2^2x_3+3x_1x_3^2+6x_1x_2x_4)+$

$p^4(3x_2x_3^2+3x_2^2x_4+6x_1x_3x_4)+p^3(x_3^3+6x_2x_3x_4+3x_1x_4^2)+p^2(3x_3^2x_4+3x_2x_4^2)+p^1(3x_3x_4^2)+p^0(x_4^3)$----(1)

Step2: Input the values of x_1, x_2, x_3, x_4 in (1)

Step3: The coefficients of the p^{th} powers starting from the highest power to the lowest power becomes the result of $(x_1x_2x_3x_4)^3$ when 10=p.

Illustration Id:

Compute Q= $(1023)^3$, Using the Modified Detached Coefficients Method.

Solution steps:

Step 0: Convert Q to a multinomial form Q= $(p^3x_1+p^2x_2+p^1x_3+p^0x_4)^3$

Step 1: $(p^3x_1+p^2x_2+p^1x_3+p^0x_4)^3 = p^9(x_1^3) + p^8(3x^2{}_1x_2) + p^7(3x_1x_2^2+3x_1^2x_3) + p^6(x_2^3+6x_1x_2x_3+3x_1^2x_4) + p^5(3x_2^2x_3+3x_1x_3^2+6x_1x_2x_4) + p^4(3x_2x_3^2+3x_2^2x_4 + 6x_1x_3x_4) + p^3(x_3^3+6x_1x_3x_4+3x_1x_4^2) + p^2(3x_3^2x_4+3x_1x_4^2) + p^1(3x_3x_4^2) + p^0(x_4^3)$----(1)

Step2: Input the values of $x_1=1,x_2=0,x_3=2,x_4=3$ in (1), we have;

$(1023)^3 = p^9(1)+p^8(0)+p^7(6)+p^6(9)+p^5(12)+p^4(36)+p^3(35)+p^2(36)+p^1(54)+ p^0(27)$

$= p^9(1)+p^8(0)+p^7(6)+p^6(9)+p^5(10+2)+p^4[3(10)+6]+p^3[3(10)+5]+p^2[3(10)+6]+ p^1[5(10)+4]+p^0[2(10)+7]$

$= p^9(1)+p^8(0)+p^7(6)+p^6(9)+p^5(p+2)+p^4[3(p)+6]+p^3[3(p)+5]+p^2[3(p)+6]+p^1[5(p)+4]+ p^0[2(p)+7]$

$= 1p^9+0p^8+6p^7+9p^6+p^6+2p^5+3p^5+6p^4+3p^4+5p^3+3p^3+6p^2+5p^2+4p^1+ 2p^1+7p^0$

$=1p^9+0p^8+6p^7+10p^6+5p^5+9p^4+8p^3+11p^2+6p^1+7p^0$

$=1p^9+0p^8+6p^7+10p^6+5p^5+9p^4+8p^3+(10+1)p^2+6p^1+7p^0$

$=1p^9+0p^8+6p^7+pp^6+5p^5+9p^4+8p^3+(p+1)p^2+6p^1+7p^0$

$=1p^9+0p^8+6p^7+p^7+5p^5+9p^4+8p^3+p^3+1p^2+6p^1+7p^0$

$=1p^9+0p^8+7p^7+5p^5+9p^4+9p^3+1p^2+6p^1+7p^0$

$=\mathbf{1}p^9+\mathbf{0}p^8+\mathbf{7}p^7+\mathbf{0}p^6+\mathbf{5}p^5+\mathbf{9}p^4+\mathbf{9}p^3+\mathbf{1}p^2+\mathbf{6}p^1+\mathbf{7}p^0$

The coefficients are appended in their order of placement.

Therefore $(1023)^3$ = 1070599167

4.1.5 Computing the cubes of integers, using the modified detached coefficients method (5digits).

Problem: Evaluate $Q = (x_1x_2x_3x_4x_5)^3$ using the modified detached coefficient method.

Algorithm.

Step 0: Convert Q to a multinomial form $Q = (p^4x_1+p^3x_2+p^2x_3+p^1x_4+p^0x_5)^3$

Step 1: Use the multinomial expansion to expand $(p^4x_1+p^3x_2+p^2x_3+p^1x_4+p^0x_5)^3$

$(p^4x_1+p^3x_2+p^2x_3+p^1x_4+p^0x_5)^3 =$

$P^{12}(x_1^3)+p^{11}(3x^2{}_1x_2)+p^{10}(3x_1x_2^2+3x_1^2x_3)+p^9(x_2^3+6x_1x_2x_3+3x_1^2x_4)+p^8(3x_2^2x_3+3x_1^2x_5+6x_1x_2x_4)+$

$p^7(3x_2x_3^2+3x_2^2x_4$

$+6x_1x_3x_4+6x_1x_2x_5)+p^6(x_3^3+6x_2x_3x_4+3x_1x_4^2+6x_1x_3x_5+3x_2^2x_5)+p^5(3x_3^2x_4+3x_2x_4^2$

$+6x_2x_3x_5+6x_1x_4x_5)+p^4(3x_3x_4^2+3x_3^2x_5+6x_2x_4x_5+3x_1x_5^2)+p^3(x_4^3+6x_3x_4x_5+3x_2x_5^2)+$

$p^2(3x_4^2x_5+3x_3x_5^2)+p^1(3x_4x_5^2)+p^0(x_5^3)$-------(1)

Step 2: Input the values of x_1, x_2, x_3, x_4, x_5 in (1)

Step 3: The coefficients of the p^{th} powers starting from the highest power to the lowest power becomes the result of $(x_1x_2x_3x_4x_5)^3$ when 10=p

Illustration Ie:

Compute $(10011)^3$, Using the Modified Detached Coefficients Method.

Solution steps:

Step1: $(p^4x_1+p^3x_2+p^2x_3+p^1x_4+p^0x_5)^3 = p^{12}(x_1^3) + p^{11}(3x^2{}_1x_2) + p^{10}(3x_1x_2^2+3x_1^2x_3) +$ $p^9(x_2^3+6x_1x_2x_3+3x_1^2x_4) + p^8(3x_2^2x_3+3x_1^2x_5+6x_1x_2x_4) +$ $p^7(3x_2x_3^2+3x_2^2x_4+6x_1x_3x_4+6x_1x_2x_5) + p^6(x_3^3+6x_2x_3x_4+3x_1x_4^2+6x_1x_3x_5+3x_2^2x_5) +$

$p^5(3x_3^2x_4+3x_2x_4^2+6x_2x_3x_5+6x_1x_4x_5) + p^4(3x_3x_4^2+3x_3^2x_5+6x_2x_4x_5+3x_1x_5^2) + p^3(x_4^3+6x_3x_4x_5+3x_2x_5^2)+ p^2(3x_4^2x_5+3x_3x_5^2) + p^1(3x_4x_5^2) + p^0(x_5^3)$------(1)

Step2: Input the values of $x_1=1,x_2=0,x_3=0,x_4=1,x_5=1$ into equation (1),we have

$(10011)^3=(1p^4+0p^3+0p^2+1p^1+1p^0)^3$

$=P^{12}(1)+p^{11}(0)+p^{10}(0)+p^9(3)+p^8(3)+p^7(0)+p^6(3)+p^5(6)+p^4(3)+p^3(1)+p^2(3)+p^1(3)+p^0(1)$

$= \mathbf{1}P^{12}+\mathbf{0}p^{11}+\mathbf{0}p^{10}+\mathbf{3}p^9+\mathbf{3}p^8+\mathbf{0}p^7+\mathbf{3}p^6+\mathbf{6}p^5+\mathbf{3}p^4+\mathbf{1}p^3+\mathbf{3}p^2+\mathbf{3}p^1+\mathbf{1}p^0$

The coefficients appended are in their order of placement.

Therefore $(10011)^3 = 1003303631331$

4.2 COMPUTING THE FIFTH POWER OF INTEGERS, USING THE STAIRCASE METHOD (2digits).

Problem: Evaluate $Q=(x_1x_2)^5$ using the Modified Detached Coefficients method.

Algorithm

Step0**:** Convert Q to a binomial, $Q=(\mathbf{p^1x_1+p^0x_2})^5$

Step1: Use the multinomial expansion to expand $(\mathbf{p^1x_1+p^0x_2})^5$

$(p^1x_1 + p^0x_2)^5 = p^5 (x_1^5)+p^4(5x_1^4 x_2)+p^3(10x_1^3x_2^2)+ p^2(10x_1^2x_3^3)+ p^1(5x_1x_2^4)+p^0(x_2^5)$….(1)

Step 2: Input the values of x_1,x_2 into equation (1) of step (1).

Step 3: Arrange the coefficients of the p^{th} powers in the staircase form.

Step 4: Add staircase digit wise, one digit to the left and placing them successively beneath the previous coefficient.

Illustration IIa:

Computing $Q = (11)^5$,Using the Staircase Method.

Solution steps:

Step 0: $Q = (11) = (p^1x_1+p^0x_2)^5$

Step 1: Use the multinomial expansion to expand $(p^1x_1+p^0x_2)^5$

$$\mathbf{(p^1x_1 + p^0x_2)^5 = p^5 (x_1^5) + p^4(5x_1^4 x_2) + p^3(10x_1^3x_2^2) + p^2(10x_1^2x_3^3) + p^1(5x_1x_2^4) + p^0(x_2^5)} \ldots.(1)$$

Step2: Input the values of $x_1=1, x_2=1$ in step (1)

$(11)^5 = (1p^1+1p^0)^5 = p^5(1)+p^4(5)+p^3(10)+p^2(10)+p^1(5)+p^0(1)$

	=	01
		+05
		+10
		+05
		+01
$(11)^5$	=	165051

4.2.1 Computing the squares of integers, using the staircase method (4digits).

Problem: Evaluate $Q = (x_1x_2x_3x_4)^2$ by Staircase Method.

Algorithm.

Step 0 : Convert Q to a multinomial form $Q= (p^3x_1+p^2x_2+p^1x_3+p^0x_4)^3$,when p=10

Step 1: Use the staircase multinomial expansion to expand $(p^3x_1+p^2x_2+p^1x_3+P^0x_4)^2$

$(p^3x_1+p^2x_2+p^1x_3+p^0x_4)^2 =$

$=$

$p^6(x_1^2)+p^5(2x_1x_2)+p^4(x_2^2+2x_1x_3)+p^3(2x_1x_4+2x_2x_3)+p^2(2x_2x_4+x_3^2)+p^1(2x_3x_4)+p^0(x_4^2)$---(1)

Step 2: Input the values of x_1, x_2, x_3, x_4 in step(1)

Step 3: Arrange the coefficients of the p^{th} powers in the staircase form

Step 4: Add staircase digit wise, one digit to the left and placing them successively beneath the previous coefficient.

Illustration IIb

Compute $Q=(1023)^2$, Using the staircase method.

Solution steps:

Step 0: Convert Q to a multinomial form $Q= (p^3x_1+p^2x_2+p^1x_3+p^0x_4)^3$,when p=10

Step 1: $(p^3x_1+p^2x_2+p^1x_3+p^0x_4)^2 =$

$$= p^6(x_1^2)+p^5(2x_1x_2)+p^4(x_2^2+2x_1x_3)+p^3(2x_1x_4+2x_2x_3)+p^2(2x_2x_4+x_3^2)+p^1(2x_3x_4)+p^0(x_4^2) \text{---}(1)$$

Step 2: Input the values of $x_1=1$,$x_2=0$,$x_3=2$,$x_4=3$ in (1), we have;

$(1023)^2 = p^6(1)+p^5(0)+p^4(4)+p^3(6)+p^2(4)+p^1(12)+p^0(9)$

$$
\begin{array}{rr}
= & 09 \\
 & +12 \\
 & +04 \\
 & +06 \\
 & +04 \\
 & +00 \\
 & +01 \\
\hline
(1023)^2 = & 1046529
\end{array}
$$

4.2.2 Computing the squares of integers, using the staircase method (5digits).

Problem: Evaluate Q = $(x_1x_2x_3x_4x_5)^2$ using the modified detached coefficient method.

Algorithm.

Step 0 : Convert Q to a multinomial form Q= $(p^3x_1+p^2x_2+p^1x_3+p^0x_4)^2$

Step 1: Use the staircase multinomial expansion to expand $(p^4x_1+p^3x_2+p^2x_3+P^1x_4+p^0x_5)^2$

$(p^4x_1+p^3x_2+p^2x_3+p^1x_4+p^0x_5)^2 =$

$p^8(x_1^2)+p^7(2x_1x_2)+p^6(x_2^2+2x_1x_3)+p^5(2x_1x_4+2x_2x_3)+p^4(2x_1x_5+2x_2x_4+x_3^2)+p^3(2x_2x_5+2x_3x_4)+$

$p^2(2x_3x_5+x_4^2)+p^1(2x_4x_5)+p^0(x_5^2)$ ---(1)

Step 2: Input the values of x_1,x_2,x_3,x_4,x_5 in (1)

Step 3: Arrange the coefficients of the p^{th} powers in the staircase form.

Step 4: Add staircase digit wise, one digit to the left and placing them successively beneath the previous coefficient.

Illustration IIc:

Compute Q= $(10011)^2$, Using the staircase method (5 digits).

Solution steps:

Step 0: Convert Q to a multinomial form Q= $(p^3x_1+p^2x_2+p^1x_3+p^0x_4)^3$

Step 1: Use the staircase multinomial expansion to expand

$(p^4x_1+p^3x_2+p^2x_3+P^1x_4+p^0x_5)^2$

$(p^4x_1+p^3x_2+p^2x_3+p^1x_4+p^0x_5)^2 =$

=

$p^8(x_1^2)+p^7(2x_1x_2)+p^6(x_2^2+2x_1x_3)+p^5(2x_1x_4+2x_2x_3)+p^4(2x_1x_5+2x_2x_4+x_3^2)+p^3(2x_2x_5+2x_3x_4)+$

$p^2(2x_3x_5+x_4^2)+p^1(2x_4x_5)+p^0(x_5^2)$ ---(1)

Step 2: Input the values of $x_1=1$, $x_2=0$, $x_3=0$, $x_4=1$, $x_5=1$ in (1)

$(10011)^2 = (1p^3+0p^2+0p^1x_3+1p^0)^2$

$= p^8(1)+p^7(0)+p^6(0)+p^5(2)+p^4(2)+p^3(0)+p^2(1)+p^1(2)+p^0(1)$

01

+02

+01

+00

+02

+02

+00

+00

+01

$(10011)^2 =$ 100220121

4.2.3 Computing the cubes of integers, using the staircase method (4digits).

Problem: Evaluate $Q = (x_1x_2x_3x_4)^3$ using the Staircase Method.

Algorithm.

Step 0: Convert Q to a multinomial form $Q= (p^3x_1+p^2x_2+p^1x_3+p^0x_4)^3$,when p=10

Step 1: Use the multinomial expansion to expand $(p^3x_1+p^2x_2+p^1x_3+p^0x_4)^3$

$(p^3x_1+p^2x_2+p^1x_3+p^0x_4)^3 =$

$p^9(x_1^3)+p^8(3x^2{}_1x_2)+p^7(3x_1x_2^2+3x_1^2x_3)+p^6(x_2^3+6x_1x_2x_3+3x_1^2x_4)+p^5(3x_2^2x_3+3x_1x_3^2+6x_1 x_2x_4)+$

$p^4(3x_2x_3^2+3x_2^2x_4 \quad +6x_1x_3x_4)+p^3(x_3^3+6x_1x_3x_4+3x_1x_4^2)+p^2(3x_3^2x_4+3x_1x_4^2)+p^1(3x_3x_4^2)+ p^0(x_4^3)$----(1)

Step 2: Input the values of x_1,x_2,x_3,x_4 in step(1), result is polynomial in p

Step 3: Arrange the coefficients of the p^{th} powers in the staircase form

Step 4: Add staircase digit wise, one digit to the left and placing them successively beneath the previous coefficient.

Illustration IId:

Compute $Q=(1023)^3$,Using the staircase method

Solution steps:

Step 0: Convert Q to a multinomial form $Q= (p^3x_1+p^2x_2+p^1x_3+p^0x_4)^3$

Step 1: $(p^3x_1+p^2x_2+p^1x_3+p^0x_4)^3 = p^9(x_1^3) + p^8(3x^2{}_1x_2) + p^7(3x_1x_2^2+3x_1^2x_3) + p^6(x_2^3+6x_1x_2x_3+3x_1^2x_4) + p^5(3x_2^2x_3+3x_1x_3^2+6x_1x_2x_4) + p^4(3x_2x_3^2+3x_2^2x_4 + 6x_1x_3x_4) + p^3(x_3^3+6x_1x_3x_4+3x_1x_4^2) + p^2(3x_3^2x_4+3x_1x_4^2) + p^1(3x_3x_4^2) + p^0(x_4^3)$-----(1)

Step 2: Input the values of $x_1=1,x_2=0,x_3=2,x_4=3$ into equation in step (1), we have;

$(1023)^3= (1p^3+0p^2+2p^1+3p^0)^3 = p^9(1)+p^8(0) + p^7(6)+p^6(9) + p^5(12) + p^4(36) + p^3(35) + p^2(36) + p^1(54) + p^0(27)$

=		27
		+54
		+36
		+35
		+36
		+12
		+09
		+06
		+00
		+01
$(1023)^3$	=	1070599167

4.2.4 Computing the cubes of integers, using the staircase method (5digits).

Problem: Evaluate $Q = (x_1x_2x_3x_4x_5)^3$ using the Staircase method.

Algorithm.

Step 0 : Convert Q to a multinomial form $Q= (p^4x_1+p^3x_2+p^2x_3+p^1x_4+p^0x_5)^3$

Step 1: Use the staircase multinomial expansion to expand $(p^4x_1+p^3x_2+p^2x_3+p^1x_4+p^0x_5)^3$

$(p^4x_1+p^3x_2+p^2x_3+p^1x_4+p^0x_5)^3 = p^{12}(x_1^3) + p^{11}(3x^2{}_1x_2) + p^{10}(3x_1x_2^2+3x_1^2x_3) + p^9(x_2^3+6x_1x_2x_3+3x_1^2x_4) + p^8(3x_2^2x_3+3x_1x_3^2+6x_1x_2x_4) + p^7(3x_2x_3^2+3x_2^2x_4 + 6x_1x_3x_4+6x_1x_2x_5) + p^6(x_3^3+6x_2x_3x_4+3x_1x_4^2+6x_1x_3x_5+3x_2^2x_5) + p^5(3x_3^2x_4+3x_2x_4^2+ 6x_2x_3x_5+6x_1x_4x_5) + p^4(3x_3x_4^2+3x_3^2x_5+6x_2x_4x_5+3x_1x_5^2) + p^3(x_4^3+6x_3x_4x_5+3x_2x_5^2) + p^2(3x_4^2x_5+3x_3x_5^2) + p^1(3x_4x_5^2) + p^0(x_5^3)$------(1)

Step 2: Input the values of x_1,x_2,x_3,x_4,x_5 in step (1)

Step 3: Arrange the coefficients of the p^{th} powers in the staircase form

Step 4: Add staircase digit wise, one digit to the left and placing them successively beneath the previous coefficient.

Illustration IIe:

Compute Q = $(10011)^3$,Using the staircase method.

Solution steps:

Step 0: Convert Q to a multinomial form Q= $(p^4x_1+p^3x_2+p^2x_3+p^1x_4+p^0x_5)^3$

Step 1: $(p^4x_1+p^3x_2+p^2x_3+p^1x_4+p^0x_5)^3 = p^{12}(x_1^3) + p^{11}(3x^2_1x_2) + p^{10}(3x_1x_2^2+3x_1^2x_3) + p^9(x_2^3+6x_1x_2x_3+3x_1^2x_4) + p^8(3x_2^2x_3+3x_1x_3^2+6x_1x_2x_4) + p^7(3x_2x_3^2+3x_2^2x_4 + 6x_1x_3x_4+6x_1x_2x_5) + p^6(x_3^3+6x_2x_3x_4+3x_1x_4^2+6x_1x_3x_5+3x_2^2x_5) + p^5(3x_3^2x_4+3x_2x_4^2 + 6x_2x_3x_5+6x_1x_4x_5) + p^4(3x_3x_4^2+3x_3^2x_5+6x_2x_4x_5+3x_1x_5^2) + p^3(x_4^3+6x_3x_4x_5+3x_2x_5^2) + p^2(3x_4^2x_5+3x_3x_5^2) + p^1(3x_4x_5^2) + p^0(x_5^3)$------(1)

Step 2: Input the values of $x_1=1,x_2=0,x_3=0,x_4=1,x_5=1$ in (1),we have

$(10011)^3=P^{12}(1)+p^{11}(0)+p^{10}(0)+p^9(3)+p^8(3)+p^7(0)+p^6(3)+p^5(6)+p^4(3)+p^3(1)+p^2(3)+p^1(3)+p^0(1)$

$= 1P^{12}+0p^{11}+0p^{10}+3p^9+3p^8+0p^7+3p^6+6p^5+3p^4+1p^3+3p^2+3p^1+1p^0$

$$
\begin{array}{lr}
= & 01 \\
 & +03 \\
 & +03 \\
 & +01 \\
 & +03 \\
 & +06 \\
 & +03 \\
 & +00 \\
 & +03 \\
 & +03 \\
 & +00 \\
 & +00 \\
 & +01 \\
\hline
(10011)^3 = & 1003303631331
\end{array}
$$

4.3 COMPARING THE MODIFIED DETACHED COEFFICIENTS METHOD WITH THE STAIRCASE METHOD.

The comparison involves comparing the steps in computing positive integers raised to any Positive nth power by using both the modified detached coefficients method and the staircase method. Some examples are used to illustrate the steps involve in solving powers of positive integers using the methods mentioned above.

4.3.1 Computing the cubes of integers, using the modified detached coefficients method.

Problem: Evaluate Q = $(x_1x_2x_3x_4)^3$ using the modified detached coefficient method.

Algorithm.

Step 0: Convert Q to a multinomial form Q= $(p^3x_1+p^2x_2+p^1x_3+p^0x_4)^3$

Step1: Use the staircase multinomial expansion to expand $(p^3x_1+p^2x_2+p^1x_3+p^0x_4)^3$

$(p^3x_1+p^2x_2+p^1x_3+p^0x_4)^3 = p^9(x_1^3) + p^8(3x^2{}_1x_2) + p^7(3x_1x_2^2+3x_1^2x_3) + p^6(x_2^3+6x_1x_2x_3+3x_1^2x_4) + p^5(3x_2^2x_3+3x_1x_3^2+6x_1x_2x_4) + p^4(3x_2x_3^2+3x_2^2x_4 + 6x_1x_3x_4) + p^3(x_3^3+6x_1x_3x_4+3x_1x_4^2) + p^2(3x_3^2x_4+3x_1x_4^2) + p^1(3x_3x_4^2) + p^0(x_4^3)$-----(1)

Step 2: Input the values of x_1,x_2,x_3,x_4 in equation (1) of step (1).

Step 3: The coefficients of the p^{th} powers starting from the highest power to the lowest power becomes the result of $(x_1x_2x_3x_4)^3$ when 10=p.

Illustration IIIa:

Compute Q= $(1023)^3$, Using the Modified Detached Coefficients Method

Solution steps:

Step 0: Convert Q to a multinomial form Q= $(p^3x_1+p^2x_2+p^1x_3+p^0x_4)^3$

Step: $(p^3x_1+p^2x_2+p^1x_3+p^0x_4)^3 = p^9(x_1^3) + p^8(3x^2{}_1x_2) + p^7(3x_1x_2^2+3x_1^2x_3) + p^6(x_2^3+6x_1x_2x_3+3x_1^2x_4) + p^5(3x_2^2x_3+3x_1x_3^2+6x_1x_2x_4) + p^4(3x_2x_3^2+3x_2^2x_4 + 6x_1x_3x_4) + p^3(x_3^3+6x_1x_3x_4+3x_1x_4^2) + p^2(3x_3^2x_4+3x_1x_4^2) + p^1(3x_3x_4^2) + p^0(x_4^3)$-----(1)

Step 2: Input the values of x_1=1,x_2=0,x_3=2,x_4=3 in (1), we have;

$(1023)^3 = p^9(1)+p^8(0)+p^7(6)+p^6(9)+p^5(12)+p^4(36)+p^3(35)+p^2(36)+p^1(54)+ p^0(27)$

$= p^9(1)+p^8(0)+p^7(6)+p^6(9)+p^5(10+2)+p^4[3(10)+6]+p^3[3(10)+5]+p^2[3(10)+6]+ p^1[5(10)+4]+ p^0[2(10)+7]$ $= p^9(1) + p^8(0) + p^7(6) + p^6(9) + p^5(p+2) + p^4[3(p) + 6]+p^3[3(p) + 5]+p^2[3(p)+6] + p^1[5(p)+4] + p^0[2(p)+7]$

$= 1p^9+0p^8+6p^7+9p^6+p^6+2p^5+3p^5+6p^4+3p^4+5p^3+3p^3+6p^2+5p^2+4p^1+2p^1+7p^0$

$=1p^9+0p^8+6p^7+10p^6+5p^5+9p^4+8p^3+11p^2+6p^1+7p^0$

$=1p^9+0p^8+6p^7+10p^6+5p^5+9p^4+8p^3+(10+1)p^2+6p^1+7p^0$

$=1p^9+0p^8+6p^7+pp^6+5p^5+9p^4+8p^3+(p+1)p^2+6p^1+7p^0$

$=1p^9+0p^8+6p^7+p^7+5p^5+9p^4+8p^3+p^3+1p^2+6p^1+7p^0$

$=1p^9+0p^8+7p^7+5p^5+9p^4+9p^3+1p^2+6p^1+7p^0$

$=\mathbf{1}p^9+\mathbf{0}p^8+\mathbf{7}p^7+\mathbf{0}p^6+\mathbf{5}p^5+\mathbf{9}p^4+\mathbf{9}p^3+\mathbf{1}p^2+\mathbf{6}p^1+\mathbf{7}p^0$

The coefficients appended are in their order of placement.

Therefore $(1023)^3$ = 1070599167

4.3.2 Computing the cubes of integers, using the staircase method (4digits).

Problem: Evaluate $Q = (x_1x_2x_3x_4)^3$ by Staircase Method

Step 0: Convert Q to a multinomial form $Q= (p^3x_1+p^2x_2+p^1x_3+p^0x_4)^3$,when p=10

Step 1: Use the staircase multinomial expansion to expand $(p^3x_1+p^2x_2+p^1x_3+p^0x_4)^3$

$(p^3x_1+p^2x_2+p^1x_3+p^0x_4)^3 = p^9(x_1^3) + p^8(3x^2{}_1x_2) + p^7(3x_1x_2^2+3x_1^2x_3) + p^6(x_2^3+6x_1x_2x_3+3x_1^2x_4) + p^5(3x_2^2x_3+3x_1x_3^2+6x_1x_2x_4) + p^4(3x_2x_3^2+3x_2^2x_4 + 6x_1x_3x_4) + p^3(x_3^3+6x_1x_3x_4+3x_1x_4^2) + p^2(3x_3^2x_4+3x_1x_4^2) + p^1(3x_3x_4^2) + p^0(x_4^3)$----(1)

Step 2: Input the values of x_1,x_2,x_3,x_4 in equation (1) of step(1).

Step 3: Arrange the coefficients of the p^{th} powers in the staircase form

Step 4: Add staircase digit wise, one digit to the left and placing them successively beneath the previous coefficient.

Illustration IIIb:

Computing $Q=(1023)^3$,Using the staircase method

Solution steps:

Step 0: Convert Q to a multinomial form $Q= (p^3x_1+p^2x_2+p^1x_3+p^0x_4)^3$

Step1: $(p^3x_1+p^2x_2+p^1x_3+p^0x_4)^3 = p^9(x_1^3) + p^8(3x^2{}_1x_2) + p^7(3x_1x_2^2+3x_1^2x_3) + p^6(x_2^3+6x_1x_2x_3+3x_1^2x_4) + p^5(3x_2^2x_3+3x_1x_3^2+6x_1x_2x_4) + p^4(3x_2x_3^2+3x_2^2x_4 + 6x_1x_3x_4) + p^3(x_3^3+6x_1x_3x_4+3x_1x_4^2) + p^2(3x_3^2x_4+3x_1x_4^2) + p^1(3x_3x_4^2) + p^0(x_4^3)$----(1)

Step 2: Input the values of $x_1=1,x_2=0,x_3=2,x_4=3$ into equation in step (1), we have;

$(1023)^3= (1p^3+0p^2+2p^1+3p^0)^3 = p^9(1) + p^8(0) + p^7(6) + p^6(9) + p^5(12) + p^4(36)+ p^3(35) + p^2(36) + p^1(54) + p^0(27)$

= 27

+54

+36

+35

+36

+12

+09

+06

+00

+01

$(1023)^3$ = 1070599167

4.4 DISCUSSION:

It can be seen from the illustrations shown in section 4.3.1 and section 4.3.2, the steps used in finding Q^3 in section 4.3.2 is lesser than the steps used in section 4.3.1.

Comparatively, the Staircase method has lesser steps than the Modified Detached Coefficients method and therefore the Staircase method will be faster than Modified Detached Coefficients method in terms of computation.

However, the method of Ward (2007), basically looked at powers of numbers that produces binomial and multinomial coefficients but the modified detached coefficients method generally produces results of positive integers raised to any positive power.

The Modified Detached Coefficients Method and the Staircase Method would always produce the same results of any positive integer or integers raised to any positive power as shown in Table 4.1 below.

Table 4.1: Summary of results.

METHOD	NUMBER,($d_1d_2\ldots d_m$)	POWER,(n)	RESULT
Staircase	(123)	2	15129
Modified Detached Coefficient	(123)	2	15129
Staircase	(1023)	3	1070599167
Modified Detached Coefficient	(1023)	3	1070599167
Staircase	(10011)	2	100220121
Modified Detached Coefficient	(10011)	2	100220121

From table 4.1 above , it is clear that both the Modified Detached Coefficients Method and

Staircase Method produce the same results.

CHAPTER 5

CONCLUSION AND RECOMMENDATION

5.1 CONCLUSION:

This research is based on improving the method of Detached Coefficients proposed by Ward (2007). The Staircase Method and the Modified Detached Coefficients Method are used to resolve the problem of Ward (2007).

Also, comparison of the Modified Detached Coefficients Method with the Staircase method were made in terms of computing powers of positive integers.

Examples shown in section 4.3.2 and section 4.3.1 clearly indicates that the Staircase Method has lesser steps than the Modified Detached Coefficients Method in terms of computation.

Therefore, the Staircase Method will be faster in terms of computation than the Modified Detached Coefficient Method. And also they produce same results.

Also, the modified detached coefficients method does not only produce the results of power numbers which are binomial coefficients or multinomial coefficients as illustrated in sections

4.0 (i) and 4.0(ii) but it can be used to find the result of any positive integer(s) raised to any positive power.

5.2 RECOMMENDATION

Since the Modified Detached Coefficients Method can be used to find any positive integer to the nth power and based on it's simple procedure I therefore recommend that researchers should research into this method to include negative numbers and fractions.

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