KWAME NKRUMAH UNIVERSITY OF SCIENCE AND TECHNOLOGY,

KUMASI

MODELING VEHICLE TRAFFIC FLOW BETWEEN TRAFFIC LIGHT NODES WITH PARTIAL DIFFERENTIAL EQUATIONS

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BY

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DECLARATION

I hereby declare that this submission is my own work towards Master of Science degree and that, to the best of my knowledge, it contains no material previously published by another person nor material which has been accepted for the award of any other degree of the University, except where due acknowledgement has been made in the text.

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DEDICATION

This work is dedicated to, my wife Dorothy and my children: Afua Obenewah, Agyenim Boateng, Nana Kwame and Isaac, all my brothers and sisters and my parents especially Mr .and Mrs. Boateng and to all prayer members

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ABSTRACT

Vehicle traffic flow has become a challenge in this modern world of automobiles due to constantly accelerated influx of vehicles on the road network. The challenge of traffic flow has motivated many researchers to model traffic flow at both the macroscopic and microscopic levels. In order to ensure free flow of vehicles in the cities, road engineers have developed and implemented traffic lights to regulate the inflow and outflow of vehicles at the major intersections of road. However, the traffic lights have become one the major causes of traffic jams in the cities since they force the vehicles to stop causing a discontinuity of flow. This study seeks to model the continuous and the discontinuous behaviour of vehicles by the parameters: flow, density and velocity between two given traffic light nodes in Kumasi, Ghana with the use of partial differential Equations (PDEs) based on Lighthill, Whitham and Richard (LWR) model. Primary data was obtained from the study zone, a 100m road segment in Kumasi, during the peak hour in the morning .The flow was regressed on the density and the result indicated a quadratic dependence between flow and density confirming the fundamental relation of Greenshield. With prescribed initial conditions, the first order nonlinear homogeneous and nonlinear inhomogeneous PDE's models were solved using the method of characteristics.

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CHAPTER 1

INTRODUCTION

1.1 BACKGROUND OF THE STUDY

This chapter relating to the background can be classified into four main domains such as: overview of the challenges of urban traffic congestion, causes of urban traffic congestion, the management of urban traffic congestion and the traffic flow network in Kumasi.

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1.1.1 THE CHALLENGES OF URBAN TRAFFIC CONGESTION

Cities and traffic have developed hand -in-hand since the earliest large human settlement. The same force that draw inhabitants to congregate in large urban areas also lead to intolerable level of traffic congestion on urban streets. Urban traffic congestion has been a challenge in both developed and developing countries as demand for urban trip continue to grow relatively to population growth. For Example, according to the traffic congestion and Reliability Report (FHWA,2005), congestion has grown substantially throughout the developed and developing countries over the past twenty (20)years, regardless of city size. The 2009 urban mobility Report estimate that "In 2007, congestion caused urban American to travel 4.2 billion hours more, resulting an extra 2.8 billion of fuel for a cost of $7.2billion,an increase of more than 50 percent over the previous decade"(Schrank and Lomax, 2009). Similarly, in developing countries such as Ghana has had their share of urban traffic congestion due to increase in the volume of vehicles in road networks. The problem of traffic tie-ups in every major world city is impacting not only inconvenience to travelers, but also a significant economic hardship and if not managed correctly, could seriously hinder the development of our cities and our country. Hence we look at the causes of traffic congestion.

1.1.2 CAUSES OF URBAN TRAFFIC CONGESTION

Traffic congestion in urban areas is often the outcome of successful urban economic development, employment, and housing and cultural, policies that make people want to live and work relatively close to each other and attract firms to benefit from the gains in productivity thus derived. Urban road users are prepared to live along crowded roads so long as they derived other benefits from living and working in their cities. Other cause of traffic: is too many vehicles for given road's design(intersection capacity) ,dynamic changes in roadway capacity caused by lane-switching and car-following behavior and other indirect factors such as land use patterns, employment patterns, income levels, car ownership trends.

1.1.3 TRAFFIC CONGESTION AND TRAFFIC CONTROL

Traffic congestion on motorways is becoming an ever more pressing problem in countries all over the world. One approach to tackle congestion could be the construction of new roads to enlarge the capacity of the traffic infrastructure. This approach is very costly and it is often not possible due to environmental or societal constraints. In addition, it can only be executed on the longer term (Bellemans, et al., 2002). So there is a need for another, short-term solution. This short-term solution exists of controlling traffic in such a way that congestion is solved, reduced, or at least postponed. In most countries traffic operators in traffic control centers monitor the traffic situation based on video images and measurements from e.g. loop detectors1. Especially during rush hours the interpretation by experts of information about traffic intensities, average velocities, weather conditions, incidents, on the motorways leads to advice for the travelers and/or actions in order to solve, to prevent or to postpone congestion. Traffic models allow for a simulation of future traffic densities and average velocities. These predictions can help traffic operators to determine which control measures to take. Another advantage of simulation models is that they allow for a prediction of the traffic state that takes exceptional situations such as (partially) blocked roads into account. (Bellemans, et al., 2002).

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To fully eradicate roadway congestion is neither an affordable nor feasible goal in economically dynamic urban areas. However, much can be done to reduce its occurrence and to lessen its impacts on roadway users within large cities-congestion Effectively managing congestion requires both a holistic and integrated strategy that goes beyond the visible incidence of congestion “on the road” and extends to the management of the urban region as whole. Congestion mitigation actions are part of the broad and complex land use, urban planning and general transport master planning process unique to each urban region. Roadway congestion impacts not only road users but all urban inhabitants. Congestion management requires an integrated strategy equal to the scope and scale of the challenge. Below are some means traffic can be managed

1. Integration of planning of land use and the community objectives.
2. Delivering predictable travel times.
3. Adequate system performance.

Many traffic management initiatives provide online traffic information directed at drivers via the Internet, commercial radio, Global System for Mobile Communication

(GSM), or electronic panels. Some studies have demonstrated that if drivers had this sort of information, it would improve traffic flow.

1.1.4 TRAFFIC FLOW NETWORK IN KUMASI

In mathematics and engineering traffic flow is the study of interactions between vehicles ,drivers and infrastructure (including highways and traffic control devices), with the aim of understanding and developing an optimal road network with efficient movement of traffic and minimal traffic congestion problems (Wikipedia Encyclopedia,2012)

Nevertheless, even with the advent of significant computer processing power, to date there has been no satisfactory general theory that can be consistently applied to real flow conditions. Current traffic models use a mixture of empirical and theoretical techniques .These models are then developed into traffic forecasts, to take account of proposed local or major changes, such as increased vehicle use, change land use or change in mode of transport (with people moving from cars for example), and to identify areas of congestion where the network needs to be adjusted. With the knowledge of traffic flow we then proceed to a case study for this study in Kumasi – Ghana. Kumasi is the second largest city in Ghana and it is considered as the centre of Ghana in Ashanti Region. The road network in Kumasi is linked up by major and minor road network. The four major networks are;

Kumasi-Accra, Kumasi-Sunyani, Kumasi-Tamale and Kumasi-Cape coast. The inner and the outer road network are marked by modern traffic light and roundabout to govern the flow of vehicle. A case study would be done on Technology-Kejatia road a suburb of Kumasi. That stretch of network is marked by influx of minor network. At the various intersection of network are set of traffic light which manages the traffic flow. Heavy Congestion on the inner and outer stretch of road in Kumasi often occurs from 6.30am-10.00am and 6.30pm-10.00pm.

1.2 PROBLEM STATEMENT

Vehicle traffic has been the subject of considerable research interest in developed and developing countries, and has recently drawn the attention of both road engineers and statistical researchers. The exposure to vehicle traffic continues to be the leading source of problems in the economic growth due to the judicious productive time lost in long traffic jams and as a result needs to be managed. Transportation problem have plagued man long before the advent of the automobile, however in recent years traffic congestion has become especially acute (Haberman and Richard, 2007). Ghana as a developing country also experiences severe negative impact from long traffic jams in diverse areas which include: Economic development .In past decades much of the productive hours and goods in Ghana that maximize Economic growth are spent in long traffic jam that amount to delay at work place and eventually affect economic growth negatively. The motivation for this study came from the observation that most capital cities in Ghana are experiencing some levels of congestion on their freeway networks. Accra and Kumasi are experiencing congestion between three to eight hours per day. In addition to this, the vehicle traffic jam imposes environmental hazard due to excessive emission of poisonous gases in the cities. This further imposes health threat and stress to both the people living in the cities and the travelers. Traffic congestion has become a matter of concern in Ghana because about 95% transportation is by road. However, existing road infrastructure has not been improved significantly and in most cases, it is not viable to extend traffic infrastructures due to costs, limited available space, and environmental impact. Further, irrespective of these challenges, as the population grows, more vehicles are imported into the system to compete the existing vehicles. Hence it is important to look at effective management of vehicle traffic flow between the existing traffic lights in Kumasi from Technology junction to Kejatia as a case study based on the continuous and the discontinuous behavior of the traffic light nodes that imposes much traffic jam on our roads in the cities in Ghana.

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1.3 OBJECTIVES OF THE STUDY

This study seeks to model the traffic flow of vehicles in the realistic urban dynamic traffic situation of long traffic congestion on the limited number of road on the peak hours. Hence the main objectives of the study are as follows:

1. To use partial differential Equations (PDEs) based on Lighthill and Whitham (1955) and Richard (1956) (LWR) model to model traffic flow between two traffic light nodes on the KNUST - Kejetia stretch of road in Kumasi, Ghana as the study zone.
2. Use regression analysis to obtain the quadratic relationship between the flux(flow) and the density of vehicles within traffic light nodes
3. To use method of characteristics to solve the both nonlinear homogeneous and nonlinear inhomogeneous PDE

1.4 METHODOLOGY

In this study LWR macroscopic model would be employed to generate first order nonlinear partial differential equations. The method of characteristics would be used to solve the systems of PDEs. A primary data would be collected between the stadium junction traffic light node and the Amakom traffic light node about 100metres interval during the day from 7.00 am and 10.30am. Data count such as density and time would be obtained from tagged cars that travel from one traffic light node to the other by stand-by observers. Microsoft excel software would be needed for running the data to establish some fundamental relationship among the traffic parameter such as density, flow (flux) and speed. The internet and both the university and department library would of good help in the studies

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1.5 JUSTIFICATION

Effective traffic management is essential to the long-term success of any country's development; since it promotes economic development and prevents most countries from going bankrupt. If not properly managed, it may lead to the collapsing of the economic market and most companies. It is therefore justifiable because it helps traffic engineers to verify whether traffic properties and characteristics such as speed(velocity), Density and flow among others determines the effectiveness of traffic flow. Hence the LWR model can be used by engineers of road network to plan ahead to prevent excessive traffic jam.

W ↄ SANE NO BADWENMA NYANSAPↄ

1.6 LIMITATION

There were several challenges relating to the success of the study. The following are few among others;

1. Non- availability of modern road detector to record accurate road information in the citiess
2. Limited access to extensive data set of variables.

3. Non- availability of some powerful mathematical and econometric software's for data analysis

1.7 THESIS ORGANIZATION

The first chapter deals with the general introduction of traffic flow and background of study. Chapter 2 contains the literature review and the contributions of other researchers regarding vehicle traffic optimization and the applications on real world networks. In Chapter 3 the methodology is presented as mathematical treatment and logical presentation (mathematical theory) of formulation and models of solution. Chapter 4 deals with the data collection, analysis and result. Chapter 5 synthesis the whole study (thesis) and presents the conclusion, summary and recommendation.

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CHAPTER 2

LITERATURE REVIEW

2.0 INTRODUCTION

This chapter deals with the general overview of research done on basic freeway traffic flow theory and control tools.

2.1 BASIC FREEWAY TRAFFIC FLOW THEORY

According to Austroads (2008), a freeway is generally known as an uninterrupted facility because traffic on the mainline is not interrupted by a control device such as a traffic signal. Traffic flow on a freeway segment is therefore relatively easier to model analytically, especially in free-flow or uncongested conditions. This section provides the basic analytical framework for freeway traffic models and also briefly reviews the tools currently used for automatic freeway control. It provides the context for subsequent sections on reviews of the characteristics of flow breakdowns.

2.1.1 TRAFFIC FLOW THEORY

Traffic Flow theory comprises the study of the movement of individual drivers and vehicles between two points and the interactions they make with one another and plays a vital role in the progress of overall social productivity. In the 1950s James Lighthill and Gerard Whitham, two experts in fluid dynamics, (and, independently P. Richards), modelled the flow of car traffic along a single road using the same equations describing the flow of water (Lighthill and whitman,(1955) and Richard ,(1956). The basic idea is to consider traffic on a large scale so that cars are taken as small particles and to assume the conservation of the number of cars. The LWR model is described by a single conservation law, a special partial differential equation where the dependent variable, the car density, is a conserved quantity, i.e. a quantity which can neither be created nor destroyed. Traffic flow when likened to fluid flow has several parameters associated with it. These parameters could then provide information regarding the nature of traffic flow, which would help a traffic analyst or modeler in detecting any variation in flow characteristics. Thus understanding traffic behavior would require a thorough knowledge of these traffic stream parameters and their mutual relationships. The first section of the paper describes the key ideas of macroscopic traffic flow theory. The LWR model is presented in some detail including a description of an analytical solution using the method of characteristics.

In formulating a mathematical model for a continuum traffic flow, there are basic steps that are often used as giving by Haight (1963); Haberman (1977); Banks (1992); Michalopoulos, et al. (1993); Bellomo, et al.(2002) and Bellemans, et al. (2002) below:

1. Identify appropriate conservation laws (e.g. mass, momentum, energy, etc) and their corresponding densities and fluxes.
2. Write the corresponding equations using conservation law and Close the system of equations by proposing appropriate relationships between the fluxes and the densities.

2.2 HISTORY OF TRAFFIC MODEL

Attempts to produce a mathematical theory of traffic flow dates back to the 1920s, when Frank Knight first produced an analysis of traffic equilibrium, which was refined into first and second principles of equilibrium by Wardrop (1952).

2.2.1 TRAFFIC FLOW MODEL

There are several traffic flow models, which can be mostly divided in 3 categories: macroscopic, mesoscopic and microscopic (listed with growing level of details). Below are some types of traffic flow model:

2.2.2 MACROSCOPIC TRAFFIC FLOW MODELS AND PARAMETERS

Leutzbach (1988) said that a macroscopic traffic flow theory relates traffic flow, running speed, and density. Analogizing traffic to a stream, it has principally been developed for limited access roadways. The fundamental relationship "q=kv" (flow (q) equals density (k) multiplied by speed (v)) is illustrated by the fundamental diagram. Many empirical studies have quantified the component bivariate relationships (q vs. v, q vs. k, k vs. v), refining parameter estimates and functional forms by Gerlough and Huber (1975); Pensaud, et al. (1991); Hall, et al.(1992); Gilchrist and Hall (1992); Disbro and Frame (1992).

The most widely used model is the Greenshields model, which depicts that the relationships between speed and density are linear. These were most appropriate before the advent of high-powered computers enabled the use of microscopic models. Macroscopic properties like flow and density are the product of individual (microscopic) decisions. The macroscopic properties of traffic.Flow rate, speed(velocity) and density are the three most important macroscopic traffic flow parameters used to describe the state of an uninterrupted traffic stream:

Flow, q, is defined as an equivalent hourly rate at which vehicles pass over a given point of a roadway during a during a given time interval in less than one hour. Flow rate is expressed in vehicles per hour (veh/hr)

.Speed, u , is defined as the rate of motion expressed as distance per unit of time or kilometres per hour (km/hr). Generally the term speed means the average travel speed of all vehicles on a road segment in a time period. The speed is calculated by dividing the segment length by average travel time of the vehicle. Density, ρ , is defined as the number of vehicles occupying a given length of a roadway at a particular instant. It is expressed in vehicles per kilometre (veh/km)

2.2.3 KINEMATIC WAVE MODEL

An extension of the fundamental relationships is to consider speed, flow and density as functions of time (t) and space (x), and they are not independent parameters. For example, flow is a function of density k, which is a function of time t. A model that considers the traffic process in time and space is the kinematic wave model of Lighthill and Whitham (1955), which is more suitable for high density conditions and therefore has its place in analysing flow breakdowns.

The kinematic model assumes that high density traffic will behave like a continuous fluid (hence also called a continuum model). Consider the flow in and out of a short length of road Δx. The condition of continuity requires that if the density of vehicles has increased it must have been due to a difference in the amounts flowing in at one end and out at the other, or:

$$\frac{\partial k}{\partial t} + \frac{\partial q}{\partial x} = 0 \qquad \text{(a)}$$

where q is the flow (veh/h)

k is the density (veh/km)

x is distance (km)

t is time (h) to travel a distance of x km.

With q as a function of density k, Lighthill and Whitham further into the LW model as follows:

$$\frac{\partial k}{\partial t}+\frac{\partial q}{\partial k}\frac{\partial k}{\partial x}=0$$

Define below a wave speed U that represents the speed of waves carrying continuous changes of vehicle flow in a traffic stream:

$$U = \frac{\partial q}{\partial k}$$

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Because q = v k the wave speed:

Because speed decreases with density, $\frac{\partial v}{\partial k}$ is always negative and the wave speed U is therefore always less than the space mean speed v. The relationship between space mean speed (v) and wave speed (U) are illustrated in the flow-density diagram in Figure 2.1, which also shows the shock wave speed (U_{SW}). The following observations can be made (Wohl & Martin, 1967):

- At low densities when vehicle-to-vehicle interactions are minimal, $\frac{\partial v}{\partial k}$ is almost zero and the wave speed is similar to the space mean speed. The wave moves forward relative to the road.
- At the maximum flow and critical density, the wave is stationary. At densities higher than the critical density (k_c), the wave moves backward relative to the road.
- The wave speed changes with density and a traffic stream can have different densities at different sections of a freeway. A section of light traffic could follow a section of high density due to a decrease in lanes, an accident or on-

ramp traffic. The wave in the low density traffic moves forward (relative to the freeway) at a speed faster than the wave in the high density traffic.

- When the two waves meet, a new wave called a shock wave will be formed. All three waves move forward for the situation shown in Figure 2.1. The shock wave speed U_{sw} is given by:

$$U_{sw} = \frac{q_2 - q_1}{k_2 - k_1}$$

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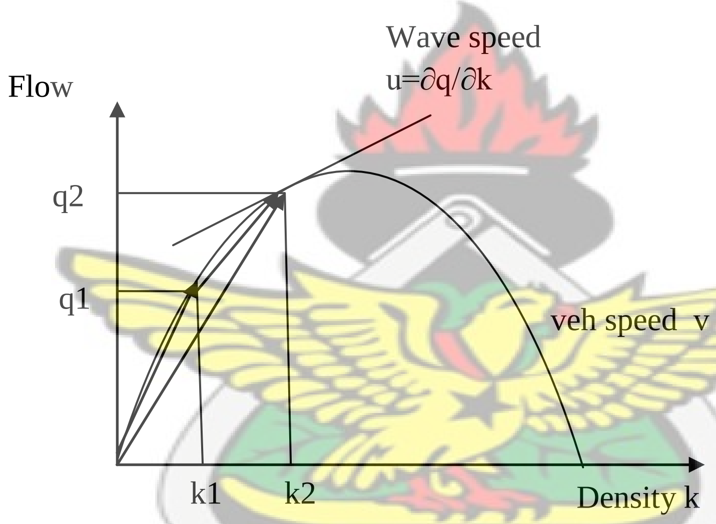

Figure 2.1: The relationship between vehicle speed, wave speed and shock wave speed (Arnold, 2008).

The density at the bottleneck entry suddenly increases from the density at C to the density at E in Figure 2.4. The wave speed at E is negative with respect to the freeway and will be reflected from the bottleneck back to the approach section. The reflected wave will meet the oncoming wave corresponding to the slope at C. A shock wave of negative speed relative to the freeway is formed. The effect of the bottleneck will be reflected along the entire approach section if the arrival flow remains constant, with a consequent loss of maintaining capacity flow (q_m).

Edie and Foote (1958) reported how shock waves were generated at an upgrade leading to the Holland Tunnel exit in New York. The shock waves propagated backward towards the tunnel entry with inefficient traffic flow. The solution was to control the entry of vehicles into the tunnel so that the entry flow did not exceed the capacity of the bottleneck section. The vehicles entered in short platoons of about 40 vehicles every 2 min with a 10s gap between platoons. Maximum capacity q_m CE Shock wave speed Speeds of vehicles in queue Speed of vehicles approaching queue Bottleneck capacity q_b Speed of approaching wave Flow q Density k Speed of reflected wave.

The kinematic model can be solved using the finite difference (or finite element) method and has continued to be an interesting area of research as recommended by Leo & Pretty (1992); Michalopoulos (1988); Ngoduy, et al. (2006); Papageorgiou (1983) and Payne (1971). At the University of Queensland, Leo and Pretty were able to model the propagation of congested density upstream in a freeway lane drop situation and platoon movements in a pair of coordinated signals at very small, discrete levels of time (0.5 to 1 s) and space (about 15 m) but have not further pursued their research since the early 1990s.

The LW model is a first order model with limitations such as (Papageorgiou, 1998):

- Assume that vehicle speeds can change instantaneously, i.e. large values of acceleration and deceleration rates are assumed possible at a bottleneck

- Predict that the tail-end of a platoon on arterial roads will speed up to catch up with the main platoon when it is more common to observe a dispersed rear-end.
- Assume that outflow (q_b) at a freeway bottleneck is best achieved with some congestion at the bottleneck entry. This is equivalent to assuming that the outflow cannot be increased by avoiding mainline congestion, i.e. no control. The reality is that some control of a bottleneck (if possible) can improve throughput.

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Second order LW models have been proposed by Daganzo (2006); Papageorgiou, et al. (1990); Payne (1971), Schonhof and Helbing (2007) to overcome these limitations. Kinematic models can contribute towards the understanding of freeway flow breakdowns and will be revisited as the project progresses.

2.2.4 TWO PHASE HCM MODEL

A conventional understanding of the formation of congested flow conditions is that a queue would form upstream of a bottleneck due to conditions such as lane drop, merge area, weaving section or upgrade. The trailing edge of the queue moves upstream at a rate depending on demand and capacity conditions. When the tail of this queue reaches any upstream location, freeway operation moves from the uncongested regime to the congested regime, at approximately the same flow. The HCM 2000 and the earlier 1986 edition have advocated the need to consider maximum flows or capacities of a freeway segment in two regimes or phases. Two maximum flow rates can be identified as follows:

- Maximum flow when flow is stable – this is the maximum flow before the formation of a queue at a bottleneck, i.e. the maximum pre-queue flow.
- Maximum queue discharge flow – this is the maximum flow after a queue is formed and is associated with a speed drop, and has been found to be less than the pre-queue maximum flow rate. A possible reason for this decrease in flow rate is driver caution – departures from a freeway queue require more care because drivers may not be aware of conditions downstream. This is in contrast to a start-up queue at a signalised approach where maximum flow is achieved even though different vehicles have different acceleration rates.

There have been debates on where the maximum flows should be measured. Hall and Agyemang-Duah (1991) argued that the two phases are observable only if detectors are located at some distance upstream of a bottleneck and that there is only one congested regime if they are at *a* bottleneck.

In a study of a bottleneck on a four-lane freeway near San Diego, Banks (1990) measured the above two maximum flow rates. The frequency distribution polygons of the counts on the fast lane were used. The results clearly showed that there is a statistically significant difference between the two flow rates. Hall, Hurdle and Banks (1992) also suggested that much more research is needed in understanding freeway congested flow.

2.2.5 THREE PHASE MODEL

Kerner and Rehborn (1996) first proposed the classification of freeway traffic flow into three phases based on time series of flow, occupancy, and average speed. Kerner

(2004) later completed the three-phase traffic theory based on earlier work. In the three-phase traffic theory, there are two traffic phases in congested traffic, synchronised flow and wide moving jam, defined as follows:

- A synchronised flow is a congested traffic state and the downstream front of this flow is often fixed at a freeway bottleneck. Within the downstream front of synchronised flow, vehicles accelerate from lower speeds in synchronised flow to higher speeds in free-flow.
- A wide moving jam is a moving jam that maintains the mean velocity of the downstream jam front, even when the jam propagates through any other traffic states or freeway bottlenecks.

The three traffic phases are therefore free-flow (F), synchronised flow (S) and widemovingjam (J). The three-phase traffic theory explains the complexity of traffic phenomena based on phase transitions among these three traffic phases. For example, transitions can be spontaneous F→ S or induced F→ S, and their complex nonlinear spatio-temporal features. In Kerner's three-phase theory, a transition from F→ S is a flow breakdown (Kerner 2004); Kerner ,et al.(2005). An induced F→ S transition is caused by a short-term external disturbance in traffic flow. This traffic flow can be related to the propagation of a moving spatio-temporal congested pattern that initially occurs at a different freeway location.

This breakdown phenomenon or F→ S transition is caused by an internal local disturbance (e.g. an on-ramp bottleneck) in traffic flow. There are no external disturbances in traffic flow responsible for this phase transition

The F→ S transition or breakdown phenomenon usually occurs at the same freeway bottleneck. These bottlenecks are called effectual bottlenecks in Kerner's model.Based on different combinations of traffic phases, different congested patterns are formed. Kerner studied traffic flow on the freeway over a large number of days and found that the spatio-temporal structure of congestion patterns exhibits predictable features. These features can be used to forecast freeway congestion and develop effective freeway control tools.

Lindgren (2005) also investigated a 30 km section of freeway north of Frankfurt and found some similar traffic patterns that match Kerner's three traffic phases. In Lindgren's freeway study, traffic flows were observed in which speeds across all lanes were notably lower than in free-flowing conditions, and they were more consistent across all lanes. This phenomenon was observed in congested flows upstream of the bottleneck following activation. This pattern matched Kerner's synchronised flow phase. Lindgren also revealed several occurrences of congested patterns in which a relatively short duration traffic disturbance travelled several kilometres upstream. This pattern matched Kerner's wide moving jam.

Lindgren's study represented some of the first apparent independent validation of Kerner's traffic phase findings Lindgren (2005); Lindgren, et al.(2006). However, Lindgren also offered different analysis techniques and comments on Kerner's work.

Further, Brilon, et al. (2005) also showed that three traffic flow states exist in a freeway: fluent traffic state, congested traffic state and a transient state that occurs in each breakdown and recovery of traffic flow.

2.2.6 EMPIRICAL PROBABILISTIC NATURE OF TRAFFIC BREAKDOWN

Kerner (2004, 2007a, 2007c) found that the traffic breakdown exhibits a probabilistic nature. At a given flow rate, traffic breakdown at a freeway bottleneck can occur but it may not necessarily occur.

The probability for an F→ S transition, i.e. a traffic breakdown, (P_{FS}) at a bottleneck is an increasing function of the flow downstream of the bottleneck q_{sum}. There is a threshold flow rate q_{th} and a critical flow rate q_{max}. Regardless of free-flow control application there is a range when $q_{th} \leq q_{sum} \leq q_{max}$ within which traffic flow breakdowns can occur with probability $P_{FS} > 0$.

A flow breakdown, if due to a speed disturbance in free flow in the neighbourhood of a bottleneck, occurs only when the speed decreases below a critical speed. The critical speed depends on the q_{sum}. The smaller the $q_{sum,}$ the lower the critical speed required for breakdown. The probability for traffic breakdown P_{FS} is the probability of random critical speed disturbances appearing at the bottleneck. Disturbances with small amplitudes in free flow at the bottleneck do not lead to breakdown. However, if a random short-term speed disturbance in free flow at the bottleneck exceeds some critical values, traffic breakdown occurs. Kerner (2007a) stated that empirical fundamental features of probabilistic traffic breakdown cannot be explained and cannot be predicted by earlier freeway flow models.

2.3 TRAFFIC CONGESTION

Levinson(2003) said ,Traffic congestion management has the goal to optimize transportation flow of people and goods particularly in the metropolitan area .To define what is meant by traffic congestion, the executives report of Organization of Economic co-operation and Development (OCED) and European conference of ministers of Transport (ECMT), January, 2004 stated that there is no single definition for congestion since congestion takes on many faces occurs in many different contexts and caused by many different process. However, below are some of their definitions of congestion:

I. Congestion is a situation in which demand for road spacing exceeds supply

II. Congestion is the impedance vehicles impose on each other due to the speed flow relationship, in condition where the use of a transport system approaches capacity.

III. Congestion is essentially a relative phenomenon that is linked to difference between the road system performance that users expect and how the system actually performs

Susan Grant-Muller and Laird (2007), describes Traffic congestion as a widely recognized transport cost and is a significant factor in transport system performance evaluation which affects transport planning decisions. To individual motorists, congestion is a cost they bear, but each motorist also imposes congestion on other road users. The existence of congestion can be explained by the fact that each additional vehicle imposes more total delay on others than they bear, resulting in economically excessive traffic volumes. Congestion can be recurrent (regular, occurring on a daily, weekly or annual cycle) or non-recurrent (traffic incidents, such as accidents and disabled vehicles) as revealed by ECMT(January,200).. Some congestion costs analyst only consider recurrent, others include both. Economist William and Vickrey (1969), identified six types of congestion as follows:

2.3.1 TYPES OF TRAFFIC CONGESTION

1. Simple interaction on homogeneous roads: where two vehicles travelling close together
1. delay one another.
2. Multiple interactions on homogeneous roads: where several vehicles interact.
3. Bottlenecks: where several vehicles are trying to pass through narrowed lanes.
4. "Trigger neck" congestion: when an initial narrowing generates a line of vehicles interfering with a flow of vehicles not seeking to follow the jammed itinerary.
5. Network control congestion: where traffic controls programmed for peak-hour traffic inevitably delay off-peak hour traffic.
6. Congestion due to network morphology, or polymodal polymorphous congestion: where traffic congestion reflects the state of traffic on all itineraries and for all modes. The cost of intervention for a given segment of roadway increases through possible interventions on other segments of the road, due to the effect of triggered congestion. Most congestion cost analysis concentrates on the second and third types of congestion: congestion arising from interactions between multiple vehicles on homogeneous road section and bottleneck congestion. Others types are overlooked or assumed to be included in the types that are measured. Another often overlooked factor that complicates economic analysis is that congestion reduces some costs.

Moderate highway congestion reduces traffic speeds to levels that maximize vehicle throughput and vehicle fuel efficiency, and although congestion tends to increase crash rates per vehicle-mile, the crashes that occur tend to be less severe, reducing injuries and deaths. (Vickrey, 1969)

2.4 FUNDAMENTAL RELATION

There is unique relation among the three macroscopic traffic flow parameters density, flow and speed.Under uninterrupted flow conditions, these traffic variables are all related by

$q = \rho \cdot u$ (1)(This relationship represents the fundamental equation of traffic flow).

2.4.1 FUNDAMENTAL DIAGRAMS

Fundamental Diagrams give pictorial view of the experimental relationships between the traffic flow parameters. It can be illustrated in three form as follows; Greenshields and Weids (1952). Fundamental diagrams.

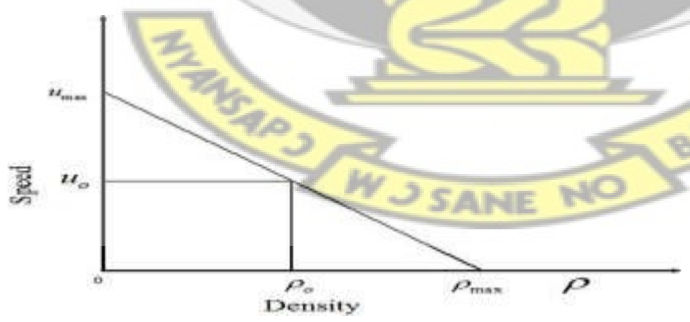

Figure 2.2: Speed-density relationship (Greenshields and Weids, 1952)

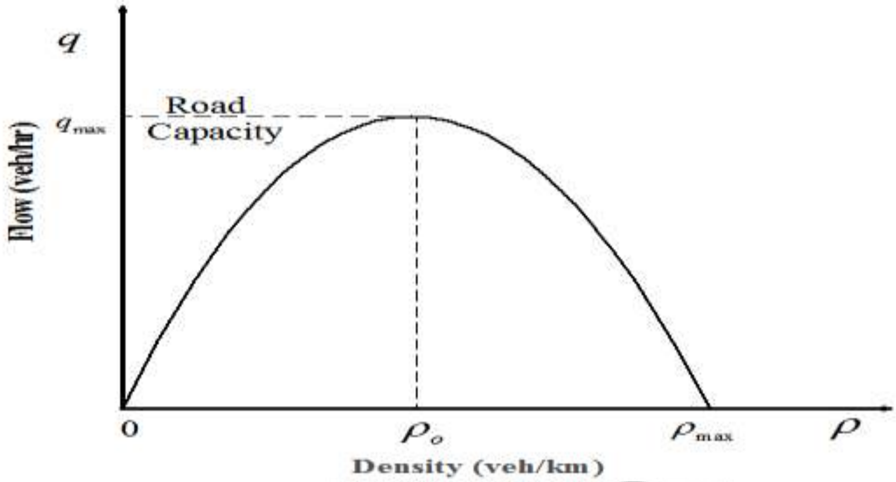

Figure 2.3: Rate of Flow- Density relationships (Greenshields and Weids, 1952)

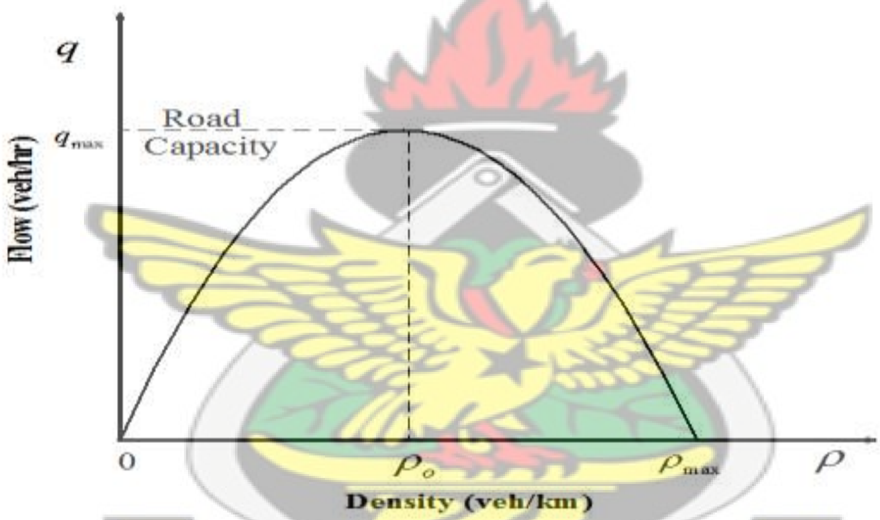

Figure 2.4: Rate of Flow -Speed relationships (Greenshields and Weids, 1952)

Figure 2.5 illustrates what are commonly known as the three fundamental diagrams relating speed, flow and density and can found in most textbooks on classical traffic flow theory, e.g. May (1990) and Taylor, et al. (2000). The relationships in Figure 2.5 have been useful for the basic understanding of traffic flow and control, especially for uninterrupted facilities such as freeways.

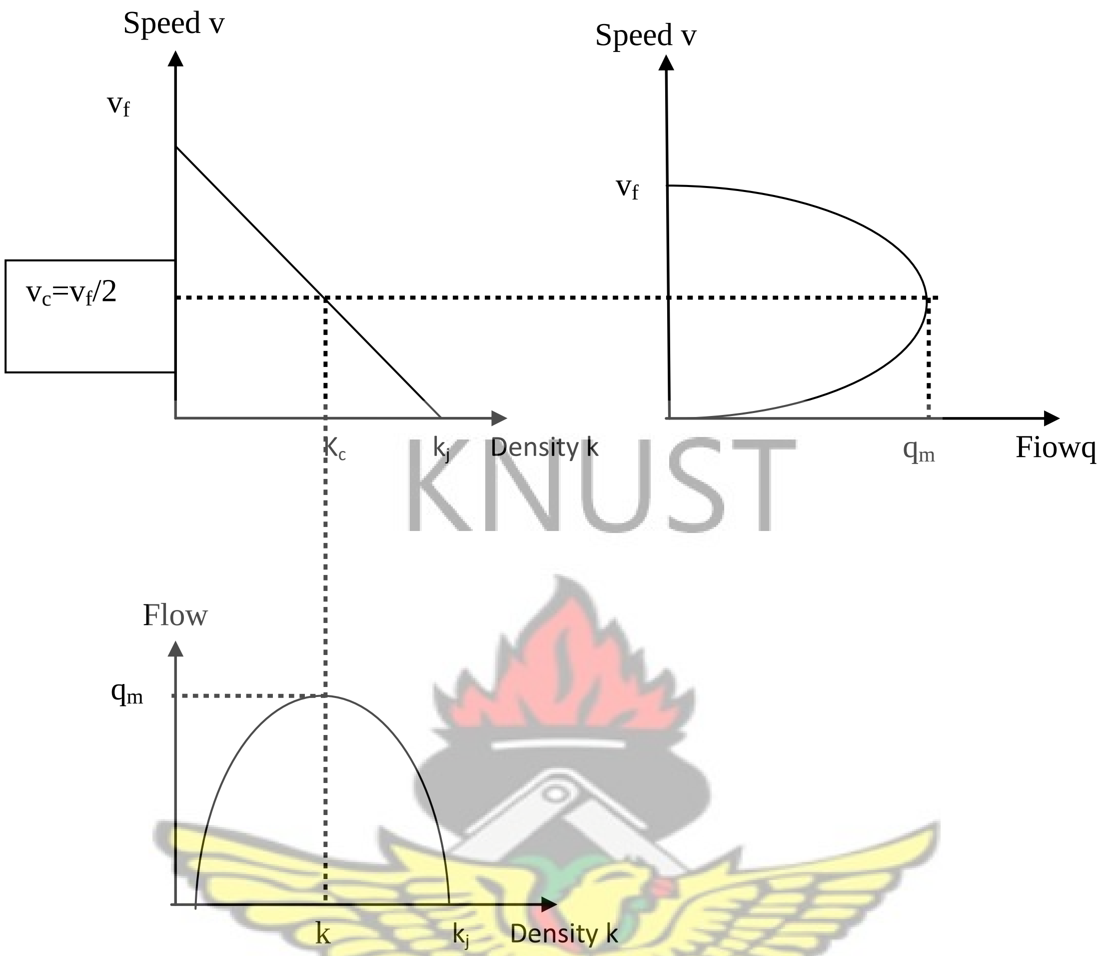

Figure 2.5: Fundamental diagrams assuming linear speed-density relationship (Arnold, 2008)

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These relationships are called fundamental diagrams. They are obtained from measurements and can be represented mathematical as:

$$u(\rho) = u_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right) ,$$

$$q(\rho) = u_{\max}\left(\rho - \frac{\rho^2}{\rho_{\max}}\right) \quad \text{and}$$

$$q(u) = \rho_{\max}\left(u - \frac{u^2}{u_{\max}}\right) \qquad \text{Respectively}$$

2.5 MEASURMENT OF TRAFFIC STREAM PROPERTIES

Traffic flow is generally constrained along a one-dimensional pathway (e.g. a travel lane). A time-space diagram provides a graphical depiction of the flow of vehicles along a pathway over time. Time is measured along the horizontal axis, and distance is measured along the vertical axis. Traffic flow in a time-space diagram is represented by the individual trajectory lines of individual vehicles. Vehicles following each other along a given travel lane will have parallel trajectories, and trajectories will cross when one vehicle passes another. Time-space diagrams are useful tools for displaying and analyzing the traffic flow characteristics of a given roadway segment over time (e.g. analyzing traffic flow congestion). WikiProject Mathematics (*June 2011*)

There are three main variables to visualize a traffic stream: speed (v), density (k), and flow(q)

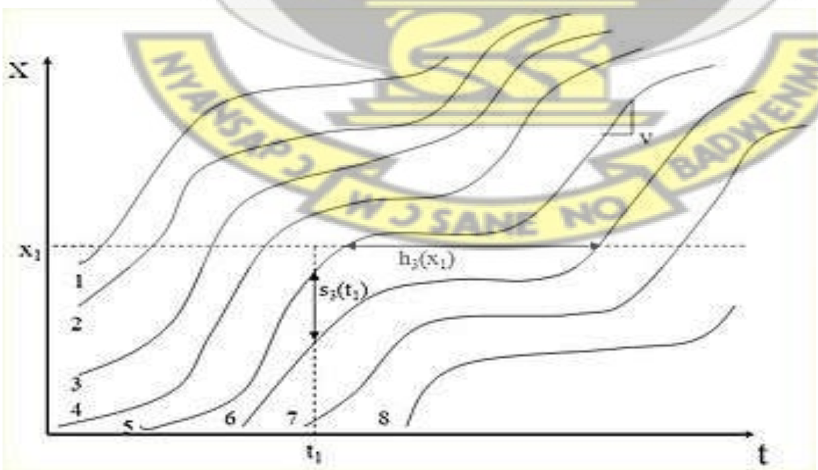

Figure 2.6: The time space diagram (http://en.wikipedia.org/wiki/file)

2.5.1 SPEED

Speed in traffic flow is defined as the distance covered per unit time. The speed of every individual vehicle is almost impossible to track on a roadway; therefore, in practice, average speed is based on the sampling of vehicles over a period of time or area and is calculated and used in formulas. If speed is measured by keeping time as reference it is called time mean speed, and if it is measured by space reference it is called space mean speed.

- Time mean speed is measured by taking a reference area on the roadway over a fixed period of time. In practice, it is measured by the use of loop detectors. Loop detectors, when spread over a reference area, can record the signature of vehicles and can track the speed of each individual vehicle. However, average speed measurements obtained from this method are not accurate because instantaneous speeds averaged among several vehicles cannot account for the difference in travel time for the vehicles that are traveling at different speeds over the same distance
- $V_t = \left(1/m\right)\sum_{i=1}^{m} v_i$ Where m represents the number of cars passing the fixed point.

Space mean speed is the speed measured by taking the whole roadway segment into account. Consecutive pictures or video of a roadway segment track the speed of individual vehicles, and then the average speed is calculated. It is considered more accurate than the time mean speed. The data for space calculating space mean speed may be taken from satellite pictures, a camera, or both. WikiProject Mathematics(*June 2011*)

$$V_s = \left(n / \sum_{i=1}^{n}\left(\frac{1}{v_i}\right)\right)$$

Where n represent the number of vehicle passing the roadway segments The time mean speed is always greater than space mean speed.

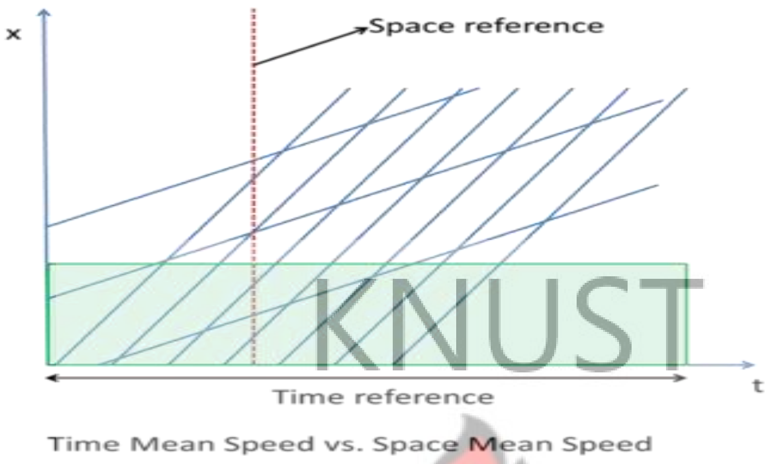

Figure 2.7 : Space-and Time speeds (http://en.wikipedia.org/wiki/file)

In a time-space diagram, the instantaneous velocity, $v = dx/dt$, of a vehicle is equal to the slope along the vehicle's trajectory. The average velocity of a vehicle is equal to the slope of the line connecting the trajectory endpoints where a vehicle enters and leaves the roadway segment. The vertical separation (distance) between parallel trajectories is the vehicle spacing (s) between a leading and following vehicle. Similarly, the horizontal separation (time) represents the vehicle headway (h). A time-space diagram is useful for relating headway and spacing to traffic flow and density, respectively. WikiProject Mathematics (*June 2011*)

2.5.2 DENSITY

Density (k) is defined as the number of vehicles per unit area of the roadway. In traffic flow, the two most important densities are the critical density (k_c) and jam density (k_j). The maximum density achievable under free flow is k_c, while k_j is minimum density achieved under congestion. In general, jam density is seven times the critical density. Inverse of density is spacing (s), which is the distance between two vehicles.

k =1/s   WikiProject Mathematics (*June 2011*)

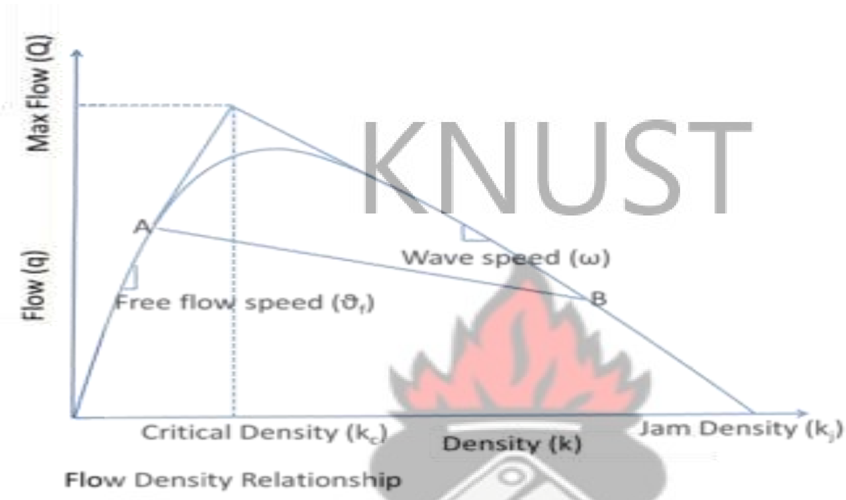

Figure 2.8: Flow density relation (http://en.wikipedia.org/wiki/file)

The density (k) within a length of roadway (L) at a given time (t1) is equal to the inverse of the average spacing of the n vehicles.

$$K(L,t_1) = \frac{n}{L} = \frac{1}{\bar{s}(t_1)}$$

In a time-space diagram, the density may be evaluated in the region A.

$$k(A) = \frac{n}{L} = \frac{ndt}{Ldt} = \frac{tt}{|A|}$$

where tt is the total travel time

2.5.3 FLOW

Flow (q) is the number of vehicles passing a reference point per unit of time, and is measured in vehicles per hour. The inverse of flow is headway (h), which is the time that elapses between the i^{th} vehicle passing a reference point in space and the i+1 vehicle. In congestion, h remains constant. As a traffic jam forms, h approaches infinity.

$$q = kv$$

$$q = 1 / h$$

The flow (q) passing a fixed point (x1) during an interval (T) is equal to the inverse of the average headway of the m vehicles.

$$q(T, x_1) = \frac{m}{T} = \frac{1}{\bar{h}(x_1)}$$

In a time-space diagram, the flow may be evaluated in the region B.

$$q(B) = \frac{m}{T} = \frac{mdx}{Tdx} = \frac{td}{|B|}$$

Where

Td is the total distance traveled in B.

2.5.4 CONGESTION SHOCKWAVE

In addition to providing information on the speed, flow, and density of traffic streams, time-space diagrams may also illustrate the propagation of congestion upstream from a traffic bottleneck (shockwave). Congestion shockwaves will vary in propagation length, depending upon the upstream traffic flow and density. However, shockwaves will generally travel upstream at a rate of approximately 20 km/h. WikiProject Mathematics(*June 2011*)

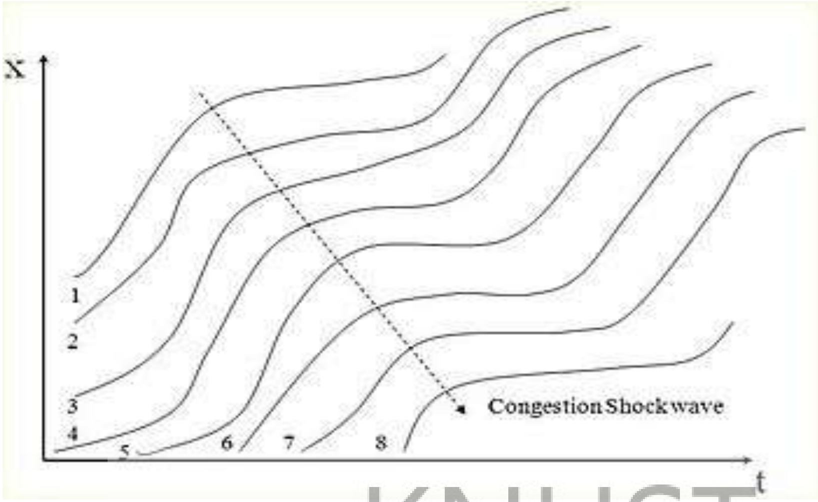

Figure 2.9: Congested shockwaves (http://en.wikipedia.org/wiki/file)

2.6 RELATED WORK IN TRAFFIC LIGHT CONTROL

2.6.1 FUZZY LOGIC.

Tan, et al. (1995) describe a fuzzy logic controller for a single junction that should mimic Human intelligence. Fuzzy logic offers a formal way of handling terms like “more”, “less” ,“Longer” etc., so rules like “f there is more traffic from north to south, the lights should stay Green longer” can be reasoned with. The fuzzy logic controller determines the time that the Traffic light should stay in a certain state, before switching to the next state. The order of states is predetermined, but the controller can skip a state if there is no traffic in a certain direction. The amount of arriving and waiting vehicles are quantized into fuzzy variables, like many, medium and none. The activation of the variables in a certain situation is given by a membership function, so when there are 5 cars in the queue, this might result in an activation of 25% of ’many’ and 75% of ’medium’. Fuzzy rules are used to determine if the duration of the current state should be extended. In experiments the fuzzy logic controller showed to be more flexible than fixed controllers and vehicle actuated controllers, allowing traffic to flow more smoothly, and reducing waiting time. A disadvantage of the controller seems to be its dependence on the preset quantification values for the fuzzy variables. They might cause the system to fail if the total amount of traffic varies. Furthermore, the system was only tested on a single junction. Lee, et al. (1995) studied the use of fuzzy logic in controlling multiple junctions. Controllers received extra information about vehicles at the previous and next junctions, and were able to promote green waves. The system outperformed a fixed controller, and was at its best in either light or heavy traffic. The controller could easily handle changes in traffic flow, but required different parameter settings for each junction. Choi ,et al. (2002) also use fuzzy logic controllers, and adapted them to cope with congested traffic flow. Comparisons with fixed fuzzy-logic traffic light controllers indicated that this enhancement can lead to larger traffic flow under very crowded traffic conditions.

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CHAPTER 3

METHODOLOGY

3.0 INTRODUCTION

This chapter deals with modeling traffic flow using partial differential equation (PDE) from the macroscopic approach hence below is a summary of PDE

3.1 NOTATION AND TERMINOLOGY

Let n denote a function of several independent variables: say, $u = u(x, y, z, t)$,the partial deferential(PDE) of u with respect to x is defined by

$$\frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{u(x+h, y, z, t) - u(x, y, z, t)}{h} \tag{3.1}$$

Provided the limit exists. We use subscript notation as follows:

$$\frac{\partial u}{\partial x} \equiv u_x \ , \ \frac{\partial u}{\partial y} \equiv u_y , \frac{\partial^2 u}{\partial x^2} \equiv u_{xx} , \frac{\partial^2 u}{\partial y^2} \equiv u_{yy} , \frac{\partial^2 u}{\partial x \partial y} \equiv u_{xy} , \ldots$$

If all partial derivatives of u up through order m are continuous in some region Ω ,we say that u is the class $c^m(\Omega)$ or u is c^m in Ω. Spiegel (1991) defines PDE as:

A **Partial differential Equation** (PDE**)** is an equation involving one or more partial derivatives of an unknown function of several variables.

The **Order** of a PDE is the order of the highest-order derivative that appears in the equation.

Linear PDE: The PDE $F(x, y, z, t; u, u_x, u_y, u_z, u_t, u_{xx}, u_{xy}, ...)$ =0 is said to be linear if the function F algebraically linear in each of the variable $u, u_x, u_y .....$ and if the coefficient of u and its derivatives are functions only of the independent variables; otherwise it is nonlinear.

A nonlinear PDE is said to be quasilinear if it is linear in the highest –order derivatives.

3.1.1 CLASSIFICATION OF FIRST ORDER PDE

The general quasilinear system of n first-order PDEs in n functions of two independent variables is of the form:

$$\sum_{j=1}^{n} a_{ij} \frac{\partial u_j}{\partial x} + \sum_{j=1}^{n} b_{ij} \frac{\partial uj}{\partial y} = c_i, (i = 1,2,3....n) \qquad (3.2)$$

Where aij, bij, and, ci may depend on $x, y, u_1, u_2, ...u_n$.If each a_{ij} and b_{ij} is independent of $u_1, u_2, u_3, ..., u_n$

Example, the system

$$\begin{aligned} (\rho u)_x + \rho_t &= 0 \\ uu_x + u_t &= -\frac{1}{\rho} p_x \\ up_x + p_t &= -\gamma pu_x \end{aligned} \qquad (3.$$

(Duchateau, 1986)

is called almost linear. If in addition each c_i depends linearly on $u_1, u_2, u_3, ..., u_n$, the system is said to be linear .The example (3.3) of systems of equation above is a quasilnear. In term of the n x n matrix $A = |a_{ij}|$ and $B = |b_{ij}|$,and the column vectors $u = (u_1, u_2, ..., u_n)'$ and $c = (c_1, c_2, ..., c_n)'$ the system of equations can be expressed as :

$$Au_x + Bu_y = c \qquad (3.4)$$

If A or B is nonsingular, it is usually possible to classify the systems of first-order according to types as follows: suppose $\det(B) \neq 0$ and define a polynomial of degree n in λ by

$$P_n(\lambda) \equiv \det(A^T - \lambda B^T) = \det(A - \lambda B) \qquad (3.5)$$

Hence systems of first- order PDEs is classified as follows:

The PDE is said to be:

I. Elliptic, if $Pn(\lambda)$ has no real zeros.

II. Hyperbolic, if $Pn(\lambda)$ has n real, distinct zero, or if $Pn(\lambda)$ has n real roots, at least one of which is repeated and the generalized eigenvalue problem $(A^T - \lambda B^T)t = 0$ yields n linearly independent eigenvectors t

IV. Parabolic, if $Pn(\lambda)$ has n real zeros, at least one of which is repeated, and the above generalized eigenvalue problem yields fewer than n linearly independent eigenvectors. However an exhaustive classification cannot be carried out when $Pn(\lambda)$ has both real and complex zeros.

(Duchateau, 1986)

The spatial variables in PDE are usually restricted to some open Ω with boundary S; the union of Ω and S is a closure of Ω and denoted by $\overline{\Omega}$ .If present the time variable is considered to run over an interval $t_1 < t < t_2$.

A function

$$u = u(x, y, z, t) \tag{3.6}$$

is the solution for a given $m^{th} - order$ PDE if ,for (x, y, z) in Ω and $t_1 < t < t_2$ .u is c^m and satisfy the PDE.

3.1.2 AUXILIARY CONDITIONS; WELL-POSED PROBLEM

The PDEs that model physical system usually have infinitely many solutions. To select the single function that represents the solution to the physical problem one must impose certain auxiliary conditions that further characterize the system being modeled. The two categories are:

Boundary Conditions: These are conditions that must be satisfied at points on the boundary S of the spatial region Ω in which PDE holds. Three forms of (BC) are:

Dirichlet condition:

$$u = g \tag{3.7}$$

Neumann (or the flux) condition:

$$\frac{\partial u}{\partial n} = g \tag{3.8}$$

Mixed (or Robin or radiation):

$$\alpha u + \beta \frac{\partial u}{\partial n} = g \,, \tag{3.9}$$

in which $g, \alpha, and \beta$ are function prescribed on S

Initial conditions: These are conditions that must be satisfied throughout Ω at the instant when consideration of the physical system begins. A typical initial condition prescribes some combination of u and its time derivatives. The prescribed initial and boundary-condition functions and any inhomogeneous term in PDE are said to comprise the data in the problem modeled by the PDE. The solution is said to depend continuously on the data if small change in the data produce correspondingly small changes in the solution. Hence a problem is said to be well posed if:

I. A solution of the problem exists

II. The solution is unique

The solution depends continuously on the data.

(Duchateau, 1986)

3.2 THE PDE MODEL BASED ON LWR MODEL

The macroscopic traffic model developed first by Lighthill and Whitham (1955) and Richard (1956) shortly called LWR model is most suitable for correct description of traffic flow; details can be seen in (Haberman, 1977). In this model, vehicles in traffic flow are considered as particles in fluid: further the behaviour of traffic flow is modeled by the method of fluid dynamics and formulated by hyperbolic partial differential equation (PDE) .The macroscopic traffic flow model is used to study traffic

flow by collective variables such as traffic flow rate (flux) $q(x,t)$,traffic speed $V(x,t)$ and traffic Density $\rho(x,t)$ all of which are functions of space, $x \in \Re$ and time, $t \in \Re^{+}$

There is unique relation among the three macroscopic traffic flow parameters density, flow and speed.Under uninterrupted flow conditions, these traffic variables are all related by the equation:

$$q = \rho u \tag{3.1}$$

This relationship is called the fundamental equation of traffic flow.

3.2.1 CONSERVATION OF CARS

Considering a section of road, if there is no on-ramp and off-ramp, then the number of cars coming in equals the number of cars going out of the segment (conservation law), and the variety rate of density will be equal to the difference of inflow and outflow. Mathematical model in traffic flow an mechanical vibration (Haberman and Richard,1977)

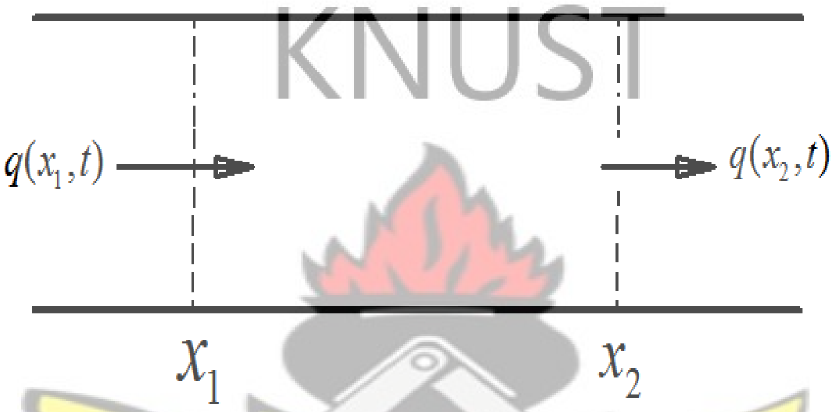

Figure 3.1: Inflow and outflow on a section of road

Let ρ denotes traffic density, u denotes space mean speed, then the well-known Lighthill, Witham and Richards (LWR) model is formulated by employing the conservation equation and is expressed as:

$$\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial q(x,t)}{\partial x} = 0 \qquad (3.2)$$

It is assumed by LWR theory that there is a relationship between u and ρ :

$$u = u(\rho) \qquad (3.3)$$

This is called equilibrium speed-density relationship.

Crucial to the approach of Lighthill, Witham and Richards (LWR) was the fundamental hypothesis, i.e. flow is a function of density and speed as in Eqn. (3.1). Thus traffic flow from equation (3.2) can be expressed in terms of the traffic density and the traffic speed

$$\frac{\partial \rho}{\partial t} + \frac{\partial \left(\rho u(\rho) \right)}{\partial x} = 0 \qquad (3.4.1)$$

$$\frac{\partial \rho}{\partial t} + \frac{dq}{d\rho}\frac{\partial \rho}{\partial x} = 0 \qquad (3.4.2)$$

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Thus, Equation (3.4.2) gives the conservation equation where the flux changes with the density. . Mathematical model in traffic flow an mechanical vibration (Haberman , 1977)

3.2.2 METHOD OF CHARACTERISTICS

The analytical method of characteristics is used to solve the macroscopic LWR traffic model in conjunction with Greenshield's flow density relation.

Let consider the function $\rho(x,t)$ at each point of the (x,t) plane. Along any curve in the (x,t) plane, x and ρ can be considered as functions of t, therefore the total derivative of $\rho(x(t),t)$ is:

$$\frac{d\rho(x(t),t)}{dt} = \frac{\partial \rho}{\partial t} + \frac{dx}{dt}\frac{\partial \rho}{\partial x} \qquad (3.5)$$

From Equations (3.4.2) and (3.5) we obtain

$$\frac{d\rho}{dt}=0 \quad \text{and} \quad \frac{dx}{dt}=\frac{dq}{d\rho}=c(\rho) \tag{3.6}$$

Thus there exists a certain curve x along which the solution ρ is constant. Such curves are called characteristics of the non- linear hyperbolic equation Greenberg (1959); John (1991); Sarra(2003); Coleman (2009).

Therefore, the governing equation for the system can be expressed as

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$$\frac{\partial\rho(x,t)}{\partial t}+c(\rho)\frac{\partial\rho(x,t)}{\partial x}=0 \tag{3.7}$$

3.3 TRAFFIC LIGHT CONTROL ON TRAFFIC FLOW

In order to maximize the flow of traffic on a given roadway we look at some traffic light control mechanism .The largest flow: $q=\rho u$ would occur if cars were bumper to bumper thus:

$\rho=\rho_{\max}$ moving at the speed limit: $u=u_{\max}$ .The hypothesis that if $\rho=\rho_{\max}$,then the cars would be bumper to bumper and would not move yielding a minimum traffic flow namely zero due to the man's driving habit. Here we assume that the road is homogeneous such that the cars velocity depends on density and not on time or position along the road. Since the traffic flow(number of cars per hour) equals density times velocity, the flow also depends on the density .Thus:

$q=\rho u(\rho)$.

(Haberman , 1977)

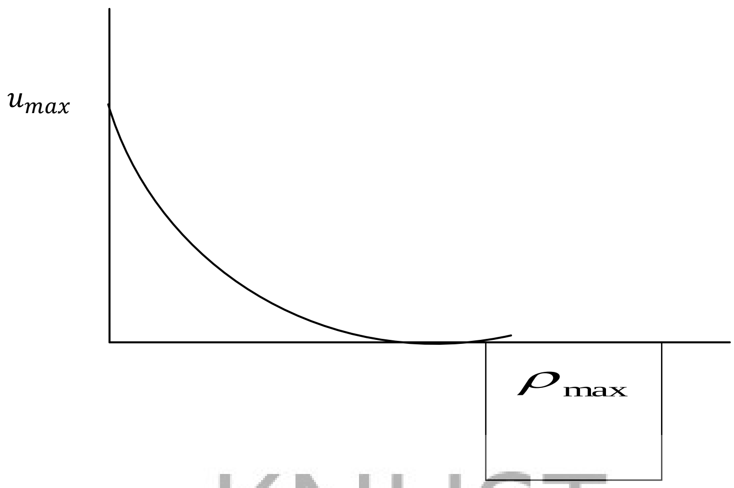

Figure 3.2 : Relationship between $u_{\max}$ and $\rho_{\max}$ (Haberman , 1977)

To improve traffic efficiency, we explain the velocity –curve by analyzing drivers decision. The model for analysis was postulate on the assumption that an individual car motion depends on the car ahead such that the car's acceleration is proportional to the relative velocity. Thus:

$$\frac{d^2 x_n(t)}{dt^2} = -\lambda\left(\frac{dx_n(t)}{dt} - \frac{dx_{n-1}(t)}{dt}\right) \tag{3.8}$$

Equation (3.8) suggests that acceleration or deceleration occurs instantaneously and λ measures the sensitivity of the cars.

Let T be delay time for reaction to the change in relative velocity. Hence we obtain a system of delay differential equations from equation (3.8) as:

$$\frac{d^2 x_n(t+T)}{dt^2} = -\lambda\left(\frac{dx_n(t)}{dt} - \frac{dx_{n-1}(t)}{dt}\right) \tag{3.9}$$

Integrating equation (3.9) yields:

$$\frac{dx_n(t+T)}{dt} = -\lambda(x_n(t) - x_{n-1}(t)) + d_n \tag{3.10}$$

Equation (3.9) depicts the velocities of cars in later times with respect to their distance between them. In steady- state situation where all cars are equidistant apart and in effect moving with the same velocity, we can we write:

$$\frac{dx_n(t+T)}{dt} = \frac{dx_n(t)}{dt} \quad \text{and letting} \quad d_n = d \qquad (3.11)$$

From equation (3.10) and (3.11) we obtain:

$$\frac{dx_n(t)}{dt} = -\lambda(x_n(t) - x_{n-1}(t)) + d \qquad (3.12)$$

By the definition of density $x_{n-1}(t) - x_n(t) = \frac{1}{\rho}$ equation (3.12) yields

$$u = \frac{\lambda}{\rho} + d$$

We choose one arbitrary constant such that at the maximum density u=0. Therefore at u=0, we write

$$0 = \frac{\lambda}{\rho_{\max}} + d$$

$$\Rightarrow d = -\frac{\lambda}{\rho_{\max}}$$

Hence from equation (3.12) the velocity- density relation is obtained as:

$$u = \lambda\left(\frac{1}{\rho} - \frac{1}{\rho_{\max}}\right) \qquad (3.13)$$

Equation (3.12) and (3.7) is reasonable for large density near $\rho = \rho_{\max}$ since it predict an infinity velocity at $\rho = 0$ (or $\rho_{\max} = 0$) for two reasons;

I. For small density the changes in speed of a car are not influence by the car in front.

II. For small density U is perhaps is limited by the speed limit $u = u_{\max}$

(Haberman ,1977)

Thus:

$$u = \begin{cases} u_{\max}, \rho < \rho_c \\ \lambda\left(\dfrac{1}{\rho} - \dfrac{1}{\rho_{\max}}\right), \rho > \rho_c \end{cases} \tag{3.14}$$

The critical density ρ_c is chosen such that the velocity is continuous function of density as shown below. The flow $q = \rho u$ is

$$q = \begin{cases} u_{\max}\rho, \rho < \rho_c \\ \lambda\left(\dfrac{1}{\rho} - \dfrac{1}{\rho_{\max}}\right), \rho > \rho_c \end{cases} \tag{3.15}$$

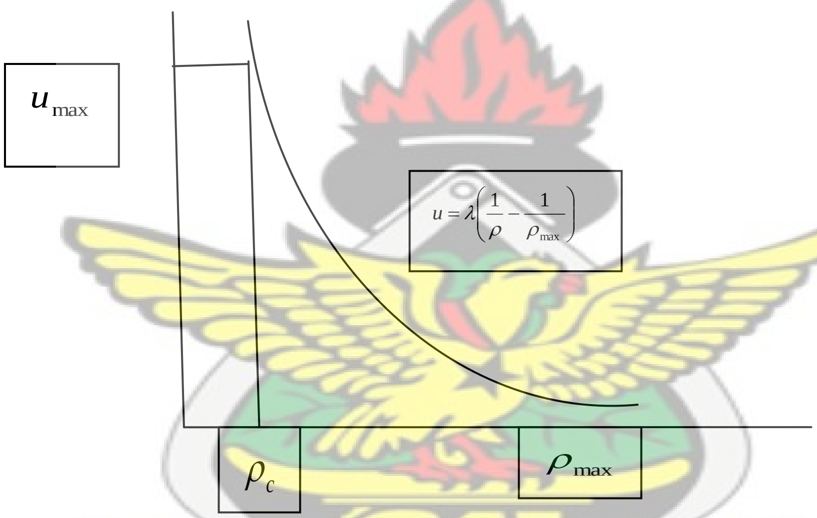

Figure 3.3: Relationship between $u_{\max}$ and $\rho_{\max}$ (Haberman , 1977)

To rectify the abnormality of infinite velocity at small density, we find the inverse relation between the sensitivity and the distance .Thus:

$$\lambda = \frac{c}{x_{n-1}(t) - x_n(t)} \tag{3.16}$$

From (3.9) and (3.16) we obtain:

$$\frac{d^2 x_n(t+T)}{dt^2} = c\,\frac{\dfrac{dx_n(t)}{dt} - \dfrac{dx_{n-1}(t)}{dt}}{x_n(t) - x_{n-1}(t)} \qquad (3.17)$$

Equation (3.17) is integrated to yield

$$\frac{dxn(t+T)}{dt} = cIn\left|x_n(t) - x_{n-1}(t)\right| + d_n \qquad (3.18)$$

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At the steady state situation

$$u = -cIn\rho + d$$

Once more the integration constant is chosen such that at maximum density

$$u = -c\,\frac{In\rho}{p_{\max}}$$

Difficulties such as

$$\rho \to 0$$

are again avoided by assuming that for low densities,

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$$u = u_{\max}$$

The constant c is chosen (perhaps by a least-square fit) so that the formula agrees well with the observed data. The constant c has a simple interpretation in the velocity as:

$$q = \rho u$$

$$\Rightarrow q = -c\rho In\frac{\rho}{\rho_{\max}}, and \qquad (3.19)$$

$$0 = \frac{dq}{d\rho} = -c\left(In\frac{\rho}{\rho\max} + 1\right)$$

From the (3.19) it can be inferred that the maximum traffic flow occurs at $\rho = \rho_{\max} / e$,

in which case the velocity at the maximum flow is c,

$$u\left(\frac{\rho_{\max}}{e}\right) = c \qquad (3.20)$$

3.3.1 EFFECT OF TRAFFIC LIGHT ON FLOW OF VEHICLES WHEN THE TRAFFIC LIGHT TURNS FROM RED TO GREEN

Suppose that traffic is lined up behind a red traffic light we call the position of the traffic

$$x = 0$$

Since the cars are bumper to bumper behind the traffic light:

$$\rho = \rho_{\max} \quad \text{For } x < 0$$

Assuming that the cars are lined up indefinitely and are not moving (very long finite line up of vehicles) then the analysis here is limited to the time and places at which the effect of thinning of the waiting line can be ignored. If the light stop traffic for a long enough, then we may assume that there is no traffic ahead of the light. Thus:

$\rho = 0, \forall x > 0$

Again suppose that at t=0, the traffic light turn from red to green ,then the density propagation of cars for all times is described by solving the PDE describing conservation of cars, thus:

$$\frac{\partial \rho}{\partial t} + \frac{dq}{d\rho}\frac{\partial \rho}{\partial t} = 0 \quad \text{. With initial condition} \qquad (3.21)$$

$$\rho(x,0) = \begin{cases} \rho_{\max}, x < 0 \\ 0, x > 0 \end{cases}$$

(Haberman ,1977)

Since the initial condition is discontinuous, the solution of equation (3.21) may be obtained by the method of characteristic as follows:

$$\frac{dx}{dt} = \frac{dq}{d\rho}$$

$$\Rightarrow \frac{d\rho}{dt} = \left(\frac{\partial \rho}{\partial t}\right) + \left(\frac{dx}{dt}\right)\left(\frac{\partial \rho}{\partial t}\right) = 0$$

Thus the traffic density $\rho(x,t)$ is constant along the characteristic which are given by

$$\frac{dx}{dt} = \frac{dq(\rho)}{d\rho} = \rho \frac{du}{d\rho} + u \qquad (3.22)$$

The density propagates at the velocity $\frac{dq}{d\rho}$ since ρ remains constant. The density moves at a constant -velocity .The characteristics are straight lines in the x-t plane

$$x = \frac{dq}{d\rho}(\rho)t + k \qquad (3.23)$$

(Haberman ,1977)

CHAPTER 4

DATA ANALYSIS AND RESULTS

4.0 INTRODUCTION

This chapter deals with the collection of data and their analysis. The data to be used in this study is primary data collected on the KNUST - Kejetia road of Kumasi in the Ashanti Region.

4.1 DATA COLLECTION

The data were collected on May 20, 2012 . The field data were collected during morning peak period. Peak hour was taken generally from 7.00am to 10.30 am, where the civil servants and traders from their home are going to work. The traffic count was done manually using the stand-by Observer method (SOM) along 100m way of Kumasi-Accra road (Ghana) between (stadium traffic junction and Amakom traffic junction). The site is a level grade straight segment between two traffic nodes. In all a one hour twenty-six minutes data were collected for the study using the stand-by Observer method.

Basically in the stand-by observer method, observer (s) stand by the roadside where vehicle travelling along a known section of road in one direction and records the number of vehicles entering and leaving the road segment.Samples of vehicle are tagged and the time they enter

and leave the traffic nodes are recorded. The speed and flow (fiux) were obtained based on the relations:

$$u = \frac{l}{t}$$

$$q = \rho u$$

4.2 ANALYSIS

The essential part of any data collection process is to be analyzed and presented in a format that is easily understandable. Below is an illustration of a simplified manual counts data analysis, as transformed from field data collection. In all, twelve (12) runs by the observer's was done to collect the data and the results are presented in Tables 4.1.

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Table 4.1: Typical Analysis of Manual Data Count

Density ρ (No of cars passing)	***Time taken t(s)***	***Velocity u(m/s)***	***Flow Q (ρ u)***
0	*60*	*1.666670*	*0*
60	*93*	*1.075270*	*64.5162*
59	*73*	*1.369860*	*80.6217*
49	*65*	*1.538460*	*75.3845*
55	*103*	*0.970874*	*53.3981*
65	*87*	*1.149430*	*74.7130*
68	*106*	*0.943396*	*64.1509*
72	*100*	*1.000000*	*72.0000*
82	*113*	*0.884956*	*72.5664*
82	*74*	*1.351350*	*52.7027*
38	*72*	*1.38889 0*	*52.7778*
161	*3600*	*0.027778*	*4.47222*

4.3 RESULTS

The result of regression of the model for data in Table 4.1 on two variables while holding the other constant is shown in Figure4.1 – 4.3 below.

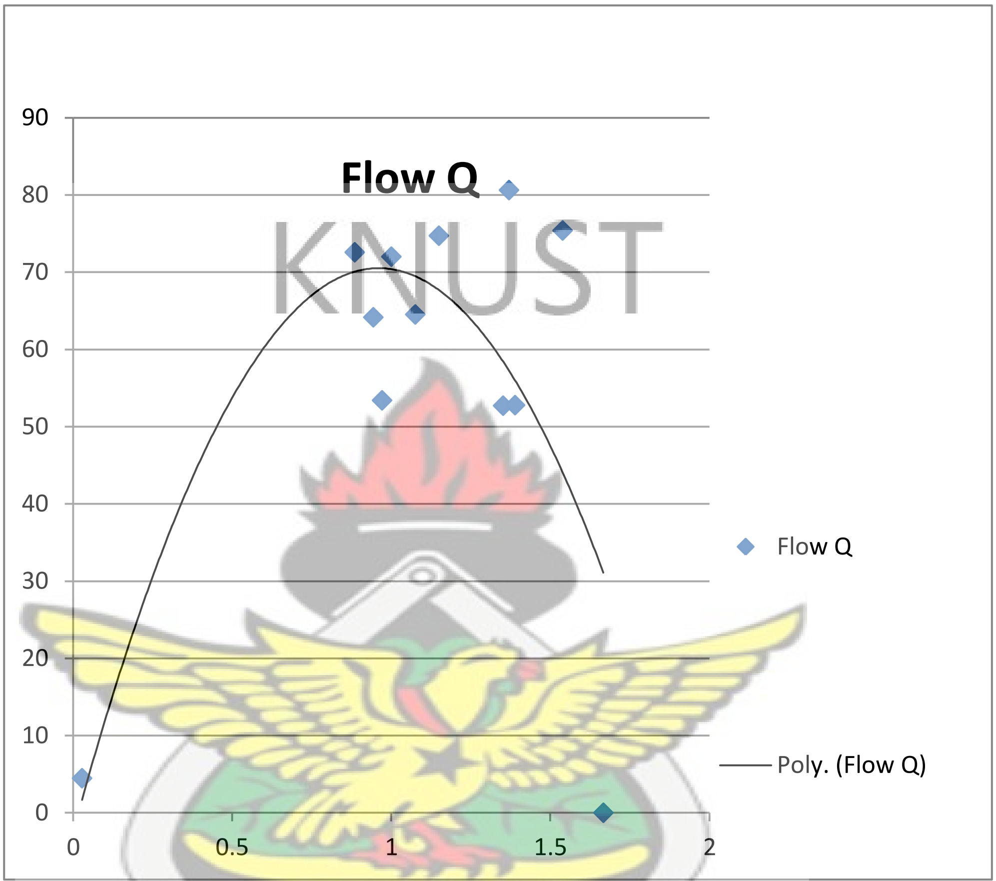

Figure 4.1: Velocity- Flow (q) curve

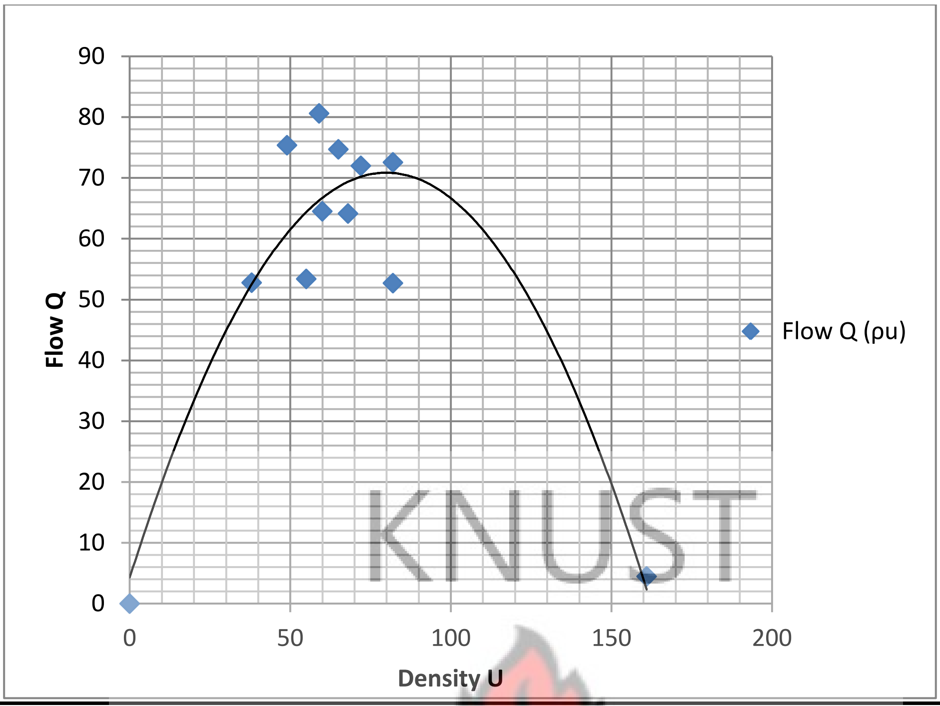

Figure 4.2: Density - Flow Curve

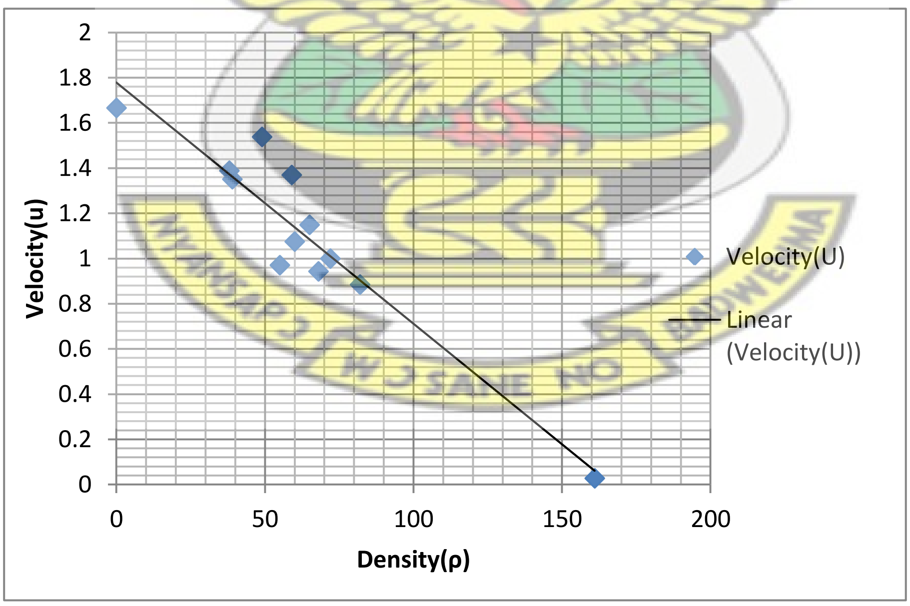

Figure 4.3: Linear Velocity- Density Graph

4.4 DISCUSSION OF RESULTS

This part of study discusses the flow – density relation from the fundamental diagrams

4.5 FLOWS VERSUS DENSITY

For the (LWR) model, It was determined that a model for predicting densities on the basis of flow would be the most effective procedure for predicting traffic operations in the basic freeway section of the roadways.

The model of the result of regression for the flow rate versus density is indicated as follows:

Using the fundamental equation;

$$q(\rho) = u_{\max}\left(\rho - \frac{\rho^2}{\rho_{\max}}\right)$$

$$=1.76\left(\rho - \frac{\rho^2}{160}\right)$$

$$q(\rho) = 1.76\rho - 0.011\rho^2 \qquad (5)$$

$$q(\rho) = \rho(1.76 - 0.011\rho) \qquad (6)$$

Comparing Eqn. (6) with the LWR flow-density relation, we have

$u_{\max} = 1.76\ (\text{m/min})\ \rho_{\max} = 160\ (\text{veh/m})$

$q_{\max} = 75\ (\text{veh/min})$

4.6 NONLINEAR HOMOGENEOUS PARTIAL DIFFERENTIAL EQUATION

Now differentiating Eqn. (5) and Substituting into the governing equation

$\frac{\partial \rho(x,t)}{\partial t} + c(\rho)\frac{\partial \rho(x,t)}{\partial x} = 0$, from the LWR model, we have

$$\frac{\partial \rho}{\partial t} + \left(1.76 - 0.022\rho\right)\frac{\partial \rho}{\partial x} = 0 \qquad (7)$$

Since the solution is $\rho = \rho\left(x,t\right)$ taken the derivate with respect to t we get

$$\frac{d\rho\left(x\left(t\right),t\right)}{dt} = \frac{\partial \rho}{\partial t} + \frac{dx}{dt}\frac{\partial \rho}{\partial x} \qquad (8)$$

Comparing the Eqn. (7) and Eqn. (8)

$\frac{d\rho}{dt} = 0$ and $\frac{dx}{dt} = \frac{dq}{d\rho} = 1.76 - 0.022\rho$

Thus on a certain curve $x\left(t\right)$ the solution $\rho\left(x,t\right)$ of Eqn. (5) is a constant.

4.7 CHARACTERISTIC CURVES

The graph of $x\left(t\right)$ to the ordinary equation

$$\frac{dx}{dt} = \frac{dq}{d\rho} = 1.76 - 0.022\rho$$

is called a characteristic curve.

The solution $\rho\left(x,t\right)$ is thus constant along the characteristic curves.

Now applying the method of characteristics to Eqn. (7) along with the initial condition

$\rho(x,0) = 120x$ at $t = 0$

The model yielded

$x = (1.76 - 2.64x_0)t + x_0$

As the characteristic curve which starts at

$x = x_0$ When $t = 0$

The solution of the nonlinear homogeneous is obtained by making x_0 the subject of the characteristic equation. Thus:

$$x_0 = \frac{x - 1.76t}{1 - 2.64t} .$$

However, $\rho(x,t) = F(x_0)$

Hence the solution of the homogeneous nonlinear PDE is given implicit function

$$\rho(x,t) = \frac{x - 1.76t}{1 - 2.64t}$$

4.8 NONLINEAR INHOMOGENEOUS PARTIAL DIFFERENTIAL EQUATION

Let $\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial x}\frac{dx}{dt} = \rho$

$$\frac{d\rho}{dt} = \rho$$

By integrating we obtain:

$$\ln \rho = t + c$$
$$\Rightarrow \rho = \rho_0 \ell^t$$

$$\Rightarrow \rho = A(x_0)\ell^t$$

If

$$\frac{dx}{dt} = (1.76 - 0.22\rho)$$
$$\Rightarrow \frac{dx}{dt} = (1.76 - 0.022A(x_0)\ell^t)$$

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Hence,

$$x = 1.76t - 0.022A(x_0)\ell^t t + x_0$$

Therefore

$$x_0 = x - 1.76t + 0.022A(x_0)\ell^t t$$
$$\Rightarrow x_0 = x + (0.022A(x_0) - 1.76)t$$

If $A(x_0) = \alpha$ then:

$$x_0 = x + (0.022\alpha - 1.76)t$$
$$\rho = \alpha \ell^t$$

CHAPTER 5

CONCLUSION AND RECOMMENDATIONS

5.1 CONCLUSION

To some extent, the results of the study agree with those of LWR model, which gives a quadratic relationship between flow and density

In the study of a section of the roadway between two traffic light nodes, using regression analysis, the Flow Density curve was found to be quadratic of the form

$$q(\rho) = 1.76\rho - 0.011\rho^2$$

The method of characteristics was used to solve both the non-linear homogeneous and the inhomogeneous PDE and the model yielded characteristics curve of the form

$$x = (1.76 - 2.64x_0)t + x_0$$

And

$$x = (1.76 - 0.022A(x_0)\ell^t)t + x_0$$

Consequently the **homogeneous solution** of the Partial Differential Equation was obtained as

$$\rho(x,t) = \frac{x - 1.76t}{1 - 2.64t}$$

And the **inhomogeneous solution** of the PDE is obtained as:

$$x_0 = x + (0.022\alpha - 1.76)t$$
$$\rho = \alpha\ell^t$$

5.2 RECOMMENDATIONS

5.2.1 RECOMMENDATION TO STAKEHOLDERS

It is recommended to conduct the same study on working days, during evening peak hour and on various segments of the road on the test site. In addition, a study is to be conducted to model traffic flow using round-about to identify the type that suit Kumasi metropolis .Further more it is recommended that to obtain accurate information on the count rate on the road segment, loop detectors should be installed on the various road in the cities of Ghana.

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To prevent congestion and to improve efficiency, traffic should somehow be forced to move at a density corresponding to maximum traffic flow. A signal which literally stops traffic and then permits it to go (in intervals yielding the density corresponding maximum flow) would result in an increased flow of cars on the road. Thus momentarily stopping traffic would actually result in an increase flow.

5.2.2 RECOMMENDATION TO RESEARCHERS

It is recommended to use other methods such as the finite difference method to solve the kinetic wave model by LWR instead of the method of characteristics.

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