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Approximation methods for common fixed points of non-expansive mappings in Hilbert spaces

By
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Declaration

I hereby declare that this submission is my own work towards the award of the M. Phil degree and that, to the best of my knowledge, it contains no material previously published by another person nor material which had been accepted for the award of any other degree of the university, except where due acknowledgement had been made in the text.

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Dedication

To God be the glory.

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Abstract

This thesis is an extensive exposition and review of the paper "Approximation methods of common fixed point of non-expansive mappings in a Hilbert space" in which the author, Paul-Emile Mainge proposed two numerical approaches to solving this problem by implicit and explicit viscosity like-methods. The study as obtain in the thesis was the strong convergence results of the implicit *anchor*-like algorithm and the explicit procedure for approximating the common fixed point of countable infinite family of non-expansive self-mappings. This thesis basically details the proofs of the main theorem of Paul's paper as well as detailed exposition of the mathematics involved in it. Detailed proofs of subsidiary results leading up to the proof of the main theorem of Paul's paper are also discussed. Finally, the main theorem of the paper is also demonstrated in a series of lemmas.

Contents

Chapter 1

Introduction

We give an highlight on the theory of fixed points of mappings in a general metric space, with a few historical remarks. Details of iterative methods for approximating a fixed point of non-expansive mapping on a closed convex subset of a Banach space is discussed in the second section. Particular emphasis is laid on detailed proofs of subsidiary results leading up to the proofs of the main theorem of Paul's paper.

1.1 Fixed Point Theory

Many problems arising in different areas of mathematics, such as optimisation, variational analysis and differential equations can be modelled by the equation

$$Tx = x, \tag{1.1}$$

where T is a non-linear operator on any set X into itself. The solutions to (1.1) are called *fixed points* of T. Now for a solution to (1.1) to be meaningful, the set X must allow for continuity and separation of points and both notions are captured in the concept of a *complete metric space*. Let X be a non-empty set and $d : X \times X \to \mathbb{R}_{\geq 0}$. Assume that for all x, y and z in X we have the following conditions:

1. $d(x,y) \geq 0$.

2. $d(x,y) = 0$ if and only if $x = y$.

3. $d(x,y) = d(y,x)$.

4. $d(x,y) \leq d(x,z) + d(z,y)$.

Then we call d *a metric* on X, and X together with d is called a *metric space* (X,d). Given a metric space (X,d), a countable sequence of elements $\{x_n\}_{n\geq 1}$ is a *Cauchy sequence* if for every $\varepsilon > 0$, there exists a positive integer $N(\varepsilon)$ such that for all natural numbers $m,n > N$, we have $d(x_m,x_n) < \varepsilon$. A metric space containing the convergent points of all its Cauchy sequences is termed as a *complete metric space*. The set of real numbers is a typical example.

A self-map T on a complete metric space (X,d) is *uniformly continuous* on the space if for every $\varepsilon > 0$, there exists a real number $\delta > 0$, such that whenever $d(x,y) < \delta$, then $d(Tx,Ty) < \varepsilon$ for all $x,y \in X$. Given a complete metric space (X,d), the most well-studied types of self-maps are referred to as *Lipschitz mappings* (or *Lipschitz maps*, for short), which are given by the metric inequality

$$d(Tx,Ty) \leq cd(x,y), \tag{1.2}$$

where $c > 0$ is a real number, usually referred to as the *Lipschitz constant* of T. The important property of (1.2) is that they are uniformly continuous. Thus, for any sequence $\{x_n\}_{n\geq 1}$ converging to x in X, we have $d(Tx_n,Tx) = 0$ as $n \to \infty$. Lipschitz mappings also maps *bounded sets to bounded sets*. That is, if T is a Lipschitz mapping and C is a subset of the domain on which T acts, then supremum norm of T on C is a finite number.

Now for technical and historical reasons, we classify Lipschitz mappings (1.2) into three categories, thus

1. Contraction mappings, that is $c < 1$.

2. Non-expansive mappings, that is $c = 1$.

3. Expansive mappings, that is $c > 1$.

The fixed point theory of Lipschitz mappings in metric spaces is centred on these two basic well-posed enquiries:

- Existence and Uniqueness of fixed points and

- Algorithms or Iterative Schemes for approximating such fixed points

This thesis focuses non-expansive type of Lipschitz mappings. These mappings are defined by the mathematical expression

$$d(Ty,Tx) \leq d(y,x) \tag{1.3}$$

where d is a metric. In some situations, *existence* of fixed points of the mapping T as defined in (1.3) may not be guaranteed in the space under consideration. For example, if $T : R \to R$ given by $Tx = x + k$ where $k \neq 0$ is any number, then T has no fixed point. In other words, the set of fixed point of a nonexpansive map T may be empty or closed and convex as shown in

Lemma 2.0.10. In situations where the fixed point of a nonexpansive map may not exist, then we must assume additional conditions on T or the underlying space to ensure the existence of fixed points. The famous fixed point theorem for nonexpansive maps were given by (Browder, 1965a), (Kirk, 1965) and (Göhde, 1965) independently in 1965. The following books also cover a good deal of fixed points theorems Agarwal et al. (2001); Granas and Dugundji (2013); Border (1989) and Rus (2001); Singh et al. (2013). These theorems are not true in general when extended to arbitrary continuous functions in infinite dimensional spaces. For instance if B is a unit ball in an infinite dimensional Hilbert Space and $T : B \to B$ is a continuous function, then T need not to have a fixed point. This was presented and analysed by (Kakutani et al., 1941). The first fixed point theorem in an infinite dimensional Banach Space was also given by (Schauder, 1930). Brouwer (1912) also presented and analysed a famous fixed point theorem in 1912.

Fixed points of nonexpansive mappings are generally not *unique*. Brouwer (1912) presented and analysed that if $T : B \to B$ is a continuous function and B is a ball in $\mathbb{R}^n$, then T has a fixed point. This theorem simply guarantees the existence of a solution, but gives no information about the uniqueness of the solution. For instance, if $T : [0,1] \to [0,1]$ is given by $Tx = x^2$, then $T(0) = 0$ and $T(1) = 1$, that is T has two fixed points which clearly demonstrates the non-uniqueness of fixed points. That is, fixed point problems of nonexpansive mappings in general are ill-posed and as such additional information must be imposed on the space or T to ensure uniqueness of fixed point. In Hilbert spaces (in general strictly convex reflexive spaces or weakly compact subsets of Banach spaces), adding information by means of projections can 'regularize' the fixed point problem as in (Maingé, 2007) so that a unique solution is achieved. Fixed points problems of contraction mappings defined in Definition 15 always exist and it is unique due to the Banach contraction principle as proved by (Banach, 1922). The most important feature of the Banach Contraction principle is that it gives the existence and uniqueness of a fixed point of a contraction mapping, which can be approximated by the Picard iterative sequence $x_{n+1} = Tx_n$. Concerning the fixed point approximation problem, the sequence of Picard iterates $\{T^n x\} := T^{n-1}(Tx)$ strongly converges for contractions on complete metric space unlike in nonexpansive mappings. This is a very useful result and it has been applied in the determination of the existence and uniqueness of many results in analysis and economics (see for instance, Border (1989) and Freeman (1976)).

Fixed point theory is a fascinating subject, with an enormous number of applications in various fields of mathematics and engineering. In number of situations, one may need to find a common fixed point of a family of mappings. In practice, a modification may be needed to turn the problem into a fixed point problem. For instance, in finding solution to a first-order differential equation defined under suitable conditions, one can modify the problem into a contraction mapping and therefore, by the Banach Contraction Theorem, has a unique solution in the space within which the differential equation is defined (see for instance Picard (1890) and Lindelöf (1894)). The fixed point problem has extensively been applied to various types of linear and non-linear problems such as equilibrium problems (see Tang and Chang (2009)), proximal point algorithm (see Solodov and Svaiter (2000)), convex optimisation and minimisation problems (see Takahashi (2000) and Takahashi (2009)) and variational inequalities (see Haugazeau (1968) and Blum and Oettli (1994)).

In practice, finding an exact closed form of a solution to a fixed point problem is almost a difficult task. For this reason, it has been of particular importance in the development of feasible iterative methods for approximating fixed points of a nonexpansive mapping. For instance, Bruck (1983) and Chidume (2009) analysed the asymptotic behaviour of nonexpansive mappings in Hilbert and Banach spaces.

1.2 Iterative algorithms for nonexpansive mappings

If T is a contraction defined on a complete metric space X, then the Banach contraction principle Banach (1922) states that for any $x \in X$, the Picard iteration $\{T^n x\}_{n=1}^{\infty}$ converges strongly to the unique fixed point of T unlike in nonexpansive mappings where certain conditions must be imposed on T to ensure the existence of fixed points of T and even where a fixed point exists, the sequence of iteration in general do not converge to the fixed point. In particular case where T is firmly nonexpansive, Picard iteration does not converge assuming existence of fixed point as analysed and presented in (Goebel and Reich, 1984). The study of iterative methods for approximating fixed points of a nonexpansive mapping T have been studied extensively in the last decades and are still the focus of a host of research. The most relevant progresses are mainly based on two types of iteratives algorithms: Mann/Ishikawa and Halpern iterations.
Basically, the iterative schemes for nonexpansive mappings can be put into three types namely:

- Implicit Iterative Methods
- Explicit Iterative Methods and
- Projection/Hybrid Methods

1.2.1 Implicit Iterative Methods

Sangago (2011) proposed and studied an Implicit Iterative Algorithm for approximating the fixed point of a non-expansive mapping. His iterative scheme is as follows:

$$x_n = \alpha_n x_{n-1} + \beta_n T(x_{n-1}) + \gamma_n T(x_n), \quad n \geq 1. \tag{1.4}$$

where $\alpha_n + \beta_n + \gamma_n = 1$ and $0 < \inf \gamma_n \leq \sup \gamma_n < 1$. The above (1.4) converges weakly to a fixed point of the non-expansive map T. The references, Browder (1965b) and Reich (1994) can also be consulted for implicit iterative schemes for approximating a fixed point of a non-expansive mapping.

1.2.2 Explicit Iterative Methods

An example of the explicit method is the Mann iteration Mann (1953) which is an algorithm defined by the recursive scheme

$$x_{n+1} = (1-a_n)x_n + a_n T x_n, \quad n \geq 0. \tag{1.5}$$

where x_0 is an arbitrary point in the domain of T usually a closed convex subset and $\{a_n\} \in [0,1]$ with the assumption that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n\geq 1} \alpha_n = \infty$. For $n \geq 1$, we have that x_n is contained in the domain of T for that fact T is a self-map. Reich (1979) presented and analysed that if the space under consideration is uniformly convex and has a Frechet differentiable norm, T has a fixed point and that $\sum_n a_n(1-a_n) = \infty$, the sequence $\{x_n\}$ defined by (1.5) converges to the fixed point of T. In an infinite-dimensional space, the Mann iteration cannot have a strong convergence as it was presented by (Genel and Lindenstrauss, 1975). These references Browder (1965b),Falset et al. (2001) and Ishikawa (1976) can also be consulted for the convergence of the Mann iteration.

Another explicit method is the Halpern iteration initially due to (Halpern, 1967) is defined by the recursive scheme

$$x_{n+1} = a_n u + (1-a_n)T x_n, \quad n \geq 0. \tag{1.6}$$

where we have x_0 and u to be arbitrary points in the domain of T and $\{a_n\} \in [0,1]$ with the following impositions:

- $\lim_{n\to\infty} \alpha_n = 0.$
- $\sum_{n\geq 1} \alpha_n = \infty.$
- Either $\sum_{n\geq 0} |\alpha_n - \alpha_{n+1}| < \infty$ or $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 0.$

Strong convergence for (1.6) can be proved given that the underlying space is smooth enough as presented in the following references (Wittmann, 1992), (Suzuki, 2007), (Reich, 1994) and (Reich, 1980; Xu, 2002). The strong convergence for (1.6) is not that clear if the underlying space is not smooth. Strong convergence for (1.5) and (1.6) are both guaranteed for all closed convex subset of a Hilbert space. Halpern demonstrated that for a Hilbert space, the first two assumptions above are necessary; but whether or not they are sufficient is not resolved.

1.2.3 The Projection Methods

Haugazeau (1968) initially proposed the projection method which was later developed by (Solodov and Svaiter, 2000). Takahashi et al. (2008) developed special type of the projection called the *Shrinking Projection Method* which was used by (Kimura, 2014). The Shrinking Projection Method can be described by the algorithm:

$$\begin{aligned} y_n &= \alpha_n x_n + (1-\alpha_n)Tx_n \\ \beta_n &= \{z \in C : ||y_n - z|| \leq ||x_n - z||\}, \quad C \quad \text{a closed convex subset} \\ C_{n+1} &= \beta_n \cap C_n \\ x_{n+1} &= P_{C_{n+1}} x_0 \end{aligned}$$

where $P_{C_{n+1}}x_0$ is the metric projection of the arbitrarily chosen point x_0 in $C := C_1$ onto the closed convex set C_{n+1}. Strong convergence for the above algorithm is always guaranteed for all closed convex subsets of a Hilbert space with the assumption that $\limsup_{n\to\infty} \alpha_n < 1$.

1.3 Scope and Structure of Thesis

In this thesis, we study and make an extensive review of the paper Maingé (2007) by Paul-Emile Mainge titled "*Approximation methods for common fixed points of non-expansive mappings in Hilbert spaces*".

The structure of the thesis is as follows:

- Chapter 1: Introduction
- Chapter 2: Preliminary Results
- Chapter 3: Main Results
- Chapter 4: Conclusion
- References

As an extensive review of Maingé (2007), the Introduction highlights on related literatures on the subject within which the paper under study is reviewed; more importantly, the study of iterative algorithms which is central to the main result of *Paul's paper*. The Preliminaries section elucidate and contain detailed proofs that were used in the paper under review. Chapter 3 gives us the breakdown on the extensive review of *Paul's paper*. That is, in this chapter we study and analyse the strong convergence of the two proposed iterative algorithms employed in *Paul's paper* for finding specific common fixed point of infinite countable families of non-expansive self-mappings in a real Hilbert space. Under suitable conditions imposed on the involved parameters, we study the convergence in norm of x_t (as $t \to 0$) defined by (3.2) and that of (x_n) (as $n \to \infty$) defined by (3.3) to the unique fixed point of the map P_S o C. The conclusion gives us a summary of Paul's paper.

There have been several viscosity-like methods for finding common fixed points of non-expansive operators. Most of them are iterative processes for approximating common fixed points of finite families of non-expansive mappings (even for more general operators such as asymptotically non-expansive or quasi non-expansive mappings) in Hilbert or Banach spaces. These implicit or nonimplicit algorithms have been investigated by several authors such of which can be found in Lions (1977), Wittmann (1992), Bauschke (1996) and Zhou and Chang (2002).

Chapter 2

Preliminary Results

In this chapter, we introduce all mathematical tools that will be used in the proof of our main theorem and discussion. As a result, we give some basic concepts in Hilbert space and outline the proofs of various lemma's, propositions and theorems that will be used in the main work.

Definition 1. *(Vector or linear Space)*
A vector space X over a scalar field $\mathbb{K}$ $(\mathbb{R}$ or $\mathbb{C})$ is the set of elements called vectors together with two operations. The first operation is addition which associates any two vectors $x,y \in X$ with a vector $x+y \in X$, the sum of x and y. The second operation is scalar multiplication which associates with any vector $x \in X$ and any scalar α a vector $\alpha x \in X$; the scalar multiple of x and α. The set X and the operations of addition and scalar multiplication are to satisfy the following axioms:

1. *X is an abelian group. That is, for all $x,y,z \in X$, then,*
    - $x+y = y+x$ *(Commutative law).*
    - $(x+y)+z = x+(y+z)$ *(Associative law).*
    - *There is a null vector $0 \in X$ such that $0+x = x$.*
    - $\forall x \in X, \exists : (-x) \in X$ *such that* $x+(-x) = 0$.
2. $\lambda \cdot (x+y) = \lambda \cdot x + \lambda \cdot y, \forall \lambda \in \mathbb{K}$.
3. $(\alpha+\beta) \cdot x = \alpha \cdot x + \beta \cdot x, \forall \alpha, \beta \in \mathbb{K}$.
4. $(\alpha\beta) \cdot x = \alpha(\beta \cdot x) \quad \forall \alpha, \beta \in \mathbb{K}$.
5. $1 \cdot x = x, \quad 1 \in \mathbb{K}$.

Definition 2. *(Normed Linear Space)*
A norm on a (real or complex) vector space X is a function $|| \cdot ||: X \longrightarrow [0,\infty)$ *such that for all* $x,y \in X, \lambda \in \mathbb{K}$*, the following conditions are satisfied:*

1. $|| x ||= 0$ *if and only if* $x = 0$.

2. $|| \lambda x ||=| \lambda ||| x ||$ *for every scalar* $\lambda \in \mathbb{K}$ *(homogeneity).*

3. $|| x+y ||\leq|| x || + || y ||$ *(triangle inequality).*

A linear or vector space with a norm defined on it is called a Normed Linear Space. A Banach Space is a complete normed linear space (complete in the metric defined by the norm).

Definition 3. *(Inner product space, Hilbert Space)*
An inner product (scalar product) on a vector space X is a mapping $\langle \cdot,\cdot \rangle : X \times X \longrightarrow \mathbb{C}$*,* $\mathbb{C}$ *the set of complex numbers with the properties that, for all vectors* $x,y,z \in X$ *and scalars* $\lambda \in \mathbb{C}$*, the following conditions are satisfied:*

1. $\langle x,x \rangle$ *is nonnegative.*
2. $\langle x,x \rangle = 0$ *if and only if* $x = 0$.

3. $\langle \alpha x,y \rangle = \alpha \langle x,y \rangle$.

4. $\langle x,y \rangle = \overline{\langle y,x \rangle} \quad (complex \quad conjugation)$.

5. $\langle x+y,z \rangle = \langle x,z \rangle + \langle y,z \rangle$.

Such a vector or linear space X is called an inner product space (or a unitary space or a pre-hilbert space) which is denoted by $(X, \langle \cdot,\cdot \rangle)$*. Consequently, an inner product space X is complete if every Cauchy sequence in X converges to an element of X. A complete inner product space is called a Hilbert Space.*

Remark 2.0.1. *In a real vector space, property (4) of the inner product space means symmetry. That is,*

$$\langle x,y \rangle = \langle y,x \rangle$$

Lemma 2.0.2. *(The Cauchy-Schwarz Inequality)*
Suppose a Hilbert (or, generally, any inner product) space H, then

$$| \langle x,y \rangle |^2 \leq \langle x,x \rangle \langle y,y \rangle ,$$

for all $x, y \in H$. The equality occurs if and only if x and y are linearly dependent.

Proof. If $y = 0$, then $\langle x, 0 \rangle = 0$ and the inequality is true. Assume $y \neq 0$ and that

$$a = -\frac{\langle x,y \rangle}{\langle y,y \rangle}.$$

Clearly a is a complex number since $\langle x,y \rangle$ is a complex number and $\langle y,y \rangle$ is a real number. Then we have,

$$\begin{aligned}
0 \leq \langle x+ay, x+ay \rangle &= \langle x, x+ay \rangle + \langle ay, x+ay \rangle \\
&= \langle x,x \rangle + \langle x,ay \rangle + \langle ay,x \rangle + \langle ay,ay \rangle \\
&= \langle x,x \rangle + \bar{a} \langle x,y \rangle + a \langle y,x \rangle + a \langle y,ay \rangle \\
&= \langle x,x \rangle + \bar{a} \langle x,y \rangle + a \langle y,x \rangle + a \overline{\langle ay,y \rangle} \\
&= \langle x,x \rangle + \bar{a} \langle x,y \rangle + a \langle y,x \rangle + a\bar{a} \langle y,y \rangle \\
&= \langle x,x \rangle + \bar{a} \langle x,y \rangle + a \langle y,x \rangle + |a|^2 \langle y,y \rangle \\
&= \langle x,x \rangle - \frac{\overline{\langle x,y \rangle}}{\langle y,y \rangle} \langle x,y \rangle - \frac{\langle x,y \rangle}{\langle y,y \rangle} \overline{\langle x,y \rangle} + \left| -\frac{\langle x,y \rangle}{\langle y,y \rangle} \right|^2 \langle y,y \rangle \\
&= \langle x,x \rangle - \frac{2\overline{\langle x,y \rangle}}{\langle y,y \rangle} \langle x,y \rangle + \frac{| \langle x,y \rangle |^2}{\langle y,y \rangle} \\
&= \langle x,x \rangle - \frac{2| \langle x,y \rangle |^2}{\langle y,y \rangle} + \frac{| \langle x,y \rangle |^2}{\langle y,y \rangle} \\
&= \langle x,x \rangle - \frac{| \langle x,y \rangle |^2}{\langle y,y \rangle}.
\end{aligned}$$

So we have that,

$$\begin{aligned}
0 &\leq \langle x,x \rangle - \frac{| \langle x,y \rangle |^2}{\langle y,y \rangle} \\
| \langle x,y \rangle |^2 &\leq \langle x,x \rangle \langle y,y \rangle \\
| \langle x,y \rangle | &\leq \sqrt{\langle x,x \rangle} \sqrt{\langle y,y \rangle} \\
| \langle x,y \rangle |^2 &\leq \langle x,x \rangle \langle y,y \rangle \quad \text{as desired.}
\end{aligned}$$

□

Proposition 2.0.3. *On a pre-hilbert space X, the function $\sqrt{\langle x,x \rangle}$ is a norm.*

Proof. We show that $\sqrt{\langle x,x \rangle}$ satisfies the axioms of Definition 2.

1. Since $\langle x,x\rangle$ is nonnegative, then clearly, $\sqrt{\langle x,x\rangle}$ is also nonnegative.

2. Assume $x = 0$, then clearly we have that $\sqrt{\langle x,x\rangle} = 0$. Conversely, if $\sqrt{\langle x,x\rangle} = 0$, then clearly, $x = 0$.

3. For $a \in \mathbb{C}$, we have that

$$\begin{aligned}\sqrt{\langle ax,ax\rangle} &= \sqrt{a\,\langle x,ax\rangle}\\ &= \sqrt{a\bar{a}\,\langle x,x\rangle}\\ &= \sqrt{|a|^2\,\langle x,x\rangle}\\ &= |a|\sqrt{\langle x,x\rangle}.\end{aligned}$$

4.

$$\begin{aligned}\langle x+y,x+y\rangle &= \langle x,x+y\rangle + \langle y,x+y\rangle\\ &= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle\\ &= \langle x,x\rangle + 2Re\,\langle x,y\rangle + \langle y,y\rangle\end{aligned}$$

By Lemma 2.0.2, we obtain

$$\begin{aligned}\langle x+y,x+y\rangle &\leq \langle x,x\rangle + 2Re|\,\langle x,y\rangle\,| + \langle y,y\rangle\\ &\leq \langle x,x\rangle + 2\sqrt{\langle x,x\rangle}\sqrt{\langle y,y\rangle} + \langle y,y\rangle\\ &= (\sqrt{\langle x,x\rangle} + \sqrt{\langle y,y\rangle})^2\end{aligned}$$

Taking the square roots on both sides, we obtain

$$\sqrt{\langle x+y,x+y\rangle} \leq \sqrt{\langle x,x\rangle} + \sqrt{\langle y,y\rangle} \quad \text{(sub-additivity of norm).}$$

This complete the proof. □

Proposition 2.0.4. *(The Parallelogram Law)*
In a Hilbert (or generally, any inner product) space H, we have the identity

$$||x+y||^2 + ||x-y||^2 = 2(||x||^2 + ||y||^2),$$

for all $x, y \in H$.

Proof. From Proposition 2.0.3, we have that,

$$
\begin{aligned}
||x+y||^2+||x-y||^2 &= \langle x+y,x+y\rangle+\langle x-y,x-y\rangle \\
&= \langle x,x+y\rangle+\langle y,x+y\rangle+\langle x,x-y\rangle-\langle y,x-y\rangle \\
&= \langle x,x\rangle+\langle x,y\rangle+\langle y,x\rangle+\langle y,y\rangle+\langle x,x\rangle-\langle x,y\rangle-\langle y,x\rangle+\langle y,y\rangle \\
&= 2\langle x,x\rangle+2\langle y,y\rangle \\
&= 2||x||^2+2||y||^2 \\
&= 2(||x||^2+||y||^2) \quad \text{as desired.}
\end{aligned}
$$

□

Proposition 2.0.5. *In a real Hilbert (or generally, any inner product) space H, we have the identity*

$$||x+y||^2-2\langle y,x+y\rangle=||x||^2-||y||^2,$$

for all $x,y\in H$

Proof. By Proposition 2.0.3, we have that

$$||x+y||^2-2\langle y,x+y\rangle=\langle x+y,x+y\rangle-2\langle y,x+y\rangle.$$

By Definition 3, the above equations becomes;

$$
\begin{aligned}
||x+y||^2-2\langle y,x+y\rangle &= \langle x+y,x+y\rangle-\langle 2y,x+y\rangle \\
&= \langle x+y,x\rangle+\langle x+y,y\rangle-\langle 2y,x\rangle-\langle 2y,y\rangle \\
&= \langle x,x\rangle+\langle y,x\rangle+\langle x,y\rangle+\langle y,y\rangle-2\langle y,x\rangle-2\langle y,y\rangle \\
&= ||x||^2+2\langle x,y\rangle-2\langle x,y\rangle+||y||^2-2||y||^2 \\
&= ||x||^2-||y||^2 \quad \text{as desired.}
\end{aligned}
$$

□

Definition 4. *A set K in a linear vector space is said to be* ***convex*** *if, given $x_1,x_2\in K$, all points of the form $\alpha x_1+(1-\alpha)x_2$ with $0\le\alpha\le 1$ are in K.*

Proposition 2.0.6. *Let τ be an arbitrary collection of convex sets. Then $\bigcap_{K\in\tau}K$ is convex.*

Proof. Let $C=\bigcap_{K\in\tau}K$. If C is empty, then the proposition is trivially proved. Let's assume that $x_1,x_2\in C$ and select α, $0\le\alpha\le 1$. Then $x_1,x_2\in K$ for all $K\in\tau$, and since K is convex, $\alpha x_1+(1-\alpha)x_2\in K$ for all $K\in\tau$. Thus, $\alpha x_1+(1-\alpha)x_2\in C$ and C is convex. □

Definition 5. *(Fixed point)*
Let $f:X\longrightarrow X$ be a map of metric space to itself. A point $x\in X$ is called a fixed point of f if $f(x)=x$.

Definition 6. *Let H be a real Hilbert Space and D be a subset of H. Then the mapping $T : D \longrightarrow D$ is*

- ***nonexpansive*** *if* $||Ty - Tx|| \leq ||y - x|| \quad \forall x, y \in D$.
  The set of fixed points of the nonexpansive operator T is given as $\text{Fix}(T) := \{x \in D | Tx = x\}$.

- ***quasi-nonexpansive*** *if* $||Ty - x|| \leq ||y - x|| \quad \forall y \in D \quad$ *and* $\quad x \in \text{Fix}(T)$.

Lemma 2.0.7. *All nonexpansive mappings with a fixed point are quasi-nonexpansive.*

Proof. From the definition of nonexpansive mapping, we have,

$$||Ty - Tx|| \leq ||y - x||.$$

Now if x is a fixed point of T, thus, $Tx = x$, then the above definition reduces to

$$||Ty - Tx|| = ||Ty - x|| \leq ||y - x|| \quad \text{as desired.}$$

□

Proposition 2.0.8. *For any nonexpansive self mapping $T : D \to D$, where D is a subset of a real inner product space such that* $\text{Fix}(T) \neq \{\emptyset\}$ *and for all* $(p, x) \in \text{Fix}(T) \times D$, *we have,*

$$||Tx - x||^2 \leq 2\langle x - Tx, x - p\rangle \quad \forall p \in \text{Fix}(T), \quad \forall x \in D.$$

Proof. For every non-expansive mapping T with $\text{Fix}(T) \neq \{\emptyset\}$, then from Lemma 2.0.7, we have,

$$\begin{aligned} ||x - p||^2 &\geq ||Tx - p||^2 \\ &= ||(Tx - x) + (x - p)||^2 \\ &= ||Tx - x||^2 + ||x - p||^2 + 2\langle Tx - x, x - p\rangle, \end{aligned}$$

so that

$$||x - p||^2 \geq ||Tx - x||^2 + ||x - p||^2 + 2\langle Tx - x, x - p\rangle.$$

Simplifying the above gives:

$$\begin{aligned} ||Tx - x||^2 + 2\langle Tx - x, x - p\rangle &\leq 0 \\ ||Tx - x||^2 &\leq -2\langle Tx - x, x - p\rangle \\ &\leq 2\langle x - Tx, x - p\rangle \quad \text{as desired.} \end{aligned}$$

□

Proposition 2.0.9. *In a real Hilbert Space H, a nonexpansive mapping T on a subset S of H is equivalently defined by the following inequality*

$$\langle Ty - Tx + y - x, Ty - Tx - y + x \rangle \leq 0,$$

for all $x, y \in S$.

Proof. Using the fact that for every nonexpansive mapping, $\| Ty - Tx \| \leq \| y - x \|$, then we have that,

$$\| Ty - Tx \|^2 - \| y - x \|^2 \leq 0.$$

let $a = Ty - Tx$ and $b = y - x$. Then we obtain:

$$\| a \|^2 - \| b \|^2 = \langle a, a \rangle - \langle b, b \rangle \leq 0. \tag{2.1}$$

Again, we have,

$$\langle a + b, a - b \rangle = \langle a, a \rangle - \langle a, b \rangle + \langle b, a \rangle - \langle b, b \rangle. \tag{2.2}$$

Since in a real Hilbert space, $\langle a, b \rangle = \langle b, a \rangle$, then (2.2) reduces to

$$\langle a + b, a - b \rangle = \langle a, a \rangle - \langle b, b \rangle \leq 0. \tag{2.3}$$

Since $a = Ty - Tx$ and $b = y - x$, then from (2.3), we have that,

$$\langle Ty - Tx + y - x, Ty - Tx - y + x \rangle \leq 0, \quad \text{as desired.}$$

□

Lemma 2.0.10. *If a nonexpansive mapping T is defined on a closed convex subset C of a real Hilbert space, thus* $T : C \longrightarrow H$*, then the set of fixed points of T,* $\text{Fix}(T) = \{x \in C : Tx = x\}$ *is also convex and closed.*

Proof. Let u and v be fixed points of T. Then we must show that $x = \alpha u + (1 - \alpha)v$ for $\alpha \in [0, 1)$ is a fixed point, that is, $Tx = x$.
By Proposition 2.0.9, we obtain,

$$\langle Tu - Tx + u - x, Tu - Tx - u + x \rangle \leq 0. \tag{2.4}$$

$$\langle Tv - Tx + v - x, Tv - Tx - v + x \rangle \leq 0. \tag{2.5}$$

Since u and v are fixed points of T, then (2.4) and (2.5) reduce to

$$\langle u - Tx + u - x, u - Tx - u + x \rangle = \langle 2u - (Tx + x), x - Tx \rangle \leq 0. \tag{2.6}$$

$$\langle v - Tx + v - x, v - Tx - v + x \rangle = \langle 2v - (Tx + x), x - Tx \rangle \leq 0. \tag{2.7}$$

Multiply (2.6) by α and (2.7) by $1-\alpha$ to obtain

$$\begin{aligned} \langle 2\alpha u - \alpha(Tx+x), x-Tx\rangle &\leq 0. && (2.8)\\ \langle 2(1-\alpha)v-(1-\alpha)(Tx+x), x-Tx\rangle &\leq 0. && (2.9)\end{aligned}$$

Adding (2.8) and (2.9) gives

$$\begin{aligned} \langle 2\alpha u + 2(1-\alpha)v - \alpha(Tx+x)-(1-\alpha)(Tx+x), x-Tx\rangle \leq 0\\ \langle 2x-Tx-x, x-Tx\rangle \leq 0\\ \langle x-Tx, x-Tx\rangle \leq 0.\end{aligned}$$

But by Definition 3, inner product are nonnegative, so we have,

$$0 \leq \langle x-Tx, x-Tx\rangle \leq 0,$$

so that by the squeeze theorem, we have $x-Tx=0$ which implies $Tx=x$, showing that the set of fixed points of T is convex.

Now we show that the set of fixed point of T is also closed. Let $\{z_n\}$ be a sequence of fixed points of T and let z be its limit. We show that z is a fixed point of T.
We have that

$$||z_n - Tz|| = ||Tz_n - Tz|| \leq ||z_n - z||.$$

This implies that,

$$\lim_{n\to\infty} ||z_n - Tz|| = 0 \quad \text{since} \quad ||z_n - z|| \to 0.$$

and consequently $z_n \to Tz$. But we have that $z_n \to z$ and so by the uniqueness of limit, we have that $Tz = z$ which shows that z is a fixed point of T, proving the fact that the set of fixed points of T is closed. □

Theorem 2.0.11. *(Minimizing Vector)*
(1) *Let H be a Hilbert Space and let $C \neq \{\emptyset\}$ be a closed convex subset of H. For every $x \in H$, there exist a unique $y \in C$ which is closest to x, in the sense that, $||\, x-y\, ||$ is minimised, that is*

$$d(x,C) = ||\, x-y\, || = \inf\{||\, x-z\, || : z \in C\}. \tag{2.10}$$

(2) *If the above C is a closed subspace Y of H, then $x-y$ is orthogonal to Y, thus, $(x-y)\perp Y$.*

Proof. **(1) Existence:**
If $x \in C$, then (2.10) holds with $y = x$. Otherwise if $x \notin C$, let $\{y_n\} \subset C$ be a minimizing sequence, that is, by the definition of infimum, there is sequence $\{y_n\}$ in C such that $||\, x-y_n\, || \to d(x,C)$. Now we show that y_n is Cauchy.

From Proposition 2.0.4, we have,

$$|| x+y ||^2 + || x-y ||^2 = 2\left(|| x ||^2 + || y ||^2\right) \Longrightarrow || x-y ||^2 = 2 || x ||^2 + 2 || y ||^2 - || x+y ||^2 .$$

Then we have,

$$\begin{aligned} || y_n - y_m ||^2 &= || (y_n - x) - (y_m - x) ||^2 \\ &= 2 || y_n - x ||^2 + 2 || y_m - x ||^2 - || (y_n - x) + (y_m - x) ||^2 \\ &= 2 || y_n - x ||^2 + 2 || y_m - x ||^2 - || y_n + y_m - 2x ||^2 \\ &= 2 || y_n - x ||^2 + 2 || y_m - x ||^2 - \left|\left| 2\left(\frac{y_n + y_m}{2} - x\right) \right|\right|^2 \\ &= 2 || y_n - x ||^2 + 2 || y_m - x ||^2 - 4\left|\left|\frac{y_n + y_m}{2} - x\right|\right|^2. \end{aligned}$$

Now since C is convex and the fact that every convex set contains it midpoint, it follows that $\frac{y_n + y_m}{2} \in C$ and consequently, $d(x,C) \leq \left|\left|\frac{y_n + y_m}{2} - x\right|\right|$ since $d(x,C)$ is the smallest distance between x and the set C.
Thus,

$$||y_n - y_m||^2 \leq 2 || y_n - x ||^2 + 2 || y_m - x ||^2 - 4d(x,C)^2.$$

As $n,m \longrightarrow \infty$, we obtain:

$$\begin{aligned} ||y_n - y_m||^2 &\leq 2d(x,C)^2 + 2d(x,C)^2 - 4d(x,C)^2 \\ ||y_n - y_m||^2 &\longrightarrow 0 \quad \text{as} \quad n,m \longrightarrow \infty. \end{aligned}$$

Hence $y_n \in C$ is a Cauchy sequence. Now since the set C is closed (complete), then y_n converges say $y_n \longrightarrow y \in C$. Since $y \in C$ and $x \notin C$, we have that $d(x,C) \leq ||x - y||$.
Also,

$$||x - y|| \leq ||x - y_n|| + ||y_n - y|| \longrightarrow d(x,C).$$

This shows that $||x - y|| = d(x,C)$.

Uniqueness:
Let suppose y and $\bar{y}$ are elements of C that satisfies (2.10). Then by the parallelogram law, we have

$$\begin{aligned} ||y - \bar{y}||^2 &= ||(y - x) - (\bar{y} - x)||^2 \\ &= 2||y - x||^2 + 2||\bar{y} - x||^2 - ||(y - x) + (\bar{y} - x)||^2 \\ &= 2d(x,C)^2 + 2d(x,C)^2 - 4\left|\left|\frac{y + \bar{y}}{2} - x\right|\right|^2. \end{aligned}$$

Since $\frac{y + \bar{y}}{2} \in C \Longrightarrow d(x,C) \leq \left|\left|\frac{y + \bar{y}}{2} - x\right|\right|$.
Thus,

$$||y-\bar{y}||^2 \leq 2d(x,C)^2 + 2d(x,C)^2 - 4d(x,C)^2 = 0.$$

Hence $y = \bar{y}$, proving the uniqueness of the element of C that satisfies (2.10).

(2) We prove by contradiction. Let assume $(x-y) \perp Y$ is not true. Then

$$\langle x-y, y_1 \rangle = \beta \neq 0 \quad \text{for some} \quad y_1 \in Y.$$

. Let $z = x - y$. Clearly, $y_1 \neq 0$ since otherwise $\langle z, y_1 \rangle = 0$. For any scalar α,

$$\begin{aligned}
||z-\alpha y_1||^2 &= \langle z-\alpha y_1, z-\alpha y_1 \rangle \\
&= \langle z, z-\alpha y_1 \rangle - \langle \alpha y_1, z-\alpha y_1 \rangle \\
&= \langle z, z \rangle - \bar{\alpha} \langle z, y_1 \rangle - \alpha[\langle y_1, z \rangle - \bar{\alpha} \langle y_1, y_1 \rangle] \\
&= \langle z, z \rangle - \bar{\alpha}\beta - \alpha[\bar{\beta} - \bar{\alpha} \langle y_1, y_1 \rangle] \quad \text{since} \quad \beta = \langle z, y_1 \rangle.
\end{aligned}$$

Take $\bar{\alpha} = \dfrac{\bar{\beta}}{\langle y_1, y_1 \rangle}$. Then the above equation becomes,

$$\begin{aligned}
||z-\alpha y_1||^2 &= \langle z, z \rangle - \frac{\bar{\beta}}{\langle y_1, y_1 \rangle} \cdot \beta \\
&= ||z||^2 - \frac{|\beta|^2}{\langle y_1, y_1 \rangle} \\
&< ||z||^2 \\
&= ||x-y||^2 \\
&= d(x,C)^2 \quad \text{from} \quad \textbf{(1)}.
\end{aligned}$$

But this is impossible because we have $z - \alpha y_1 = x - (y + \alpha y_1) = x - y_2$ where $y_2 = (y + \alpha y_1) \in Y$ because Y is a subspace and so $d(x,C) \leq ||z - \alpha y_1||$. Therefore, $||z - \alpha y_1|| < d(x,C)$ is a contraction and hence $(x-y) \perp Y$. □

Definition 7. *(weak and strong convergence)*
A sequence $\{x_n\}_{n\geq 1}$ weakly converges in a normed or Banach space X to a point $x \in X$ if for all $f \in X^*$, we have the inequality

$$\lim_{n\to\infty} f(x_n) = f(x),$$

where f is a continuous linear functional in the dual space X^* of X.

In a Hilbert space H, by the Riesz Representation Theorem, weak convergence of the sequence $\{x_n\}_{n\geq 1}$ to x is thus equivalent to the statement that

$$\langle x_n, y \rangle \longrightarrow \langle x, y \rangle \quad as \quad n \longrightarrow \infty,$$

for some unique $y \in H$.
It is denoted by $x_n \rightharpoonup x$ or $x_n \xrightarrow{w} x$.

On the otherhand, a sequence $\{x_n\}_{n\geq 1}$ in a normed or Banach space X converges strongly to a point $x \in X$ if

$$||x_n - x|| \longrightarrow 0 \quad as \quad n \longrightarrow \infty.$$

It is denoted by $x_n \longrightarrow x$.

Proposition 2.0.12. *Let X be a normed linear space. Suppose that $\{x_n\}_{n\geq 1} \subset X$ converges strongly to x with $x \in X$. Then for some unique $y \in X$, we have that $x_n \rightharpoonup x$. Thus, a strongly convergent sequence implies a weakly convergent sequence.*

Proof. Suppose that $x_n \longrightarrow x$, then we have that

$$\begin{aligned}
||x_n - y||^2 = \langle x_n - y, x_n - y\rangle &\longrightarrow ||x-y||^2 \\
||x_n||^2 + ||y||^2 - 2\langle x_n, y\rangle &\longrightarrow \langle x-y, x-y\rangle \\
||x||^2 + ||y||^2 - 2\langle x_n, y\rangle &\longrightarrow \langle x-y, x-y\rangle \\
\langle x, x\rangle + \langle y, y\rangle - 2\langle x_n, y\rangle &\longrightarrow \langle x-y, x-y\rangle .
\end{aligned}$$

Rearranging now gives us

$$\begin{aligned}
2\langle x_n, y\rangle &\longrightarrow \langle x, x\rangle + \langle y, y\rangle - \langle x-y, x-y\rangle \\
2\langle x_n, y\rangle &\longrightarrow 2\langle x, y\rangle \\
\langle x_n, y\rangle &\longrightarrow \langle x, y\rangle \quad \text{as desired.}
\end{aligned}$$

□

Remark 2.0.13. *The converse of Proposition 2.0.12 is not generally true when extended to an infinite dimensional space. Let's consider the sequence of unit vectors, $(e_n)_{n\in\mathbb{N}}$ in a Hilbert space H. Then for $x \in H$, we have*

$$\sum_n \Big| \langle e_n, x\rangle \Big|^2 \leq ||x||^2 \quad \textit{(Bessel inequality)},$$

where equality holds when $\{e_n\}$ is a Hilbert space basis. Since the series above converges, it corresponding sequence must go to zero. Therefore,

$$|\langle e_n, x\rangle|^2 \to 0.$$

That is,

$$\langle e_n, x\rangle \to 0.$$

which implies $e_n \xrightarrow{w} 0$. *However,* $(e_n)_{n\in\mathbb{N}}$ *doesn't converge strongly to 0 because* $||e_n - 0|| = 1$ *for all* $n \in \mathbb{N}$.

Definition 8. *A subset M of a metric space X is said to be*

1. ***dense*** *in X if for any point x in X, any neighbourhood of x contains at least one point from M. Thus, M has non-empty intersection with every non-empty open subset of X. Equivalently, the subset M of X is said to be dense in X if the closure of M is X.*

2. ***nowhere dense*** *in X if its closure* $\overline{M}$ *has no interior points (*$\overline{M}$ *does not contain an open ball).*

3. ***of the first category*** *in X if M is the union of countably many nowhere dense sets in X. That is,*

$$M = \bigcup_{k=1}^{\infty} A_k, \quad A_k \quad \textit{nowhere dense in} \quad X.$$

4. ***of the second category*** *in X if M is not of the first category in X.*

Theorem 2.0.14. *(Baire Category Theorem)*
If a metric space $X \neq \{\emptyset\}$ *is complete, then is of the second category in itself. Equivalently, if* $X \neq \{\emptyset\}$ *is complete and* $X = \bigcup_{k=1}^{\infty} A_k$, A_k *closed, then at least one* A_k *contains a non-empty open set (or open ball).*

Proof. Suppose that the complete metric space $X \neq \{\emptyset\}$ is of the first category. Then

$$X = \bigcup_{k=1}^{\infty} M_k, \quad \text{where} \quad M_k \quad \text{nowhere dense in} \quad X$$

We construct a Cauchy sequence p_k.

- M_1 is nowhere dense implies that, it closure, $\overline{M_1}$ does not contain a non-empty open ball but X contains so $\overline{M}_1 \neq X$. This implies it complement must be open. That is $\overline{M}_1^c = X \setminus \overline{M}_1$ is non-empty and open. So $\exists p_1 \in \overline{M}_1^c$ and an open ball $B_1 = B_1(p_1, \varepsilon_1) \subset \overline{M}_1^c, \quad \varepsilon_1 < \frac{1}{2}$.

- M_2 is nowhere dense implies that, it closure $\overline{M}_2$ does not contain an open ball. This also implies $\overline{M}_2$ does not contain the open ball $B(p_1, \frac{\varepsilon_1}{2})$ and this implies $\overline{M}_2^c \cap B(p_1, \frac{\varepsilon_1}{2})$ is non-empty and open and as a result we may choose an open ball in this set $B_2 = B(p_2, \varepsilon_2) \subset \left(\overline{M}_2^c \cap B(p_1, \frac{\varepsilon_1}{2})\right), \quad \varepsilon_2 < \frac{\varepsilon_1}{2}$

- By induction we obtain a sequence of balls;

$$B_k = B(p_k, \varepsilon_k), \quad \varepsilon_k < \frac{1}{2^k}.$$

  such that

$$B_k \cap M_k = \emptyset \quad \text{and} \quad B_{k+1} \subset B(p_k, \frac{\varepsilon_k}{2}) \subset B_k, \quad k = 1, 2, \ldots.$$

- Since $\varepsilon_k < \frac{1}{2^k}$, the sequence (p_k) of centres is Cauchy. Thus,

$$\text{for} \quad n > m \Longrightarrow d(p_n, p_m) < \frac{\varepsilon_m}{2} = \frac{1}{2}\varepsilon_m < \frac{1}{2} \cdot \frac{1}{2^m} = \frac{1}{2^{m+1}} \longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty.$$

  Since X is complete then $p_k \longrightarrow p \in X$.

- Also for $n > m$ we have $B_n \subset B(p_m, \frac{\varepsilon_m}{2})$ so that

$$d(p_m, p) \leq d(p_m, p_n) + d(p_n, p) < \frac{\varepsilon_m}{2} + d(p_n, p) \longrightarrow \frac{\varepsilon_m}{2} \quad \text{as} \quad n \to \infty.$$

  Now this implies that $p \in B_m \quad \forall m = 1, 2, \ldots$
  Since $B_m \subset \overline{M}_m^c$, then $p \notin M_m$ since $B_m \cap M_m = \{\emptyset\}$. But we also have that $p \in X$ which contradict $p \notin M_m$. Hence $X \neq \{\emptyset\}$ is of the second category.

□

Theorem 2.0.15. *(Uniform boundedness principle)*
Let $\{T_i\}$ be a family of continuous linear operators from Banach Space X to a normed linear space Y. If for all $x \in X$, there is an $M_x \geq 0$ for which $||T_i(x)|| \leq M_x$ for all T_i, then there is a $K \geq 0$, such that for all T_i, we have

$$||T_i|| \leq K.$$

Proof. Suppose that for every x in the Banach Space X, one has:

$$||T_i(x)|| \leq M_x.$$

For every integer $n \in \mathbb{N}$, let

$$A_n = \{x \in X : ||T_i(x)|| \leq n\}.$$

The set A_n is a closed set since $T_i^{'s}$ are continuous. Then by the assumption that X is complete, we have that,

$$\bigcup_{n\in\mathbb{N}} A_n = X \neq \{\emptyset\}.$$

Then from Theorem 2.0.14, the interior of A_{n_0} is non-empty for some $n_0 \geq 1$. That is for some $x_0 \in X$ and $r > 0$, we have $B(x_0, r) \subset A_{n_0}$. The set of points in $B(x_0, r)$ is equivalent to the set of points of the form $x_0 + zr$ where $z \in B(0,1)$ such that

$$||T_i(x_0 + rz)|| \leq n_0.$$

So for $z \in X$ with $||z|| \leq 1$, we have that,

$$\begin{aligned} ||T_i(z)|| &= \frac{1}{r}||T_i(x_0 + rz) - T_i(x_0)|| \quad \text{by the linearity of} \quad T_i \\ &\leq \frac{1}{r}(||T_i(x_0 + rz)|| + ||T_i(x_0)||) \\ &\leq \frac{1}{r}(n_0 + ||T_i(x_0)||) \\ &\leq \frac{n_0}{r} + M_{x_0}. \end{aligned}$$

Taking the supremum over z in the unit ball of X, we have,

$$\sup_{||z||\leq 1} ||T_i(z)|| = ||T_i|| \leq \frac{n_0}{r} + M_{x_0} = K.$$

That ends the proof. □

Proposition 2.0.16. *Every weakly convergent sequence in a Hilbert Space is bounded.*

Proof. Let l^2 be the set of real number sequences $\{a_n\}$ such that $\sum a_n^2 < \infty$. Let $\langle a_n, y\rangle \to \langle a, y\rangle$ for some $a \in l^2$ and for all $y \in l^2$. We want to show that $||a_n||$ is bounded. Now for each $n \in \mathbb{N}$, we define linear operator (in fact linear functional)

$$T_n : l_2 \longrightarrow \mathbb{C} : y \mapsto \langle y, a_n\rangle .$$

Thus,

$$T_n(y) = \langle y, a_n\rangle .$$

By Lemma 2.0.2, it follows that

$$|T_n(y)| = |\langle y, a_n\rangle| \leq ||a_n||||y||.$$

But we have that,

$$||T_n|| = \sup_{0\neq y\in l^2} \frac{|T_n(y)|}{||y||} \leq \sup_{0\neq y\in l^2} \frac{||a_n||||y||}{||y||}.$$

so that

$$||T_n|| \leq ||a_n||. \tag{2.11}$$

On the otherhand, we have that,

$$|T_n(a_n)| = \langle a_n, a_n \rangle = ||a_n||^2.$$

But we also have that,

$$||T_n|| = \sup_{0 \neq a_n \in l^2} \frac{|T_n(a_n)|}{||a_n||} \geq \frac{|T_n(a_n)|}{||a_n||} = \frac{||a_n||^2}{||a_n||},$$

so that

$$||T_n|| \geq ||a_n||. \tag{2.12}$$

From (2.11) and (2.12), it follows that

$$||T_n|| = ||a_n||. \tag{2.13}$$

Now for $y \in l_2$, then by the our assumption, we have that,

$$\lim_{n\to\infty} T_n(y) = \lim_{n\to\infty} \langle y, a_n \rangle = \langle a, y \rangle \quad \text{as} \quad n \to \infty.$$

This means that the sequence $\{T_n(y) : n \in \mathbb{N}\}$ is convergent and as the consequence is bounded. Since it is bounded, then for each $y \in l_2$, we have

$$\sup\{|T_n(y)| : n \in \mathbb{N}\} < +\infty.$$

Then from Theorem 2.0.15, it follows that,

$$\sup\{||T_n|| : n \in \mathbb{N}\} < +\infty.$$

So using (2.13), we conclude that $\{||a_n|| : n \in \mathbb{N}\}$ is bounded. □

Lemma 2.0.17. *If $x_n \rightharpoonup x$ and $y_n \longrightarrow y$, then in real Hilbert space, we have*

$$\langle x_n, y_n \rangle \longrightarrow \langle x, y \rangle .$$

Proof. We want to show that $|\langle x_n, y_n \rangle - \langle x, y \rangle| \longrightarrow 0$.

$$
\begin{aligned}
|\langle x_n, y_n\rangle - \langle x, y\rangle| &= |\langle x_n, y_n\rangle - \langle x_n, y\rangle + \langle x_n, y\rangle - \langle x, y\rangle| \\
&= |\langle x_n, y_n - y\rangle + \langle x_n - x, y\rangle| \\
&\leq |\langle x_n, y_n - y\rangle| + |\langle x_n - x, y\rangle| \quad \text{(triangle inequality).}
\end{aligned}
$$

By Lemma 2.0.2, we obtain,

$$|\langle x_n, y_n\rangle - \langle x, y\rangle| \leq ||x_n|| ||y_n - y|| + |\langle x_n, y\rangle - \langle x, y\rangle|.$$

Since $x_n \rightharpoonup x$, then from Proposition 2.0.16, we have that $\sup||x_n|| = D < \infty$, and also $y_n \longrightarrow y \Longrightarrow ||y_n - y|| \longrightarrow 0$ as $n \longrightarrow \infty$. So we have,

$$
\begin{aligned}
|\langle x_n, y_n\rangle - \langle x, y\rangle| &\leq D||y_n - y|| + |\langle x_n, y\rangle - \langle x, y\rangle| \\
&\longrightarrow 0 + 0 \quad \text{as} \quad n \longrightarrow \infty.
\end{aligned}
$$

Hence the proof is complete. □

Theorem 2.0.18. *(Demiclosedness principle)*
Suppose T is a nonexpansive mapping defined on a weakly closed subset S of a real Hilbert space H. Then if $x_n - Tx_n \longrightarrow 0$ but $x_n \rightharpoonup x$, then $Tx = x$, thus x is a fixed point of T.

Proof. We know that a nonexpansive mapping in a real Hilbert space is equivalent to (from Proposition 2.0.9),

$$\langle Ty - Tx + y - x, Ty - Tx - y + x\rangle \leq 0.$$

Replacing y by x_n, we obtain,

$$\langle Tx_n - Tx + x_n - x, Tx_n - Tx - x_n + x\rangle \leq 0.$$

Now since $x_n \rightharpoonup x$ and $x_n - Tx_n \longrightarrow 0$, then we have that $Tx_n \rightharpoonup x$ since $x_n - Tx_n \rightharpoonup 0$ by Proposition 2.0.12. So we have the following:

$$\langle (Tx_n + x_n) - (Tx + x), (Tx_n - x_n) - (Tx - x)\rangle \leq 0.$$

Since the term $Tx + x$ is fixed, we have that $(Tx_n + x_n) - (Tx + x)$ is weak likewise $(Tx_n - x_n) - (Tx - x)$ is strong since $(Tx - x)$ is also fixed.
Hence as $n \longrightarrow \infty$, then by Lemma 2.0.17, we have,

$$\langle x - Tx, x - Tx\rangle \leq 0.$$

Since by Definition 3, inner product are nonnegative, we have that

$$0 \leq \langle x - Tx, x - Tx\rangle \leq 0,$$

which implies

$$x - Tx = 0 \Longrightarrow Tx = x,$$

showing that x is a fixed point of T. $\square$

Definition 9. *(Countable Set)*
A countable set is an infinite set that has a bijection with the natural numbers. That is, we assign to each element of the set, a unique natural number, generally starting from 1 and proceeding upwards. So we have that, the set of natural numbers by definition, are countable since each natural number corresponds to itself.

Lemma 2.0.19. *The union of countably many countable sets is a countable set.*

Proof. Let the countably many sets be denoted by

$$S_1, S_2, S_3, \ldots.$$

We assume that there is a countable infinite number of sets and that each set itself is countably infinite. Consider the union

$$S = \bigcup_{i=1}^{\infty} S_i = S_1 \cup S_2 \cup S_3 \cup \ldots.$$

of these sets. To show that it is countable, we must put it into a bijection with $\mathbb{N}$.
Suppose a set S_i has elements

$$a_{i1}, a_{i2}, a_{i3}, \ldots.$$

Then we can arrange the elements of the union of the sets in a doubly-infinite array:

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{14} & \cdots \\ a_{21} & a_{22} & a_{23} & a_{24} & \cdots \\ a_{31} & a_{32} & a_{33} & a_{34} & \cdots \\ a_{41} & a_{42} & a_{43} & a_{44} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

We create a bijection by counting each element as we follow the arrows:

$$\begin{array}{ccccccc} a_{11} & \rightarrow & a_{12} & & a_{13} & \rightarrow & a_{14} \\ & \swarrow & & \nearrow & & \swarrow & \\ a_{21} & & a_{22} & & a_{23} & & \\ \downarrow & \nearrow & & \swarrow & & & \\ a_{31} & & a_{32} & & & & \\ & \swarrow & & & & & \\ a_{41} & & & & & & \\ \vdots & & & & & & \end{array}$$

In other words, we have the pairing:

$$\begin{array}{cccccccc} a_{11} & a_{12} & a_{21} & a_{31} & a_{22} & a_{13} & a_{14} & \dots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & \dots \end{array}$$

□

Lemma 2.0.20. *The set $\mathbb{Z}$ of integers is countable.*

Proof. We exhibit $\mathbb{Z}$ as a countable union of countable sets.

$$\mathbb{Z} = \{1,2,3,\dots\} \cup \{0\} \cup \{-1,-2,-3,\dots\}.$$

Hence by Lemma 2.0.19, we have that the set $\mathbb{Z}$ of integers is countable. □

Lemma 2.0.21. *The set of rational numbers is countable.*

Proof. Let Q be the set of rational numbers. For n a positive integer, let Q_n be the set of those rational numbers expressible with the denominator n. Thus Q_n is the set of all numbers $\frac{m}{n}, m$ ranging over all integers coprime to n. Hence we have that each of the sets Q_n is clearly in bijection with set of integers which is countable set. Then by Lemma 2.0.20, Q_n is countable. Since Q is the union of the sets $Q_n, n = 1,2,3,\dots$, it follows from lemma 2.0.19 that Q is countable. □

Definition 10. *(Denseness)*
A set D is said to be dense in a normed linear space X if for each element $x \in X$ and $\varepsilon > 0$, there exist $d \in D$ with $||x-d|| < \varepsilon$. That is, if D is dense in X, there are points of D arbitrarily close to each $x \in X$. Thus given x, a sequence can be constructed from D which converges to x.

Definition 11. *(Separability)*
A normed linear space is separable if it contains a countable dense set.

Lemma 2.0.22. *The space l^2, thus the space of square summable is a separable space.*

Proof. Let $L \subseteq l^2$ the set of square summable sequence of rational numbers. That is,

$$L = \{(q_n)_{n\geq 1}, q_n \in Q, \sum_{n\geq 1} q_n^2 < \infty\}.$$

Let fix a basis $e_1, e_2, \dots$ in L. let $\bigoplus_{n\geq 1}$ denote the direct sum symbol. Then we have

$$
\begin{aligned}
L &= q_1e_1+q_2e_2+q_3e_3+\dots \\
L &= \bigoplus_{n\geq 1} Q.
\end{aligned}
$$

This implies that L is the countable union of countable sets and so by Lemma 2.0.19, L is a countable set.
It remains to show that L is dense in l^2. To do this, let $x \in l^2$ where

$$x = (S_n)_{n\geq 1} \quad \text{where} \quad S_n \in \mathbb{R}.$$

Then for each n, let $r_n^{(m)}$ (sequence) be a fraction to the mth decimal place of S_n. Then let's define the sequence

$$x_m = \left(r_n^{(m)}\right)_{m\geq 1} \in L.$$

Let also fix the basis $e_1, e_2, \dots e_n$ in x_m. Then,

$$
\begin{aligned}
\lim_{m\to\infty} x_m &= \lim_{m\to\infty} \sum_{n\geq 1} r_n^{(m)} e_n \\
&= \sum_{n\geq 1} \lim_{m\to\infty} r_n^{(m)} e_n \\
&= \sum_{n\geq 1} S_n e_n \\
&= (S_n)_{n\geq 1} \\
&= x.
\end{aligned}
$$

Therefore, the space l^2 is a separable space. □

Theorem 2.0.23. *The closed linear span of every sequence in a Hilbert space is separable.*

Proof. Let $\{x_n\}_{n\geq 1}$ be the sequence. If the vectors x_n are orthogonal, then

$$
\begin{aligned}
\overline{\text{span}}(x_n) &= \sum_{n\geq 1} r_n x_n, \quad r_n \in \mathbb{R} \\
&= \sum_{n\geq 1} r_n ||x_n|| \cdot \frac{x_n}{||x_n||}.
\end{aligned}
$$

Since $\frac{x_n}{||x_n||}$ has a unit norm and the fact that all the x_n are orthogonal, then we can identify all the $\frac{x_n}{||x_n||}$ as an orthonormal basis vectors for the space and as a result the above now reduces to

$$\begin{aligned}\overline{\text{span}}(x_n) &= r_n||x_n||\cdot e_n \\ &= r_n^{'}\cdot e_n \quad \text{where} \quad r_n^{'} = r_n||x_n||.\end{aligned}$$

Since the basis vectors, $e_n = \dfrac{x_n}{||x_n||}$ are countable, then we have

$$\overline{\text{span}}(x_n) \cong l^2.$$

Therefore, by Lemma 2.0.22, we have that $\overline{\text{span}}(x_n)$ is separable.
If $\{x_n\}$ are not orthogonal, then by the Gram-Schmidt orthogonalisation theorem, we can find an orthogonal sequence $\{x_n\}_{n\geq 1}$ (possibly smaller in cardinality to $\{x_n\}_{n\geq 1}$) such that

$$\overline{\text{span}}(x_n) = \overline{\text{span}}(x_n^{'}) \subseteq l^2.$$

Then by Lemma 2.0.22, we have that $\overline{\text{span}}(x_n)$ is separable. □

Definition 12. *A set S in an inner product space E is called an orthogonal set if* $\langle x,y\rangle = 0 \quad \forall x,y \in S, x \neq 0, y \neq 0$. *The set S is orthonormal if it is an orthogonal set and* $||x|| = 1 \quad \forall x \in S$.

Lemma 2.0.24. *(Pythagorean Theorem) In a Hilbert (or, generally, any inner product) space H, we have the identity*

$$||x+y||^2 = ||x||^2 + ||y||^2,$$

provided x is orthogonal to y, that is, $\langle x,y\rangle = 0$.

Proof.

$$\begin{aligned}||x+y||^2 &= \langle x+y, x+y\rangle \\ &= \langle x, x+y\rangle + \langle y, x+y\rangle \\ &= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle \\ &= ||x||^2 + ||y||^2.\end{aligned}$$

which gives the Pythagorean Theorem. □

Definition 13. *Consider a Hilbert space H and a given subset S of H. The set of all vectors orthogonal to S is denoted by* $S^{\perp}$, *that is orthogonal complement of S and is defined by*

$$S^{\perp} = \{x \in H, \langle x,s\rangle = 0 \quad \forall s \in S\}.$$

Lemma 2.0.25. $S^{\perp}$ *is a closed vector subspace of H.*

Proof. We show that $S^{\perp}$ is a subspace. Let $s_1, s_2 \in S^{\perp}, \alpha, \beta$ scalars and $z \in S$. Then $\langle \alpha s_1 + \beta s_2, z\rangle = \alpha\langle s_1, z\rangle + \beta\langle s_2, z\rangle = 0 \quad \forall z \in S$. Hence, $\alpha s_1 + \beta s_2 \in S^{\perp}$. Therefore, $S^{\perp}$ is a subspace of H.

We now show that $S^{\perp}$ is closed. Let $\{s_n\}_{n=1}^{\infty}$ be a sequence in $S^{\perp}$ such that $s_n \to s$ as $n \to \infty$. We show that $s \in S^{\perp}$. For all $z \in S$, we have

$$\begin{aligned} |\langle s,z\rangle| &= |\langle (s-s_n)+s_n,z\rangle| \\ &= |\langle s-s_n,z\rangle + \langle s_n,z\rangle| \\ &= |\langle s-s_n,z\rangle| \quad \text{since} \quad \langle s_n,z\rangle = 0, s_n \in S^{\perp}, z \in S. \end{aligned}$$

Applying Lemma 2.0.2 to the above equation, we obtain

$$\begin{aligned} |\langle s,z\rangle| &\leq ||s-s_n||||z|| \\ &\to 0 \quad \text{since} \quad s_n \to s. \end{aligned}$$

This implies that $\langle s,z\rangle = 0 \quad \forall z \in S$. Hence $s \in S^{\perp}$. Therefore, $S^{\perp}$ is a closed subspace of H. □

Definition 14. *Let E be a vector space. Then E is said to be the direct sum of two subspaces M and N of E, written*

$$E = M \oplus N,$$

if each $x \in E$ can be represented uniquely as $x = m + n$ where $m \in M$ and $n \in N$.

Theorem 2.0.26. *Let Y be a closed subspace of Hilbert space H. Then*

$$H = Y \oplus Y^{\perp}.$$

Proof. Let $x \in H$, be arbitrary. Then by (1) of Theorem 2.0.11, there exist a unique vector $y \in Y$ such that

$$||x-y|| \leq ||x-z|| \quad \forall z \in Y.$$

Also by (2) of Theorem 2.0.11, we have that $(x-y) \in Y^{\perp}$. So let $n = x-y \in Y^{\perp}$. But

$$x = y + (x-y) = y+n, \quad y \in Y \quad \text{and} \quad n \in Y^{\perp}.$$

Then by Definition 14, we show that the representation $x = y+n$ is unique. Let suppose that

$$x = y_1 + n_1, \quad y_1 \in Y \quad \text{and} \quad n_1 \in Y^{\perp},$$

is another representation of x. Then we have that,

$$y+n = y_1+n_1,$$

so that

$$(y_1 - y) + (n_1 - n) = 0, \tag{2.14}$$

where $y_1 - y \in Y \quad \text{and} \quad n_1 - n \in Y^{\perp}$.
Hence,

$$(y_1 - y) \perp (n_1 - n). \tag{2.15}$$

Hence by Lemma 2.0.24, we have that

$$0 = ||(y_1 - y) + (n_1 - n)||^2 = ||y_1 - y||^2 + ||n_1 - n||^2,$$

implying that

$$||y_1 - y|| = 0 \quad \text{and} \quad ||n_1 - n|| = 0.$$

Hence,

$$y_1 = y \quad \text{and} \quad n_1 = n.$$

establishing the uniqueness of the representation and hence proving Theorem 2.0.26. □

Theorem 2.0.27. *(Schechter, 1971)*
Let H be a Hilbert space and let H^* denote it dual space consisting of all continuous (bounded) linear functionals from H into field $\mathbb{R}$ or $\mathbb{C}$. If x (unique) is an element of H, then the bounded linear functional f, for all y in H defined by

$$f(y) = \langle y, x \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of the Hilbert space is an element of H^*. That is, the theorem states that every element of H^* can be written uniquely in this form. Moreover, $||f|| = ||x||$.

Proof. Let $Y = \{x \in H : f(y) = 0\}$. Then from Definition 13, Y is a closed subspace of H. Since by definition, the closed subspace Y is nonempty, then we have that $Y = \{0\}$. By Theorem 2.0.26 of direct sum, we have

$$H = Y \oplus Y^{\perp}, \quad Y \cap Y^{\perp} = \{0\}.$$

(a) If $Y = H$, then we are done since we can take $x \in H$ as $x = 0$ and $f(y) = \langle y, 0 \rangle = 0$ is satisfied.
(b) If $Y \neq H$, let $z \in Y^{\perp}$ arbitrary. Then,

$$f(z) \neq 0.$$

Let $y \in H$ be arbitrary and let

$$u = y - \frac{f(y)}{f(z)} z. \tag{2.16}$$

Now applying f on (2.16), we obtain

$$\begin{aligned} f(u) &= f(y) - \frac{f(y)}{f(z)} \cdot f(z) \\ f(u) &= f(y) - f(y) = 0, \end{aligned}$$

implying that $u \in Y$. So we have that $\langle u, z \rangle = 0$. Next, we take the inner product of (2.16) with z to obtain

$$\begin{aligned} 0 = \langle u, z \rangle &= \left\langle y - \frac{f(y)}{f(z)} z, z \right\rangle \\ &= \langle y, z \rangle - \frac{f(y)}{f(z)} \langle z, z \rangle . \end{aligned}$$

This implies that,

$$\begin{aligned} \langle y, z \rangle &= \frac{f(y)}{f(z)} \langle z, z \rangle \\ f(y) &= \frac{f(z)}{\langle z, z \rangle} \langle y, z \rangle = \left\langle y, \overline{\frac{f(z)}{\langle z, z \rangle}} z \right\rangle . \end{aligned}$$

where

$$x = \overline{\left(\frac{f(z)}{\langle z, z \rangle} \right) z} = \frac{\overline{f(z)}}{\langle z, z \rangle} \cdot z, \quad z \in Y^{\perp} \quad \text{is arbitrary.}$$

This shows that x is independent of the choice of $z \in Y^{\perp}$,

$$x = \frac{\overline{f(z)}}{\langle z, z \rangle} \cdot z.$$

Uniqueness: Suppose that

$$f(y) = \langle y, z \rangle = \langle y, z_1 \rangle \quad \forall y \in H.$$

We show that $z = z_1$. But we have that, Then we have that

$$\begin{aligned} \langle y, z \rangle - \langle y, z_1 \rangle &= 0 \\ \langle y, z - z_1 \rangle &= 0 \quad \forall y \in H. \end{aligned}$$

Now for special $y = z - z_1$, we have

$$\begin{aligned} \langle z - z_1, z - z_1 \rangle &= 0 \\ ||z - z_1||^2 &= 0 \Rightarrow z = z_1 \quad \text{as desired.} \end{aligned}$$

Now we show that $||f|| = ||x||$.
Given $f(y) = \langle y, x \rangle \Rightarrow |f(y)| = |\langle y, x \rangle| \leq ||y||||x||$ by Lemma 2.0.2. But we have that

$$||f|| = \sup_{0 \neq y \in H} \frac{|f(y)|}{||y||} \leq \frac{||y||||x||}{||y||},$$

so that

$$||f|| \leq ||x||. \tag{2.17}$$

Again we have that $||x||^2 = \langle x, x \rangle = f(x) \leq |f(x)|$.
But we also have,

$$||f|| = \sup_{0 \neq x \in H} \frac{|f(x)|}{||x||} \geq \frac{||x||^2}{||x||} = ||x||,$$

so that

$$||f|| \geq ||x||. \tag{2.18}$$

By (2.17) and (2.18), we have that $||f|| = ||x||$ as desired. □

Lemma 2.0.28. *All bounded monotone sequence converge.*

Proof. Let (a_n) be a bounded nondecreasing sequence. Let S denote the set $\{a_n : n \in \mathbb{N}\}$. Since (a_n) is bounded, then its supremum exist so we let $b = \sup S$. Again, since $b = \sup S$, then for $\varepsilon > 0$, $b - \varepsilon$ cannot be an upper bound for S. This implies that there is a corresponding N such that $a_N > b - \varepsilon$. Since we have that a_n is nondecreasing, for all $n > N$, $a_n > b - \varepsilon$. But (a_n) is bounded, so we have $b - \varepsilon < a_n \leq b$. But this implies that $|a_n - b| < \varepsilon$. So $\lim\limits_{n \to \infty} a_n = b$ which implies a_n converge. □

Lemma 2.0.29. *Every sequence has a monotonic subsequence.*

Proof. First, the nth term of a sequence is dominant if it is greater than every term following it. So for the proof, we note that a sequence (a_n) may have finitely many or infinitely many dominant terms. Now, let suppose that (a_n) has infinitely many dominant terms. Form a subsequence (a_{n_k}) solely of dominant terms of (a_n). Then we have that $a_{n_{k+1}} < a_{n_k}k$ by definition of "dominant". Hence (a_{n_k}) is a decreasing (monotone) subsequence of (a_n).
For the second case, let's assume that our sequence (a_n) has only finitely many dominant

terms. Select n_1 such that n_1 is beyond the last dominant term. But since n_1 is not dominant, there must be some $m > n_1$ such that $a_m > a_{n_1}$. Select this m and call it n_2. However, n_2 is still not dominant, so there must be an $n_3 > n_2$ with $a_{n_3} > a_{n_2}$ and so on, inductively. The resulting sequence

$$a_1, a_2, a_3, \dots.$$

is monotonic (non-decreasing). This complete the proof. □

Theorem 2.0.30. *(Bolzano and Berg, 1979)*
Every bounded sequence of real numbers has a convergent subsequence.

Proof. By Lemma 2.0.29, it has a monotonic subsequence. Again by Lemma 2.0.28, the subsequence converges. □

Theorem 2.0.31. *(Royden et al., 1988)*
Let X be a separable normed space and $\{T_n\}$ a sequence in its dual space X^* that is bounded, that is, there is an $M \geq 0$ for which

$$|T_n(f)| \leq M \cdot ||f|| \quad \forall f \in X \quad \textit{and all} \quad n. \tag{2.19}$$

Then there is a subsequence $\{T_{n_k}\}$ of $\{T_n\}$ and T in X^* for which

$$\lim_{k \to \infty} T_{n_k}(f) = T(f) \quad \forall f \in X.$$

Proof. Let $\{f_j\}_{j=1}^{\infty}$ be a countable subset of X that is dense in X. We infer from (2.19), that the sequence of real numbers $\{T_n(f_1)\}$ is bounded. Then by Theorem 2.0.30(Bolzano Weierstrass), there is a strictly increasing sequence of integers $\{s(1,n)\}$ and a_1 for which

$$\lim_{n \to \infty} T_{s(1,n)}(f_1) = a_1.$$

Again, from (2.19), we conclude that the sequence of real numbers $T_{s(1,n)}(f_2)$ is bounded and again by Theorem 2.0.30, there is a subsequence $\{s(2,n)\}$ of $\{s(1,n)\}$ and a number a_2 for which

$$\lim_{n \to \infty} T_{s(2,n)}(f_2) = a_2 \quad \forall j.$$

We inductively continue this process or selection to obtain a countable collection of strictly increasing sequences of natural numbers $\{\{s(j,n)\}\}_{j=1}^{\infty}$ and a sequence of real numbers $\{a_j\}$ such that for each j, $\{s(j+1),n\}$ is a subsequence of $\{s(j,n)\}$ and

$$\lim_{n\to\infty} T_{s(j,n)}(f_j) = a_j.$$

For each index k, define $n_k = s(k,k)$. Then for each j, $\{n_k\}_{j=k}^{\infty}$ is a subsequence of $\{s(j,k)\}$ and hence, we have

$$\lim_{k\to\infty} T_{n_k}(f_j) = a_j.$$

Since every convergent sequence is Cauchy, then the above implies that $\{T_{n_k}(f)\}$ is a Cauchy sequence for each f in a dense subset of X. That is, $|T_{n_k}(f_j) - T_{n_l}(f_j)| < \varepsilon_0$ for all $k,l \geq N(\varepsilon_0)$. Now since T_{n_k} is bounded in X^*, thus, $|T_{n_k}(f)| \leq M \cdot ||f||$, and that fact that $\{T_{n_k}(f)\}$ is Cauchy for each f in a dense subset of X, then we claim that $\{T_{n_k}(f)\}$ is Cauchy for all f in X. That is, $|T_{n_k}(f) - T_{n_l}(f)| < \varepsilon$ for all $k,l \geq N(\varepsilon)$ and for all $f \in X$. Now we prove this as follows:

$$\begin{aligned}
|T_{n_k}(f) - T_{n_l}(f)| &= |T_{n_k}(f - f_j + f_j) - T_{n_l}(f - f_j + f_j)| \\
&= |T_{n_k}(f_j) - T_{n_l}(f_j) + T_{n_k}(f - f_j) - T_{n_l}(f - f_j)| \\
&\leq |T_{n_k}(f_j) - T_{n_l}(f_j)| + |T_{n_k}f - (f_j) - T_{n_l}(f - f_j)| \\
&\leq |T_{n_k}(f_j) - T_{n_l}(f_j)| + |T_{n_k}(f - f_j)| + |T_{n_l}(f - f_j)|
\end{aligned}$$

Now since $\{T_{n_k}(f)\}$ is Cauchy for each f in a dense subset of X, the above becomes

$$|T_{n_k}(f) - T_{n_l}(f)| \leq \varepsilon_0 + |T_{n_k}(f - f_j)| + |T_{n_l}(f - f_j)|$$

Again using the fact that T_{n_k} is bounded in X^*, then the above becomes

$$\begin{aligned}
|T_{n_k}(f) - T_{n_l}(f)| &\leq \varepsilon_0 + M||f - f_j|| + M||f - f_j|| \quad \text{for some } M \geq 0 \\
&= \varepsilon_0 + 2M||f - f_j||
\end{aligned}$$

But since the f_j are dense subset of X, then we can write $||f - f_j|| < \varepsilon_1$ and as a result the above becomes

$$\begin{aligned}
|T_{n_k}(f) - T_{n_l}(f)| &\leq \varepsilon_0 + 2M\varepsilon_1 \\
&= \varepsilon
\end{aligned}$$

Hence $\{T_{n_k}(f)\}$ is Cauchy for all f in X. Since real numbers are complete, we may define

$$T(f) = \lim_{k\to\infty} T_{n_k}(f) \quad \forall f \in X.$$

Since each T_{n_k} is linear, the limit functional T is linear. Since

$$\begin{aligned} |T_{n_k}(f)| &\leq M \cdot ||f|| \quad \text{for all} \quad k \quad \text{and} \quad \forall f \in X, \\ |T(f)| &= \lim_{k\to\infty} |T_{n_k}(f)| \leq M \cdot ||f|| \quad \forall f \in X. \end{aligned}$$

Therefore, T is bounded. □

Proposition 2.0.32. *Every bounded sequence in a Hilbert space has a weakly convergent subsequence.*

Proof. Let $\{h_n\}$ be a bounded sequence in H. Let define H_0 to be the closed linear span of $\{h_n\}$. Then by Theorem 2.0.23, H_0 is separable. Let $[H_0]^*$ denote the dual space of H_0. Then for each natural number n let's define $\psi_n \in [H_0]^*$ by

$$\psi_n(h) = \langle h_n, h \rangle \quad \text{for all} \quad h \in H_0. \tag{2.20}$$

Since $\{h_n\}$ is bounded, then from Lemma 2.0.2, we have that $\{\psi_n\}$ is also bounded. Then $\{\psi_n\}$ is a bounded sequence of bounded linear functionals on the separable normed space H_0. Then by Theorem 2.0.31(Helley's theorem), there is a subsequence $\{\psi_{n_k}\}$ of $\{\psi_n\}$ that converges pointwise to $\psi_0 \in [H_0]^*$. From (2.20), we may write

$$\psi_{n_k}(h) = \langle h_{n_k}, h \rangle \quad \text{for all} \quad h \in H_0.$$

But we have that $\psi_{n_k} \to \psi_0 \in [H_0]^*$ as $k \to \infty$, then by Theorem 2.0.27(Riesz Representation theorem), there is a vector $h_0 \in H_0$ for which $T(h_0) = \psi_0$. Thus we obtain

$$\langle h_0, h \rangle = \lim_{k\to\infty} \langle h_{n_k}, h \rangle \quad \text{for all} \quad h \in H_0.$$

Now let P be the orthogonal projection on H onto H_0, thus, $P(H) = \{x \in H; \langle x, h \rangle = 0, \quad \forall h \in H_0\}$. Then by Lemma 2.0.25, we have that $P(H)$ is a closed vector subspace and therefore by Theorem 2.0.26, we have that $H = P(H) + P(H)^\perp$ which implies $H - P(H) = P(H)^\perp$. Now for each index k, since $(Id - P)[H] = P(H)^\perp$, then

$$\langle h_{n_k}, (Id - P)[h] \rangle = \langle h_0, (Id - P)[h] \rangle = 0 \quad \text{for all} \quad h \in H.$$

Therefore,

$$\lim_{k\to\infty} \langle h_{n_k}, h \rangle = \langle h_0, h \rangle \quad \text{for all} \quad h \in H.$$

Thus $\{h_{n_k}\}$ converges weakly to h_0 in H. This complete the proof. □

Definition 15. *(Contraction)*
Let (K,d) be a metric space. Then the map $f : K \longrightarrow K$ is a contraction mapping of (K,d) if for some real number $0 \leq c < 1$, called the constant of contraction, we have

$$d(f(x), f(y)) \leq cd(x,y) \quad \forall x,y \in K.$$

Definition 16. *(Uniform Continuity)*
Let A and B be metric spaces and d,d' be metrics in A and B respectively. The map $f : A \longrightarrow B$ is said to be uniformly continuous if for any $\varepsilon > 0$, $\exists \delta > 0$ such that whenever $d(x,y) < \delta$, we have

$$d'(f(x), f(y)) < \varepsilon \quad \forall x,y \in A.$$

Proposition 2.0.33. *Contraction mappings are uniformly continuous.*

Proof. This is clear when $c = 0$ since by then f is a constant function. If $c > 0$ and we are given $\varepsilon > 0$, setting $\delta = \frac{\varepsilon}{c}$ implies that when $d(x,y) < \delta$, then

$$d(f(x), f(y)) \leq cd(x,y) < c \cdot \frac{\varepsilon}{c} = \varepsilon.$$

This complete the proof. □

Theorem 2.0.34. *(Banach, 1922)*
Let (K,d) be a complete metric space. If $f : K \longrightarrow K$ is a contraction, then f has a unique fixed point in K. Moreover, for any $x_0 \in K$, the sequence of Picard iterates $x_0, f(x_0), f(f(x_0)), \ldots$ converges to the fixed point of f.

Proof. **Existence:**
By the Definition 15, there exist a number $c \in [0,1)$ such that

$$d(f(x), f(y)) \leq cd(x,y). \tag{2.21}$$

We want to show that for any $x_0 \in K$, that the iterative sequence defined by $x_{n+1} = f(x_n)$ for $n \geq 0$ converges to a fixed point of our map f. Every time we iterate f, the distance is contracted as c being smaller than one is raised to higher powers. This will ensure convergence our sequence because (K,d) is complete.
Now we claim that $\{x_n\}$ is a Cauchy. To see this, first we note that for any $n \geq 1$, then by (2.21), we have

$$d(x_n, x_{n+1}) = d(f(x_{n-1}), f(x_n)) \leq cd(x_{n-1}, x_n).$$

Now defining for $d(x_{n-1}, x_n)$ and other subsequent ones, we see the following geometric pattern emerge

$$d(x_n, x_{n+1}) \leq c^1 d(x_{n-1}, x_n) \leq c^2 d(x_{n-2}, x_{n-1}) \leq \ldots c^n d(x_0, x_1).$$

Using this expression on the far right as an upper bound on $d(x_n, x_{n+1})$ shows that $x_n^{;s}$ are getting consecutively close at a geometric rate. This implies that $x_n^{,s}$ are Cauchy. For any $m > n$ using the triangle inequality several times shows that

$$\begin{aligned}
d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m) \\
&\leq c^n d(x_0, x_1) + c^{n+1} d(x_0, x_1) + \ldots + c^{m-1} d(x_0, x_1) \\
&= (c^n + c^{n+1} + \ldots + c^{m-1}) d(x_0, x_1) \\
&= (c^n + c^{n+1} + c^{n+2} + \ldots) d(x_0, x_1) \\
&= \frac{c^n}{1-c} d(x_0, x_1).
\end{aligned}$$

To prove from this bound that the $x_n^{,s}$ are cauchy, choose $\varepsilon > 0$ and then pick $N \geq 1$ such that $\frac{c^N}{1-c} d(x_0, x_1) < \varepsilon$. Then for any $m > n \geq N$ we have,

$$d(x_n, x_m) \leq \frac{c^n}{1-c} d(x_0, x_1) \leq \frac{c^N}{1-c} d(x_0, x_1) < \varepsilon.$$

This proves that $\{x_n\}$ is cauchy sequence. Since K is complete, the $x_n^{,s}$ converge to an element in K. Set $a = \lim_{n \longrightarrow \infty} x_n$ in K.
Since by Proposition 2.0.33 contraction f is uniformly continuous, when $x_n \longrightarrow a$, we will have $f(x_n) \longrightarrow f(a)$. Since $f(x_n) = x_{n+1}$, $f(x_n) \longrightarrow a$ as $n \longrightarrow \infty$. Then $f(a)$ and (a) are both limits of $\{x_n\}_{n \geq 0}$. From the uniqueness of limits, $f(a) = a$. This concludes that a is a fixed point of f.

Uniqueness:
let a and $\bar{a}$ be fixed points of f with $a \neq \bar{a}$. Then by Definition 15

$$d(a, \bar{a}) = d(f(a), f(\bar{a})) \leq c d(a, \bar{a}).$$

So we have that

$$d(a, \bar{a}) \leq c d(a, \bar{a}).$$

Since $a \neq \bar{a}$, then $d(a, \bar{a}) > 0$ so we can divide by $d(a, \bar{a})$ to get $1 \leq c$ which is false since $0 \leq c < 1$. Thus $a = \bar{a}$, showing that the fixed point of f is unique. □

Theorem 2.0.35. *Every nonexpansive self mapping on a closed bounded convex subset of a real Hilbert Space has a fixed point.*

Proof. Assume C is closed bounded convex subset and let $T : C \longrightarrow C$ be a nonexpansive mapping. Now for every natural number $n \geq 1$, choose an arbitrary point $u \in C$ and define the new mapping

$$T_n x = \frac{1}{n} u + \left(1 - \frac{1}{n}\right) T x.$$

Now,

$$\begin{aligned} T_n y - T_n x &= \frac{1}{n}u + \left(1 - \frac{1}{n}\right)Ty - \frac{1}{n}u - \left(1 - \frac{1}{n}\right)Tx \\ &= \left(1 - \frac{1}{n}\right)(Ty - Tx). \end{aligned}$$

This implies that,

$$\begin{aligned} ||T_n y - T_n x|| &= \left(1 - \frac{1}{n}\right)||(Ty - Tx)|| \\ &\leq \left(1 - \frac{1}{n}\right)||y - x|| \quad \text{since} \quad T \quad \text{is a nonexpansive mapping.} \end{aligned}$$

Since $\left(1 - \frac{1}{n}\right) < 1$ and $T_n x \in C$ for all $x \in C$, it implies T_n is a contraction self-mapping. Hence by Theorem 2.0.34, T_n has a unique fixed point say x_n. That is,

$$\begin{aligned} Tx_n = x_n &= \frac{1}{n}u + \left(1 - \frac{1}{n}\right)Tx_n \\ x_n &= \frac{1}{n}u + \left(1 - \frac{1}{n}\right)Tx_n \\ x_n - Tx_n &= \frac{1}{n}(u - Tx_n). \end{aligned}$$

This implies,

$$||x_n - Tx_n|| = \frac{1}{n}||u - Tx_n||$$

Now since C is bounded, it means $\sup\limits_{n \geq 1}||u - Tx_n|| = D < \infty$. Hence we have that

$$||x_n - Tx_n|| \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

That is $x_n - Tx_n \longrightarrow 0$. Since C is bounded then $\{x_n\}_{n \geq 1}$ is a bounded sequence. So by Proposition 2.0.32, there exist a weakly convergent subsequence $\{x_{n_k}\}_{k \geq 1}$ in C. So let $x_{n_k} \rightharpoonup x$ and since also $x_{n_k} - Tx_{n_k} \longrightarrow 0$ (every subsequence of a convergent sequence converges to the same limit), then by Theorem 2.0.18, we have that $Tx = x$, showing that x is a fixed point of T. This complete the proof. □

Proposition 2.0.36. *Let H be a real Hilbert space and let $x, y \in H$. Then $\langle x, y \rangle \geq 0$ if and only if $||x|| \leq ||x + \alpha y||$ for all $\alpha \geq 0$.*

Proof. Without loss of generality, let $\alpha > 0$.
if $||x|| \leq ||x + \alpha y||$, then we have,

$$||x||^2 \leq ||x+\alpha y||^2$$
$$||x||^2 - ||x+\alpha y||^2 \leq 0.$$

Then by Proposition 2.0.9, we have

$$\begin{aligned}
\langle x+x+\alpha y, x-x-\alpha y\rangle &\leq 0 \\
\langle 2x+\alpha y, -\alpha y\rangle &\leq 0 \\
\langle 2x, -\alpha y\rangle + \langle \alpha y, -\alpha y\rangle &\leq 0 \\
-2\alpha\langle x,y\rangle - \alpha^2\langle y,y\rangle &\leq 0 \\
-2\alpha\langle x,y\rangle \leq \alpha^2\langle y,y\rangle.
\end{aligned}$$

$$2\langle x,y\rangle \geq -\alpha\langle y,y\rangle \Longrightarrow 2\langle x,y\rangle \geq -\alpha||y||^2. \tag{2.22}$$

As $\alpha \longrightarrow 0$, we that $2\langle x,y\rangle \geq 0$ which implies $\langle x,y\rangle \geq 0$. This complete the first part of the proof.

Conversely, if $\langle x,y\rangle \geq 0$, since $\alpha \geq 0$, then from (2.22), we have,

$$\begin{aligned}
2\langle x,y\rangle &\geq -\alpha||y||^2 \\
\langle x,y\rangle &\geq \frac{-\alpha}{2}||y||^2.
\end{aligned}$$

which implies

$$\begin{aligned}
-2\alpha\langle x,y\rangle - \alpha^2\langle y,y\rangle &\leq 0 \\
\langle 2x+\alpha y, -\alpha y\rangle &\leq 0 \\
\langle x+(x+\alpha y), x-(x+\alpha y)\rangle &\leq 0 \\
||x||^2 - ||x+\alpha y||^2 &\leq 0.
\end{aligned}$$

So we have,

$$||x|| \leq ||x+\alpha y|| \quad \text{as desired.}$$

This complete the whole proof. □

Lemma 2.0.37. *Let C be a closed convex subset of a Hilbert space H, and let $x \in H$, $\bar{x} \in C$ such that*

$$\langle x-\bar{x}, \bar{x}-y\rangle \geq 0 \quad \forall y \in C.$$

Then $\bar{x} = P_C x$ where the unique element $\bar{x}$ is the metric projection of x onto C.

Proof. By Proposition 2.0.36,

$$\langle x - \bar{x}, \bar{x} - y \rangle \geq 0 \quad \forall y \in C,$$

if and only if

$$||x - \bar{x}|| \leq ||x - \bar{x} + \alpha(\bar{x} - y)|| \quad \forall y \in C, \alpha \geq 0.$$

This implies that,

$$||x - \bar{x}|| \leq ||x - (\alpha y + (1 - \alpha)\bar{x})||.$$

Now if we take or let $\alpha \in [0,1)$, then since y and $\bar{x}$ are in C, then their convex combination is also in C, so we have

$$C = \{\alpha y + (1 - \alpha)\bar{x}, \forall \alpha \in [0,1), \forall y \in C\}.$$

Hence,

$$||x - \bar{x}|| \leq ||x - C||.$$

So by Theorem 2.0.11, we have that $\bar{x} = P_C x$. □

Theorem 2.0.38. *Let $P_C : H \longrightarrow C$ be the metric projection in a real Hilbert space into a closed convex subset C. Then $P_C x$ is a nonexpansive mapping.*

Proof. Let $U = \langle P_C x - P_C y, P_C y - y \rangle$ and $V = \langle P_C y - P_C x, P_C x - x \rangle$. Then from Lemma 2.0.37, we have that $U \geq 0$, $V \geq 0$ so that $U + V \geq 0$. Now we have the evaluation

$$\begin{aligned}
0 \leq U + V &= \langle P_C x - P_C y, P_C y - y \rangle + \langle P_C y - P_C x, P_C x - x \rangle \\
&= \langle P_C x - P_C y, P_C y - y \rangle - \langle P_C x - P_C y, P_C x - x \rangle \\
&= \langle P_C x - P_C y, x - y + P_C y - P_C x \rangle \\
&= \langle P_C x - P_C y, x - y \rangle + \langle P_C x - P_C y, P_C y - P_C x \rangle \\
&= \langle P_C x - P_C y, x - y \rangle - \langle P_C x - P_C y, P_C x - P_C y \rangle \\
&= \langle P_C x - P_C y, x - y \rangle - ||P_C x - P_C y||^2.
\end{aligned}$$

Hence,

$$\langle P_C x - P_C y, x - y \rangle \geq ||P_C x - P_C y||^2.$$

But from Lemma 2.0.2, we have that

$$||P_C x - P_C y|| ||x - y|| \geq ||P_C x - P_C y||^2.$$

so that

$$||P_C x - P_C y|| \leq ||x - y|| \quad \text{as desired.}$$

□

Proposition 2.0.39. *Let X be a Banach Space. Suppose that every subsequence of an arbitrarily sequence $\{x_n\}_{n\geq 1}$ has a further subsequence converging strongly to the same limit x. Then $\{x_n\}_{n\geq 1}$ converges strongly to x.*

Proof. We prove by contradiction. Suppose $\{x_n\}_{n\geq 1}$ does not converge strongly to x. Then,

$$||x_n - x|| < \varepsilon,$$

fails for infinitely many n given a sufficiently small $\varepsilon > 0$. Thus, let $\{x_{n_k}\}_{k\geq 1}$ be the subsequence failing the above inequality; hence

$$||x_{n_k} - x|| \geq \varepsilon,$$

for all $k \geq 1$, thus contradicting the fact that the subsequence $\{x_{n_k}\}_{k\geq 1}$ must have further subsequence converging strongly to x and so the assumption that $\{x_n\}_{n\geq 1}$ did not converge strongly to x was false. Therefore $\{x_n\}_{n\geq 1}$ converges strongly to x. □

Chapter 3

Main Result

In this section, we give an extensive review of approximation methods for common fixed points of nonexpansive mappings in Hilbert spaces presented in Maingé (2007).
Let H be a real Hilbert spaced endowed with inner product $\langle \cdot, \cdot \rangle$ and induced norm $|\cdot|$. Let $(T_i)_{i\geq 0}$ be an infinite countably family of nonexpansive self-mappings defined on a closed convex subset D of H, such that $S := \cap_{i\geq 0}\text{Fix}(T_i) \neq \{\emptyset\}$, where $\text{Fix}(T_i) := \{x \in D | T_i x = x\}$ is the set of fixed points of T_i. By Lemma 2.0.10 and Proposition 2.0.6, then S is a closed convex set of D. Henceforth, we will use the notation $|\cdot|$ to mean $||\cdot||$.

From Definition 6, a mapping $T : D \longrightarrow D$ is called nonexpansive if

$$|Tx - Ty| \leq |x - y| \quad \forall x, y \in D.$$

Similarly, from Definition 15, a mapping $C : D \longrightarrow D$ is called a contraction map if for all $\rho \in [0, 1)$

$$|Cx - Cy| \leq \rho |x - y| \quad \forall x, y \in D.$$

We find a common fixed point of $(T_i)_{i\geq 0}$:

$$\text{find} \quad \bar{x} \in H \quad \text{such that} \quad T_i \bar{x} = \bar{x} \quad \text{for all} \quad i \geq 0. \tag{3.1}$$

More precisely, we study the implicit and non-implicit algorithms for computing a specific point in S where $S := \cap_{i\geq 0}\text{Fix}(T_i)$. Throughout, we denote

$$\Lambda_I := \{i \in \mathbb{N} | T_i \neq I\} \quad (I \quad \text{being the identity mapping onto} \quad H).$$

Now we consider an implicit regularization-like algorithm for approximating (3.1).

Consider the map $W(\cdot, \cdot)$ defined on $(0, 1) \times D$ by $W(t, x) := tCx + \sum_{i\geq 0} w_{i,t} T_i x$ for $(t, x) \in (0, 1) \times D$ where $w_{i,t} \geq 0$ (coefficient of the various T_i) and $t \in (0, 1)$. Since $W(t, x)$ is a convex combination of elements in D and D is a convex set, then $W(t, x)$ is a self mapping on D. The expression: $t + \sum_{i\geq 0} w_{i,t} = 1$ since $W(t, x)$ is a convex combination of elements in D.
CLAIM: For all x, y in D, $W(t, x)$ is a contraction. For,

$$
\begin{aligned}
|W(t,x)-W(t,y)| &= |tCx+(1-t)T_ix-tCy-(1-t)T_iy| \\
&= |tCx-tCy+(1-t)T_ix-(1-t)T_iy| \\
&= |t(Cx-Cy)+(1-t)(T_ix-T_iy)| \\
&\leq t|Cx-Cy|+(1-t)|T_ix-T_iy| \quad \text{triangle inequality.}
\end{aligned}
$$

But C being a contraction and T_i being nonexpansive, then from the above we deduce that

$$
\begin{aligned}
|W(t,x)-W(t,y)| &\leq t\rho|x-y|+(1-t)|x-y| \\
&\leq (1-(1-\rho)t)|x-y| \quad \text{as claimed.}
\end{aligned}
$$

Now since $W(t,x)$ is a contraction, then from Theorem 2.0.34 (Banach contraction principle), there exist a unique fixed point x_t of $W(t,x)$ which solves the equation (fixed point equation)

$$x_t = tCx_t + \sum_{i\geq 0} w_{i,t}T_ix_t, \tag{3.2}$$

where $t \in (0,1), w_{i,t} \geq 0$ for all $i \geq 0$ and $\sum_{i\geq 0} w_{i,t} = 1-t$. Moreover, when $i \in \Lambda_I$, we assume $w_{i,t} \neq 0$ for t small enough.

Now we study the strong convergence of the sequence x_t as $t \to 0$. We prove that the strong limit point of x_t is in S which is a common fixed point to all the non-expansive mappings T_i of the: implicit algorithm (3.2).

The second convergence is of the sequence x_n generated by a given initial point x_0 in D and the non-implicit iterative process

$$x_{n+1} := \alpha_n Cx_n + \sum_{i\geq 0} w_{i,n}T_ix_n, \tag{3.3}$$

for all $n \geq 0$, where $(\alpha_n) \subset (0,1), w_{i,n} \geq 0$ for all $i \geq 0$ and $\sum_{i\geq 0} w_{i,n} = 1-\alpha_n$. When $i \in \Lambda_I$, we assume $w_{i,n} \neq 0$ for n sufficiently large.

The proposed methods (3.2) and (3.3) are approximation methods that are going to be used to approximate an element in (3.1). That particular element we are approximating is the nearest element (metric projection) to the fixed point set S from an arbitrarily chosen point in the Hilbert space.

3.1 The Implicit Regularization-like method

Now we discusses the strong convergence results of the fixed point equation x_t of (3.2) as $t \to 0$. To end this, we introduce the following lemma.

Lemma 3.1.1. *The solution x_t of (3.2) is bounded (as $t \to 0$), besides it has at most one strong limit point in S.*

Proof. Pick q in S. Then from (3.2), we have the following:

$$\begin{aligned} x_t - q &= tCx_t + \sum_{i\geq 0} w_{i,t} T_i x_t - q \\ &= tCx_t + \sum_{i\geq 0} w_{i,t} T_i x_t - (tq + (1-t)q) \\ &= tCx_t - tq + \sum_{i\geq 0} w_{i,t} T_i x_t - (1-t)q \\ &= t(Cx_t - q) + \sum_{i\geq 0} w_{i,t}(T_i x_t - q) \quad \text{since} \quad (1-t) = \sum_{i\geq 0} w_{i,t}. \end{aligned}$$

Now using $x_t - q$ to compute the inner product, we have

$$\begin{aligned} |x_t - q|^2 &= \langle x_t - q, x_t - q\rangle \\ &= \left\langle t(Cx_t - q) + \sum_{i\geq 0} w_{i,t}(T_i x_t - q), x_t - q \right\rangle \\ &= \langle t(Cx_t - q), x_t - q\rangle + \left\langle \sum_{i\geq 0} w_{i,t}(T_i x_t - q), x_t - q \right\rangle, \end{aligned}$$

so that

$$|x_t - q|^2 = t\langle Cx_t - q, x_t - q\rangle + \sum_{i\geq 0} w_{i,t}\langle T_i x_t - q, x_t - q\rangle .$$

By Lemma 2.0.2, the above equality becomes

$$|x_t - q|^2 \leq t\langle Cx_t - q, x_t - q\rangle + \sum_{i\geq 0} w_{i,t}|T_i x_t - q||x_t - q|.$$

By Lemma 2.0.7, the above inequality becomes

$$\begin{aligned} |x_t - q|^2 &\leq t\langle Cx_t - q, x_t - q\rangle + \sum_{i\geq 0} w_{i,t}|x_t - q||x_t - q| \\ &= t\langle Cx_t - q, x_t - q\rangle + \left(\sum_{i\geq 0} w_{i,t}\right)|x_t - q|^2 \\ &= t\langle Cx_t - q, x_t - q\rangle + (1-t)|x_t - q|^2 \quad \text{since} \quad \sum_{i\geq 0} w_{i,t} = 1-t. \end{aligned}$$

Simplifying the above now becomes;

$$\begin{aligned} |x_t - q|^2 - (1-t)|x_t - q|^2 &\leq t\langle Cx_t - q, x_t - q\rangle \\ |x_t - q|^2(1-1+t) &\leq t\langle Cx_t - q, x_t - q\rangle \\ t|x_t - q|^2 &\leq t\langle Cx_t - q, x_t - q\rangle. \end{aligned}$$

So we obtain

$$|x_t - q|^2 \leq \langle Cx_t - q, x_t - q\rangle. \tag{3.4}$$

Since C is a contraction with $\rho \in [0,1)$, then (3.4) becomes;

$$\begin{aligned} |x_t - q|^2 &\leq \langle Cx_t - q, x_t - q\rangle \\ &= \langle Cx_t - Cq + Cq - q, x_t - q\rangle \\ &= \langle Cx_t - Cq, x_t - q\rangle + \langle Cq - q, x_t - q\rangle \\ &\leq |Cx_t - Cq||x_t - q| + \langle Cq - q, x_t - q\rangle \\ &\leq \rho|x_t - q||x_t - q| + \langle Cq - q, x_t - q\rangle \\ &= \rho|x_t - q|^2 + \langle Cq - q, x_t - q\rangle. \end{aligned}$$

Hence,

$$|x_t - q|^2 \leq \frac{1}{1-\rho}\langle Cq - q, x_t - q\rangle. \tag{3.5}$$

By Lemma 2.0.2, the above inequality becomes

$$|x_t - q|^2 \leq \frac{1}{1-\rho}|Cq - q||x_t - q|.$$

So that

$$|x_t - q| \leq \frac{1}{1-\rho}|Cq - q|.$$

which proves the boundedness of (x_t).
Let assume q_1 and q_2 are two strong limit points of (x_t) in S. Then from (3.4), we have the following two inequalities:

$$|q_1 - q_2|^2 \leq \langle Cq_1 - q_2, q_1 - q_2 \rangle \tag{3.6}$$
$$|q_2 - q_1|^2 \leq \langle Cq_2 - q_1, q_2 - q_1 \rangle = \langle q_1 - Cq_2, q_1 - q_2 \rangle. \tag{3.7}$$

Adding (3.6) and (3.7) gives us

$$2|q_1 - q_2|^2 \leq \langle (Cq_1 - Cq_2) + (q_1 - q_2), q_1 - q_2 \rangle.$$

By Lemma 2.0.2 and using the fact that C is a contraction mapping, we have that

$$\begin{aligned} 2|q_1 - q_2|^2 &\leq |(Cq_1 - Cq_2) + (q_1 - q_2)||q_1 - q_2| \\ &\leq (|Cq_1 - Cq_2| + |q_1 - q_2|)|q_1 - q_2| \quad \text{(triangle inequality)} \\ &\leq (\rho|q_1 - q_2| + |q_1 - q_2|)|q_1 - q_2| \\ &= ((\rho+1)|q_1 - q_2|)|q_1 - q_2| \\ &= (\rho+1)|q_1 - q_2|^2. \end{aligned}$$

so that we have

$$\begin{aligned} 2|q_1 - q_2|^2 - (\rho+1)|q_1 - q_2|^2 &\leq 0 \\ (2 - \rho - 1)|q_1 - q_2|^2 &\leq 0 \\ |q_1 - q_2|^2 &\leq 0. \quad (\text{since}(1-\rho) > 0) \end{aligned}$$

Since by definition, norms are non-negative, then by the squeeze theorem, we have that

$$0 \leq |q_1 - q_2|^2 \leq 0.$$

so that $|q_1 - q_2| = 0$ which implies $q_1 = q_2$ proving the uniqueness of a strong limit point of (x_t) in S. □

Lemma 3.1.2. *Let $t_n \in (0,1)$ such that $t_n \to 0$ as $n \to +\infty$ and assume the following condition (L) holds:*

$$(L) \quad \forall i \in \Lambda_I, \lim_{n \to +\infty} \frac{t_n}{w_{i,t_n}} = 0.$$

Then the solution x_t of (3.2) satisfies

$$\lim_{n\to+\infty}|x_{t_n} - T_i x_{t_n}| = 0 \quad \textit{for each} \quad i \in \Lambda_I. \tag{3.8}$$

Proof. By the definition of (3.2), we have that

$$\begin{aligned}
x_t - tx_t &= (tCx_t + \sum_{i\geq 0} w_{i,t} T_i x_t) - tx_t \\
x_t - tx_t - \sum_{i\geq 0} w_{i,t} T_i x_t &= tCx_t - tx_t \\
(1-t)x_t - \sum_{i\geq 0} w_{i,t} T_i x_t &= t(Cx_t - x_t) \\
\sum_{i\geq 0} w_{i,t} x_t - \sum_{i\geq 0} w_{i,t} T_i x_t &= t(Cx_t - x_t) \quad \left(\text{since} \quad (1-t) = \sum_{i\geq 0} w_{i,t}\right) \\
\sum_{i\geq 0} w_{i,t}(x_t - T_i x_t) &= t(Cx_t - x_t).
\end{aligned}$$

Using the fact that for all a,b and x in an inner product space, if $a = b$, then $\langle a,x\rangle = \langle b,x\rangle$ which implies $\langle a-b,x\rangle = 0$. Hence, given $q \in S$, it follows that,

$$\begin{aligned}
\left\langle \sum_{i\geq 0} w_{i,t}(x_t - T_i x_t) - t(Cx_t - x_t), x_t - q \right\rangle &= 0 \\
\left\langle \sum_{i\geq 0} w_{i,t}(x_t - T_i x_t), x_t - q \right\rangle - \langle t(Cx_t - x_t), x_t - q\rangle &= 0,
\end{aligned}$$

so that

$$\sum_{i\geq 0} w_{i,t} \langle x_t - T_i x_t, x_t - q\rangle = t\langle Cx_t - x_t, x_t - q\rangle.$$

From Proposition 2.0.8, the left-hand side of this equation results

$$\frac{1}{2}\sum_{i\geq 0} w_{i,t}|T_i x_t - x_t|^2 \leq t\langle Cx_t - x_t, x_t - q\rangle. \tag{3.9}$$

Hence, for all $i \in \Lambda_I$ and for t small enough, since $w_{i,t} \neq 0$ we have

$$|T_i x_t - x_t|^2 \leq 2\cdot\frac{t}{w_{i,t}}\langle Cx_t - x_t, x_t - q\rangle. \tag{3.10}$$

By Lemma 3.1.1, the solution x_t is bounded (as $t \to 0$), also the quantity $\langle Cx_t - x_t, x_t - q\rangle$ is bounded since from Lemma 2.0.2, we have that,

$$\begin{aligned}\langle Cx_t - x_t, x_t - q\rangle &\leq |Cx_t - x_t||x_t - q| \\ &= |Cx_t - Cq + Cq - x_t||x_t - q| \\ &\leq |x_t - q|(|Cx_t - Cq| + |Cq - x_t|) \\ &\leq \rho|x_t - q|^2 + |x_t - q||Cq - x_t| \quad \text{since} \quad C \quad \text{is a contraction mapping.}\end{aligned}$$

Now by sub-additive of norms, we obtain

$$\langle Cx_t - x_t, x_t - q\rangle \leq \rho|x_t - q|^2 + |x_t - q|(|Cq| + |x_t|).$$

Clearly, the expression on the right hand side of the above inequality is bounded, hence we have that the inner product $\langle Cx_t - x_t, x_t - q\rangle$ is bounded.
Also when the condition (L) is also satisfied, it is easily deduced from (3.10) that $|T_i x_{t_n} - x_{t_n}| \to 0$ as $t_n \to 0$, for all $i \in \Lambda$. □

Theorem 3.1.3. *Under the hypothesis of Lemma 3.1.2, the solution x_t of (3.2) satisfies*

$$\lim_{n\to+\infty} |x_{t_n} - \bar{x}| = 0, \tag{3.11}$$

where $\bar{x}$ is the unique fixed point of the contraction P_S o C, P_S being the metric projection from H onto S.

Proof. Put $y_n = x_{t_n}$. Then according to Lemma 3.1.2, we have that $\lim_{n\to+\infty} |y_n - T_i y_n| = 0$ for all $i \in \Lambda_I$ under condition (L). Since every subsequence of a convergent sequence converges to the same limit, we have that $|y_{n_k} - T_i y_{n_k}| \to 0$ as $k \to +\infty$. Also since by Lemma 3.1.1, (y_n) is a bounded sequence, then by Proposition 2.0.32 there exist a weakly convergent subsequence of (y_n) labelled $\{y_{n_k}\}_{k\geq 1}$.
By Theorem 2.0.18 of the mappings T_i, we have that $(y_{n_k}) \rightharpoonup \bar{x}$ in S so that $T_i\bar{x} = \bar{x}$.
From (3.5), we then have

$$(1-\rho)|y_{n_k} - \bar{x}|^2 \quad \leq \quad \langle C\bar{x} - \bar{x}, y_{n_k} - \bar{x}\rangle\,. \tag{3.12}$$

As $\langle C\bar{x} - \bar{x}, y_{n_k} - \bar{x}\rangle \leq |C\bar{x} - \bar{x}||y_{n_k} - \bar{x}| \to 0$ by weak convergence of y_{n_k} to $\bar{x}$, then inequality (3.12) shows that y_{n_k} strongly converges to $\bar{x}$. Since any strong cluster-point of (y_n) is in S and also by Lemma 3.5, (y_n) has a unique strong cluster-point in S, then we deduce that (y_n) converges strongly to $\bar{x}$. Then by Proposition 2.0.39, the sequence x_t converges strongly to $\bar{x}$.

It now remains to characterize the limit $\bar{x}$ of (x_{t_n}).
Let q be any element in S. Then by (3.10), since $|T_i x_t - x_t|^2$ is positive, then on the right hand side, the quantity $\langle Cx_t - x_t, x_t - q\rangle$ is also positive and as a result, we have that

$$
\begin{aligned}
\langle Cx_{t_n} - x_{t_n}, x_{t_n} - q \rangle &\geq 0 \quad \text{since} \quad t_n \in (0,1) \\
\langle x_{t_n} - Cx_{t_n}, x_{t_n} - q \rangle &\leq 0.
\end{aligned}
$$

Now passing to the limit as $t_n \to 0$, we get

$$
\langle \bar{x} - C\bar{x}, \bar{x} - q \rangle \leq 0 \quad \forall q \in S, \tag{3.13}
$$

so that $\bar{x} = P_S(C\bar{x})$, which completes the proofs. □

3.2 The Explicit iterative method

In this section, we prove the strong convergence results regarding the sequence (x_n) obtained with (3.3), by imposing the following conditions:

$$\text{(Q1)} \qquad \sum_{n=0}^{\infty} \alpha_n = \infty.$$

$$\text{(Q2)} \qquad \begin{cases} \dfrac{1}{w_{i,n}}\left|1-\dfrac{\alpha_{n-1}}{\alpha_n}\right| \to 0, \quad \text{or} \quad \displaystyle\sum_n \frac{1}{w_{i,n}}|\alpha_n-\alpha_{n-1}|<\infty \\ \dfrac{1}{\alpha_n}\left|\dfrac{1}{w_{i,n}}-\dfrac{1}{w_{i,n-1}}\right| \to 0, \quad \text{or} \quad \displaystyle\sum_n \left|\frac{1}{w_{i,n}}-\frac{1}{w_{i,n-1}}\right|<\infty \\ \dfrac{1}{w_{i,n}\alpha_n}\displaystyle\sum_{k\geq 0}\left|w_{k,n}-w_{k,n-1}\right| \to 0 \quad \text{or} \quad \sum_n \frac{1}{w_{i,n}}\sum_{k\geq 0}\left|w_{k,n}-w_{k,n-1}\right|<\infty. \end{cases}$$

$$\text{(Q3)} \qquad \frac{\alpha_n}{w_{i,n}} \to 0 \quad (\text{for all} \quad i \in \Lambda_I).$$

The next lemmas will be needed in the proof of the main result of this section.

Lemma 3.2.1. *Let $(s_n),(c_n) \subset \mathbb{R}_+,(a_n) \subset (0,1)$ and $(b_n) \subset \mathbb{R}$ be sequences such that*

$$s_{n+1} \leq (1-a_n)s_n + b_n + c_n \quad \textit{for all} \quad n \geq 0. \tag{3.14}$$

Assume $\sum_{n\geq 0}|c_n| < \infty$. Then the following claim holds:

CLAIM:

1. *if $b_n \leq \beta a_n$ (where $\beta \geq 0$, then (s_n) is a bounded sequence.*

2. *if*

$$\sum_{n=0}^{\infty} a_n = \infty \quad \textit{or equivalently} \quad \prod_{n=1}^{\infty}(1-a_n) = 0 \quad \textit{and} \quad \limsup_{n\to\infty}\frac{b_n}{a_n} \leq 0,$$

then $\lim_{n\to\infty} s_n = 0$.

Proof. Set $\gamma_{n,k} := \prod_{j=k}^{n}(1-a_j)$(for $\quad n \geq k \geq 0$) and by condition (1),

$$s_{n+1} \leq (1-a_n)s_n + \beta a_n + c_n \quad \text{since} \quad b_n \leq \beta a_n.$$

Then,

$$
\begin{aligned}
s_n &\leq (1-a_{n-1})s_{n-1}+\beta a_{n-1}+c_{n-1}. && (3.15)\\
s_{n-1} &\leq (1-a_{n-2})s_{n-2}+\beta a_{n-2}+c_{n-2}. && (3.16)
\end{aligned}
$$

We substitute (3.15) and (3.16) into the expression for s_{n+1} and inducting backwards we obtain the following:

$$
\begin{aligned}
s_{n+1} &\leq (1-a_n)[(1-a_{n-1})s_{n-1}+\beta a_{n-1}+c_{n-1}]+\beta a_n+c_n\\
&\leq (1-a_n)[(1-a_{n-1})[(1-a_{n-2})s_{n-2}+\beta a_{n-2}+c_{n-2}]+\beta a_{n-1}+c_{n-1}]+\beta a_n+c_n\\
&\vdots\\
(1-a_n) &\ldots(1-a_0)s_0+(1-a_n)(\beta a_{n-1}+c_{n-1})+(1-a_n)(1-a_{n-1})(\beta a_{n-2}+c_{n-2})+\ldots\\
+ &\quad (1-a_n)\ldots(1-a_1)(\beta a_0+c_0)+\beta a_n+c_n\\
&\leq \prod_{j=0}^{n}(1-a_j)s_0+\sum_{j=0}^{n-1}\prod_{j}(1-a_j)(\beta a_j+c_j)+\beta a_n+c_n.
\end{aligned}
$$

That is

$$
\begin{aligned}
s_{n+1} &\leq \prod_{j=0}^{n}(1-a_j)s_0+\sum_{j=0}^{n-1}\prod_{j}(1-a_j)(\beta a_j+c_j)+\beta a_n+c_n\\
&\leq (\gamma_{n,0})s_0+\sum_{j=0}^{n-1}(\gamma_{n,j+1})(\beta a_j+c_j)+\beta a_n+c_n\\
&= (\gamma_{n,0})s_0+\beta\sum_{j=0}^{n-1}(\gamma_{n,j+1})a_j+\beta a_n+\sum_{j=0}^{n-1}(\gamma_{n,j+1})c_j+c_n\\
&= (\gamma_{n,0})s_0+\beta\left(\sum_{j=0}^{n-1}(\gamma_{n,j+1})a_j+a_n\right)+\sum_{j=0}^{n-1}(\gamma_{n,j+1})c_j+c_n.
\end{aligned}
$$

We simply the term $(\gamma_{n,j+1})a_j$.
When $j=k=n$, the product symbol goes away so that we can find for a_j. That is, $\gamma_{j,j}=1-a_j \Rightarrow a_j=1-\gamma_{j,j}$. So we have that

$$
\begin{aligned}
(\gamma_{n,j+1})a_j &= (\gamma_{n,j+1})(1-\gamma_{j,j})\\
&= \gamma_{n,j+1}-(\gamma_{n,j+1})(\gamma_{j,j})\\
&= \gamma_{n,j+1}-(\gamma_{n,j}).
\end{aligned}
$$

Hence we obtain the following

$$s_{n+1} \leq (\gamma_{n,0})s_0 + \beta \left(\sum_{j=0}^{n-1} (\gamma_{n,j+1} - \gamma_{n,j}) + a_n \right) + \sum_{j=0}^{n-1} (\gamma_{n,j+1})c_j + c_n. \qquad (3.17)$$

Now working out $\sum_{j=0}^{n-1} (\gamma_{n,j+1}) - \sum_{j=0}^{n-1} (\gamma_{n,j})$ gives

$$\begin{aligned} &= (\gamma_{n,1}) + (\gamma_{n,2}) + \ldots + (\gamma_{n,n-1}) + (\gamma_{n,n}) - [(\gamma_{n,0}) + (\gamma_{n,1}) + (\gamma_{n,2}) + \ldots + (\gamma_{n,n-1})] \\ &= (\gamma_{n,n}) - (\gamma_{n,0}) \\ &= (1 - a_n) - (\gamma_{n,0}). \end{aligned}$$

So from (3.17) we obtain,

$$s_{n+1} \leq (\gamma_{n,0})s_0 + \beta(1 - a_n - (\gamma_{n,0}) + a_n) + \sum_{j=0}^{n-1} (\gamma_{n,j+1})c_j + c_n.$$

which simplifies to

$$s_{n+1} \leq (\gamma_{n,0})s_0 + \beta(1 - (\gamma_{n,0})) + \sum_{j=0}^{n-1} (\gamma_{n,j+1})c_j + c_n. \qquad (3.18)$$

Since $\gamma_{n,j} \leq 1$ for $0 \leq j \leq n$, we deduce that

$$\begin{aligned} s_{n+1} &\leq s_0 + \beta + \sum_{j=0}^{n-1} c_j + c_n \\ &= s_0 + \beta + (c_0 + c_1 + \ldots + c_{n-1}) + c_n \\ &= s_0 + \beta + \sum_{j=0}^{n} c_j. \end{aligned}$$

so that (s_n) is bounded since $\sum_{j=0}^{n} c_j < \infty$ which proves (1).

We prove (2): Let ε be any positive real number. If we have that $\limsup_{n\to\infty} \frac{b_n}{a_n} \leq 0$, then by the definition of lim sup, there exist $p = p(\varepsilon)$ in $\mathbb{N}$ such that $b_n \leq \varepsilon a_n$ for all $n \geq p$; hence by (3.18), we obtain,

$$s_{n+1} \leq (\gamma_{n,p})s_p + \varepsilon(1 - (\gamma_{n,p})) + \sum_{j=0}^{n-1} (\gamma_{n,j+1})c_j + c_n. \qquad (3.19)$$

Moreover, since $\sum_j c_j < \infty$, then there exists q_ε in $\mathbb{N}$ such that

$$q_\varepsilon \geq p \quad \text{and} \quad \sum_{j \geq q_\varepsilon + 1} c_j < \varepsilon,$$

Hence,

$$\begin{aligned} \forall n > q_\varepsilon, \quad \sum_{j=p}^{n-1} (\gamma_{n,j+1}) c_j + c_n &\leq \gamma_{n,q_\varepsilon+1} \sum_{j=p}^{q_\varepsilon} c_j + \sum_{j \geq q_\varepsilon+1}^{n-1} c_j + \varepsilon \\ &\leq \gamma_{n,q_\varepsilon+1} \sum_{j \geq 0} c_j + 2\varepsilon. \end{aligned}$$

Now combining this last inequality with (3.19), then for $n > q_\varepsilon$, we obtain

$$s_{n+1} \leq (\gamma_{n,p}) s_p + \varepsilon(1 - (\gamma_{n,p})) + \gamma_{n,q_\varepsilon+1} \sum_{j \geq 0} c_j + 2\varepsilon.$$

But we have that $\lim_{n\to+\infty} \gamma_{n,p} = 0$ and $\lim_{n\to+\infty} \gamma_{n,q_\varepsilon+1} = 0$ if $\sum_{n=0}^{\infty} a_n = \infty$; consequently, from the above, we deduce that $\lim_{n\to\infty} s_n \to 0$, that is the second part of the claim. □

Lemma 3.2.2. *The sequence (x_n) generated by the scheme (3.3) is bounded.*

Proof. Now given any $p \in S$, we have that

$$\begin{aligned} x_{n+1} - p &= \alpha_n C x_n + \sum_{i \geq 0} w_{i,n} T_i x_n - p \\ &= \alpha_n C x_n + \sum_{i \geq 0} w_{i,n} T_i x_n - (p\alpha_n + (1 - \alpha_n)p) \\ &= \alpha_n C x_n - p\alpha_n + \sum_{i \geq 0} w_{i,n} T_i x_n - p(1 - \alpha_n) \\ &= \alpha_n (C x_n - p) + \sum_{i \geq 0} w_{i,n} (T_i x_n - p) \quad \text{since} \quad (1 - \alpha_n) = \sum_{i \geq 0} w_{i,n}. \end{aligned}$$

Now taking the norm of the above, we obtain

$$\begin{aligned} |x_{n+1} - p| &= |\alpha_n (C x_n - p) + \sum_{i \geq 0} w_{i,n} (T_i x_n - p)| \\ &\leq |\alpha_n (C x_n - p)| + |\sum_{i \geq 0} w_{i,n} (T_i x_n - p)| \quad \text{triangle inequality} \\ &= |\alpha_n (C x_n - Cp + Cp - p)| + |\sum_{i \geq 0} w_{i,n} (T_i x_n - p)| \\ &\leq \alpha_n |C x_n - Cp| + \alpha_n |Cp - p| + \sum_{i \geq 0} w_{i,n} |T_i x_n - p|. \end{aligned}$$

Since $\sum_{i \geq 0} w_{i,n} = 1 - \alpha_n$, C, a contraction with modulus ρ and the fact that every nonexpansive mapping with a fixed point satisfies Lemma 2.0.7, then the above inequality becomes

$$\begin{aligned}
|x_{n+1}-p| &\le \alpha_n\rho|x_n-p|+\alpha_n|Cp-p|+(1-\alpha_n)|x_n-p| \\
&= (1-(1-\rho)\alpha_n)\underbrace{|x_n-p|}_{s_n}+\underbrace{\alpha_n|Cp-p|}_{b_n}. \\
&\le (1-(1-\rho)\alpha_n)s_n+b_n+c_n \quad \text{where } (c_n)\subset\mathbb{R}_+.
\end{aligned}$$

From the above inequality, since $\alpha_n \subset (0,1)$ and the fact that $b_n \le \beta\alpha_n$ where $\beta = |Cp-p|$ is positive, then by (1) of Lemma 3.2.1, we have that $|x_n-p|$ is bounded which implies that the (x_n) is bounded. □

Lemma 3.2.3. *If conditions (Q1)-(Q2) hold, then the sequence (x_n) given by scheme (3.3) satisfies*

$$\frac{1}{w_{i,n}}|x_{n+1}-x_n| \to 0 \quad \textit{for all} \quad i\in\Lambda_I.$$

Proof. By the definition of scheme (3.3), we have

$$\begin{aligned}
x_{n+1}-x_n &= (\alpha_nCx_n+\sum_{i\ge0}w_{i,n}T_ix_n)-(\alpha_{n-1}Cx_{n-1}+\sum_{i\ge0}w_{i,n-1}T_ix_{n-1}) \\
&= \alpha_nCx_n-\alpha_{n-1}Cx_{n-1}+\sum_{i\ge0}w_{i,n}T_ix_n-\sum_{i\ge0}w_{i,n-1}T_ix_{n-1} \\
&= \alpha_nCx_n-\alpha_nCx_{n-1}+\alpha_nCx_{n-1}-\alpha_{n-1}Cx_{n-1} \\
&\quad +\sum_{i\ge0}w_{i,n}T_ix_n-\sum_{i\ge0}w_{i,n}T_ix_{n-1}+\sum_{i\ge0}w_{i,n}T_ix_{n-1}-\sum_{i\ge0}w_{i,n-1}T_ix_{n-1} \\
&= \alpha_n(Cx_n-Cx_{n-1})+(\alpha_n-\alpha_{n-1})Cx_{n-1} \\
&\quad +\sum_{i\ge0}w_{i,n}(T_ix_n-T_ix_{n-1})+\sum_{i\ge0}(w_{i,n}-w_{i,n-1})T_ix_{n-1}.
\end{aligned}$$

The operators T_i being nonexpansive, C being a contraction with modulus ρ, and since $\sum_{i\ge0}w_{i,n}=1-\alpha_n$, then after taking norms of the above equation, we obtain

$$\begin{aligned}
|x_{n+1}-x_n| &\le (1-(1-\rho)\alpha_n)|x_n-x_{n-1}|+|\alpha_n-\alpha_{n-1}|\times|Cx_{n-1}| \\
&\quad +\sum_{i\ge0}|w_{i,n}-w_{i,n-1}|\times|T_ix_{n-1}|.
\end{aligned}$$

Thus, for all $i\in\Lambda_I$ and n large enough, we get

$$
\begin{aligned}
\frac{1}{w_{i,n}}|x_{n+1}-x_n| &\leq (1-(1-\rho)\alpha_n)\frac{1}{w_{i,n-1}}|x_n-x_{n-1}| \\
&+(1-(1-\rho)\alpha_n)\left(\frac{1}{w_{i,n}}-\frac{1}{w_{i,n-1}}\right)|x_n-x_{n-1}| \\
&+\frac{1}{w_{i,n}}|\alpha_n-\alpha_{n-1}|\times|Cx_{n-1}|+\frac{1}{w_{i,n}}\sum_{i\geq 0}|w_{i,n}-w_{i,n-1}|\times|T_ix_{n-1}|.
\end{aligned}
$$

By Lemma 3.2.2, the sequence (x_n) is bounded, hence the Lipschitz mappings T_i and C are also bounded since

$$
\begin{aligned}
|Tx_n| &= |Tx_n-Ty+Ty| \\
&\leq |Tx_n-Ty|+|Ty| \quad \text{by the sub-additivity of norm} \\
&\leq |x_n-y|+|Ty| \quad \text{non-expansiveness of} \quad T_i \\
&\leq |x_n|+|y|+|Ty| \quad \text{by the sub-additivity of norm.}
\end{aligned}
$$

Since (x_n) is bounded and the fact that $|y|$ and $|Ty|$ are fixed, consequently, we have that the family $(T_ix_n)_{i,n\geq 0}$ and (Cx_n) are also bounded. Consequently, there exist a positive constant M (maximum) such that

$$
\begin{aligned}
\frac{1}{w_{i,n}}\underbrace{|x_{n+1}-x_n|}_{s_{n+1}} &\leq \underbrace{(1-(1-\rho)\alpha_n)}_{(1-a_n)}\underbrace{\frac{1}{w_{i,n-1}}|x_n-x_{n-1}|}_{s_n}+\underbrace{M\left|\frac{1}{w_{i,n}}-\frac{1}{w_{i,n-1}}\right|}_{b_n} \\
&+M\underbrace{\left(\frac{1}{w_{i,n}}|\alpha_n-\alpha_{n-1}|+\frac{1}{w_{i,n}}\sum_{i\geq 0}|w_{i,n}-w_{i,n-1}|\right)}_{c_n}.
\end{aligned}
$$

From the above inequality, we deduce the following
Since we have from (Q1) $\sum_{n=0}^{\infty}\alpha_n=\infty$ and also the fact that $\frac{b_n}{\alpha_n}=\frac{1}{\alpha_n}\left|\frac{1}{w_{i,n}}-\frac{1}{w_{i,n-1}}\right|\to 0$ from (Q2), then we have that $\limsup_{n\to\infty}\frac{b_n}{\alpha_n}\leq 0$. This implies that s_n in the above inequality which is $\frac{1}{w_{i,n-1}}|x_n-x_{n-1}|\to 0$. Now the sequence c_n in the above inequality which is bounded can also be simplified to

$$
\alpha_n\left[\left(\frac{1}{w_{i,n}}\left|1-\frac{\alpha_{n-1}}{\alpha_n}\right|\right)+\frac{1}{w_{i,n}\alpha_n}\sum_{i\geq 0}|w_{i,n}-w_{i,n-1}|\right]\to 0 \quad \text{by} \quad (Q2).
$$

Hence we have that $\frac{1}{w_{i,n}}|x_{n+1}-x_n|\to 0 \quad \text{for all} \quad i\in\Lambda_I$. □

Lemma 3.2.4. *Assume the conditions $(Q1)-(Q3)$ hold. Then (x_n) given by scheme (3.3) satisfies*

$$\lim_{n\to\infty}|x_n - T_i x_n| = 0 \quad \forall i \in \Lambda_I.$$

Proof. From (3.3), we have

$$\begin{aligned} x_{n+1} - x_n &= \alpha_n C x_n + \sum_{i\geq 0} w_{i,n} T_i x_n - x_n \\ &= \alpha_n C x_n + \sum_{i\geq 0} w_{i,n} T_i x_n - (\alpha_n x_n + (1-\alpha_n)x_n) \\ &= \alpha_n C x_n - \alpha_n x_n + \sum_{i\geq 0} w_{i,n} T_i x_n - (1-\alpha_n)x_n \\ &= \alpha_n (C x_n - x_n) + \sum_{i\geq 0} w_{i,n}(T_i x_n - x_n) \quad \text{since} \quad (1-\alpha_n) = \sum_{i\geq 0} w_{i,n}. \end{aligned}$$

so that

$$\sum_{i\geq 0} w_{i,n}(x_n - T_i x_n) = \alpha_n (C x_n - x_n) + (x_n - x_{n+1}).$$

Using the fact that for all a,b and x in an inner product space, if $a = b$, then $\langle a,x\rangle = \langle b,x\rangle$ which implies $\langle a-b,x\rangle = 0$. Therefore, for any q in S, we have that

$$\begin{aligned} \left\langle \sum_{i\geq 0} w_{i,n}(x_n - T_i x_n) - [\alpha_n (C x_n - x_n) + (x_n - x_{n+1})], x_n - q \right\rangle &= 0 \\ \left\langle \sum_{i\geq 0} w_{i,n}(x_n - T_i x_n), x_n - q \right\rangle - \langle \alpha_n (C x_n - x_n) + (x_n - x_{n+1}), x_n - q\rangle &= 0 \\ \left\langle \sum_{i\geq 0} w_{i,n}(x_n - T_i x_n), x_n - q \right\rangle - \langle \alpha_n (C x_n - x_n), x_n - q\rangle - \langle x_n - x_{n+1}, x_n - q\rangle &= 0, \end{aligned}$$

so that

$$\sum_{i\geq 0} w_{i,n} \langle x_n - T_i x_n, x_n - q\rangle = \alpha_n \langle C x_n - x_n, x_n - q\rangle + \langle x_n - x_{n+1}, x_n - q\rangle. \tag{3.20}$$

Since each $T_i (i \geq 0)$ is nonexpansive, then by Proposition 2.0.8, we have that $|T_i x_n - x_n|^2 \leq 2\langle x_n - T_i x_n, x_n - q\rangle$. This together with (3.20) gives us

$$\frac{1}{2}\sum_{i\geq 0} w_{i,n}|T_i x_n - x_n|^2 \leq \alpha_n \langle C x_n - x_n, x_n - q\rangle + \langle x_n - x_{n+1}, x_n - q\rangle,$$

so that, for all $i \in \Lambda_I$, we have

$$|T_i x_n - x_n|^2 \leq 2 \cdot \frac{\alpha_n}{w_{i,n}} \langle Cx_n - x_n, x_n - q \rangle + 2 \cdot \frac{1}{w_{i,n}} \langle x_n - x_{n+1}, x_n - q \rangle .$$

By Lemma 2.0.2, the above inequality becomes

$$|T_i x_n - x_n|^2 \leq 2 \cdot \frac{\alpha_n}{w_{i,n}} \langle Cx_n - x_n, x_n - q \rangle + 2 \cdot \frac{1}{w_{i,n}} |x_n - x_{n+1}||x_n - q|.$$

So by Lemma 3.2.2, we have that $\langle Cx_n - x_n, x_n - q \rangle$ and $|x_n - q|$ are bounded. Hence there exists a positive constant M_1 (Maximum) such that

$$|T_i x_n - x_n|^2 \leq 2 \cdot M_1 \left(\frac{\alpha_n}{w_{i,n}} + \frac{1}{w_{i,n}} |x_n - x_{n+1}| \right).$$

Applying Lemma 3.2.3 and condition (Q3) to the above inequality completes the proof. □

Now the main the result of this section is given by the following theorem.

Theorem 3.2.5. *Under the assumptions (Q1)-(Q3), the sequence (x_n) given by scheme (3.3) converges strongly to $\bar{x}$ the unique fixed point of P_S o C, where P_S is the metric projection from H onto S.*

Proof. By scheme (3.3), we have that

$$\begin{aligned} x_{n+1} - \bar{x} &= \alpha_n C x_n + \sum_{i \geq 0} w_{i,n} T_i x_n - \bar{x} \\ &= \alpha_n C x_n + \sum_{i \geq 0} w_{i,n} T_i x_n - (\alpha_n \bar{x} + (1 - \alpha_n)\bar{x}) \\ &= \alpha_n C x_n + \sum_{i \geq 0} w_{i,n} T_i x_n - \alpha_n \bar{x} - (1 - \alpha_n)\bar{x} \\ &= \alpha_n (C x_n - \bar{x}) + \sum_{i \geq 0} w_{i,n} (T_i x_n - \bar{x}) \quad \text{since} \quad (1 - \alpha_n) = \sum_{i \geq 0} w_{i,n} \\ &= \alpha_n (C x_n - C\bar{x} + C\bar{x} - \bar{x}) + \sum_{i \geq 0} w_{i,n} (T_i x_n - \bar{x}), \end{aligned}$$

so that

$$x_{n+1} - \bar{x} = \left(\alpha_n (C x_n - C\bar{x}) + \sum_{i \geq 0} w_{i,n} (T_i x_n - \bar{x}) \right) + \alpha_n (C\bar{x} - \bar{x}). \tag{3.21}$$

By Proposition 2.0.5, we have that

$$|a+b|^2 - 2\langle b, a+b\rangle = |a|^2 - |b|^2 \le |a|^2. \tag{3.22}$$

Now let $(a+b) = x_{n+1} - \bar{x}$ and $b = \alpha_n(C\bar{x} - \bar{x})$. This implies that $a = x_{n+1} - \bar{x} - \alpha_n(C\bar{x} - \bar{x})$. Then from (3.22), we have the following

$$|x_{n+1} - \bar{x}|^2 - 2\alpha_n \langle C\bar{x} - \bar{x}, x_{n+1} - \bar{x}\rangle \quad \le \quad |x_{n+1} - \bar{x} - \alpha_n(C\bar{x} - \bar{x})|^2. \tag{3.23}$$

But from (3.21), we have that

$$(x_{n+1} - \bar{x}) - \alpha_n(C\bar{x} - \bar{x}) = \left(\alpha_n(Cx_n - C\bar{x}) + \sum_{i\ge 0} w_{i,n}(T_i x_n - \bar{x})\right)$$

With the above substitution, (3.23) becomes

$$\begin{aligned} |x_{n+1} - \bar{x}|^2 - 2\alpha_n \langle C\bar{x} - \bar{x}, x_{n+1} - \bar{x}\rangle &\le |\alpha_n(Cx_n - C\bar{x}) + \sum_{i\ge 0} w_{i,n}(T_i x_n - \bar{x})|^2 \\ &= |\alpha_n(Cx_n - C\bar{x}) + (1-\alpha_n)(T_i x_n - \bar{x})|^2. \end{aligned}$$

Since $||a+b||^2 \le (||a|| + ||b||)^2$, then the above becomes

$$|x_{n+1} - \bar{x}|^2 - 2\alpha_n \langle C\bar{x} - \bar{x}, x_{n+1} - \bar{x}\rangle \le \left(\alpha_n|Cx_n - C\bar{x}| + (1-\alpha_n)|T_i x_n - \bar{x}|\right)^2.$$

Since C is a contraction with modulus ρ and the fact that T_i is a quasi nonexpansive (see Lemma 2.0.7), then the above inequality becomes

$$\begin{aligned} |x_{n+1} - \bar{x}|^2 - 2\alpha_n \langle C\bar{x} - \bar{x}, x_{n+1} - \bar{x}\rangle &\le (\alpha_n\rho|x_n - \bar{x}| + (1-\alpha_n)|x_n - \bar{x}|)^2 \\ &= (1-(1-\rho)\alpha_n)^2|x_n - \bar{x}|^2. \end{aligned}$$

Since $(1-(1-\rho)\alpha_n)^2 \le (1-(1-\rho)\alpha_n)$, then from the above, we obtain

$$|x_{n+1} - \bar{x}|^2 - 2\alpha_n \langle C\bar{x} - \bar{x}, x_{n+1} - \bar{x}\rangle \le (1-(1-\rho)\alpha_n)|x_n - \bar{x}|^2,$$

so that

$$\underbrace{|x_{n+1} - \bar{x}|^2}_{s_{n+1}} \le \underbrace{(1-(1-\rho)\alpha_n)}_{(1-a_n)}\underbrace{|x_n - \bar{x}|^2}_{s_n} + 2\alpha_n \underbrace{\langle C\bar{x} - \bar{x}, x_{n+1} - \bar{x}\rangle}_{b_n}.$$

Lemma 3.2.4, shows that any weak limit point of (x_n) is in S by Theorem 2.0.18 of each operator T_i.
By Lemma 2.0.37, we have that $b_n = \langle C\bar{x} - \bar{x}, x_{n+1} - \bar{x}\rangle \le 0$ which implies that $\bar{x} = P_S(C\bar{x})$.

Since by (Q1), we have $\sum_{n=0}^{\infty} \alpha_n = \infty$, then

$$\limsup_{n\to\infty} \frac{b_n}{\alpha_n} \leq 0,$$

which by Lemma 3.2.1 implies that $s_n = |x_n - \bar{x}|^2 \to 0$ as $n \to \infty$. Hence we conclude that (x_n) converges strongly to $\bar{x}$ which completes the proof. $\square$

Chapter 4

Conclusion

In this thesis, an extensive work has been done on "Approximation methods of common fixed point of non-expansive mappings in a Hilbert space". Detailed proofs of subsidiary results leading up to the proof of the main theorem were offered.
In literature much research has been done on a finite family of non-expansive mappings with errors and it is hoped that most of the results in literature could be extended to cover infinite family of non-expansive mappings with errors.

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