



On the dynamics of the Tent function-Phase diagrams

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Abstract

This paper focuses on the study of a one dimensional topological dynamical system, the tent function. We give a good background to the theory of dynamical systems while establishing the important asymptotic properties of topological dynamical systems that distinguishes it from other fields by way of an example - the tent function. A precise definition of the tent function is given and iterates are clearly shown using the phase diagrams. By this same method, chaos in the tent map is shown as iterates become higher. We also show that the tent map has infinitely many chaotic orbits and also express some important features of chaos such as topological transitivity, boundedness and sensitivity to change in initial conditions from a topological viewpoint.

Keywords: Topological dynamical systems, tent function, phase diagrams, chaos.

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1. Introduction

This may be an elementary introduction to a very important concept in dynamical systems: chaos. Henri Poincaré is regarded by many as the founder of topological dynamical systems [11]. A dynamical system is a concept [20] in mathematics where a fixed rule describes the time dependence of a point in a geometrical space in [14]. Examples include the mathematical models that describe the swinging of a clock pendulum in Holm et al. 2001, the flow of water in a pipe in [17], the continuous striking of a function on a scientific calculator in [5] and the number of fish each spring in a lake in [7]. A current definition of topological Dynamics says that it is 'the study of transformation groups with respect to those topological properties whose prototype occurred in classical dynamics [9].

At any given time a dynamical system has a state given by a set of real numbers, a vector that can be represented by a point in an appropriate state space which is a geometrical manifold [15]. A phase space,

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which is interpreted as the set of all possible states of the system together with a rule of evolution (time-evolution law) which describes how each state assumed by the system changes with time in [4]. The evolution rule of the dynamical system is a fixed rule that describes what future states follow from the current state in [16]. The rule is deterministic; in other words, for a given time interval [19]; only one future state follows from the current state. The main aim of the study of dynamical systems is to understand the long term behavior of states in a system for which there is a deterministic rule for how a state evolves. A dynamical system is deterministic in the sense that the evolution of the system is described by a specific map, so that the present (the initial state) completely determines the future (the forward orbit of the state). At the same time, dynamical systems often appear to be chaotic in that they have sensitive dependence on initial conditions, i.e., minor changes in the initial state lead to dramatically different long-term behavior [3]. In topological dynamics, qualitative long term (asymptotic) properties of dynamical systems are studied from the viewpoint of general topology. The phase space in this theory is a topological space that is either metrizable compact or locally compact. [12]. To determine the state for all future times requires iterating the relation many times—each advancing time a small step. The iteration procedure is referred to as solving the system or integrating the system in [18]. Once the system can be solved, given an initial point it is possible to determine all its future points, a collection known as a trajectory or orbit in [21]. Some trajectories may be periodic, whereas others may wander through many different states of the system. The classification of all possible orbits has led to the qualitative study of dynamical systems [6], that is, properties that are invariant under coordinate changes. A dynamical system as a continuous self-map of a compact metric space [22]. It is noteworthy that topological dynamics studies the iterations of such a periodic map or equivalently the trajectories/orbits of points of the phase space which in the case of this paper is the interval.

In this paper, we intend to extend the study of one dimensional dynamical systems by examining a specific example, the tent map. The main objective is to establish that the tent map has chaotic orbits and show this using the phase diagrams of the map as we successively perform higher iterations. A phase diagram is a graphical representation of the different states of a system. It describes the different phases that the system undergoes in its phase space. The phase diagram for every physical system is unique. Movement along the lines of the phase diagram - orbits of the system describes and provides useful information about the system. Typical examples include the simple pendulum, the simple harmonic oscillator, the Van der Pol oscillator, the Duffing oscillator and the Bifurcation diagram. Phase diagrams can in many physical systems enables us to predict the behavior of the system as time increases from the initial point. Familiarity with concepts such as topological dynamical systems, topological conjugacy and transitivity, elements of general topology are assumed. In a dynamical system, it is a geometric representation of the trajectories in a phase plane. The phase diagram of the tent function is as the name implies in the form of a triangular tent that is defined by a step function. The tent map has an identical behavior to the logistic map under iteration and exhibits a dynamical behavior that ranges from predictable to chaotic and this is clearly shown in this work by the phase diagrams. In the chaotic regime, small differences in initial conditions produce widely diverging outcomes in [13]. Chaos theory largely studies the sensitive dependence of a system on initial conditions even though these systems are deterministic - their future behavior can be fully determined by their initial conditions.

2. Iterating functions

In this section, our focus is on dynamical systems defined by repeated application of a given function that maps a space to itself.

Assume a topological space X , and

$$f : X \longrightarrow X$$

a function that maps X to itself. For every positive integer n , we define the composite function

$$f^n(x) = f \circ f \circ f \circ \dots \circ f(x)$$

where $f^n(x)$ is the composition of n copies of the function f . In effect, we start with x , then apply f to x to get $f(x)$, then apply f to $f(x)$ to obtain $f^2(x)$, then $f^3(x), f^4(x)$ in that order. This iterative process is continued until we obtain $f^n(x)$.

The dynamical system defined by $f : X \rightarrow X$ is the family of functions $\{f^n\}_{n \in \mathbb{Z}_+}$ with each f^n mapping X to X .

Example 2.1. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = x/2$. The first iteration is given by

$$f(f(x)) = f^2(x) = \frac{x}{2} = \frac{x}{2^2} = \frac{x}{4}.$$

Continuing the process for $f^3(x), f^4(x), f^5(x), \dots$ the family of functions f describing the dynamical system is given by $f^n(x) = \frac{x}{2^n}$.

For a dynamical system defined by an iterating a function $f : A \rightarrow A$, we look at $f(a)$ as describing the new state of the system a unit of time after it was at state a . A typical instance is easily identified in the modeling of a bacteria population growing by the hour, we could have a function $f(x)$ representing the population size that results one hour after the population was x . Also, if we model the position and velocity of a rocket, we may have a function $f(x, v)$ representing the position and velocity of the rocket one second after it had position and velocity (x, v) [1].

Example 2.2. Consider the following functions defined on \mathbb{R} with their phase diagrams below:

- (1) $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = -4x$,
- (2) $g : \mathbb{R} \rightarrow \mathbb{R}$ defined by $g(x) = \frac{1}{3}x$,
- (3) $h : \mathbb{R} \rightarrow \mathbb{R}$ defined by $h(x) = -x$,
- (4) $k : \mathbb{R} \rightarrow \mathbb{R}$ defined by $k(x) = 0$.

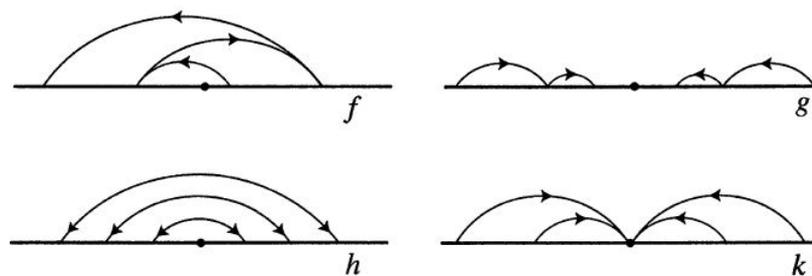


Figure 1: Phase diagrams for f, g, h and k [1].

It is important to note that each of these functions results in 0, upon evaluation at $x = 0$. Assume a particular non-zero point x_0 . A continuous repeated application of f to itself gives $f^n(x_0) = (-4)^n x_0$. That is, a repeated iteration of f on x_0 results in values that move further and further away from 0, bouncing between positive and negative values as the iteration proceeds. The qualitative depiction of the iterative behavior of this function f on \mathbb{R} as well as g, h and k is shown in the figure above. This qualitative depiction is obviously called the phase diagram for the corresponding dynamical systems. The next function g has iteration on x_0 resulting in values that move ever closer to 0, and eventually approaching 0 in the limit. h shows a different dynamic picture, in that with any the resulting dynamics is one of oscillation between positive and negative values of x_0 . The dynamics depicted by k is rather simple. All points go to 0 on application of k and this is clearly depicted in the figure above.

Phase diagrams can be produced for almost every physical phenomenon that is dynamic in nature. In the fields of biology, chemistry, economics, business through engineering and so on, phase diagrams can be used to describe effectively the dynamics of several activities. For instance, if an investor decides to invest into a clothing line at a said compound interest say I , he would want to know how much money he would be making at the end of n years. If we assume that the investor leaves his money untouched throughout this period, we can generate a simple iterative process or a dynamical system to describe how his money increases with time. Our aim is to determine A_n , the amount at the end of the n th-year considering A_0 as the initial amount invested. The dynamics of the ongoing can be described by the phase diagram below

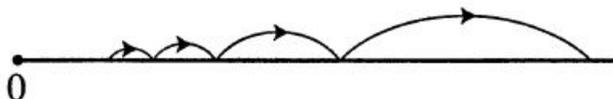


Figure 2: The dynamics of the investment [1].

The dynamics of the investment as depicted by the phase diagram shows a continual increment in the amount over the course of time as would be expected by the investor.

3. The Tent function

The most characteristic feature of the theory of dynamical systems that distinguishes it from other areas of mathematics is the emphasis on asymptotic or long term behavior: properties related with the behavior of the system as time goes to infinity. One of the most efficient ways of explaining what significant asymptotic properties are is to examine specific examples of dynamical systems and to determine the most characteristic features of their behavior, the behavior of which in our case is the behavior of the tent map.

Consider a function $T : [0, 1] \rightarrow [0, 1]$ defined by setting $T(x)$ equal to the new position of a particle that was initially at x . The tent map is defined as $T(x) = 1 - |2x - 1|$. The interval $[0, 1]$ is considered as the phase space since the function maps the interval unto itself. It is important to note that the orbits of the tent map are bounded (for positive k values) since the interval $[0, 1]$ is positively invariant under T . The definition of the Lyapunov exponent which uses the derivative of the function can only be for those orbits that avoid the point $\frac{1}{2}$ where T is not differentiable. The tent function is precisely defined as

$$T(x) = \begin{cases} 2x, & \text{if } x \in [0, \frac{1}{2}] \\ 2 - 2x, & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

with diagram

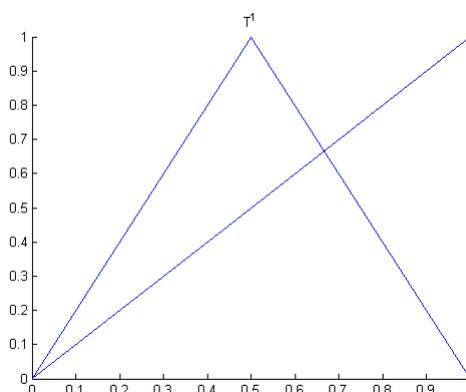


Figure 3: The tent function.

Now consider T^2 , the graph of which is shown in the figure below along with the graph of $y = x$.

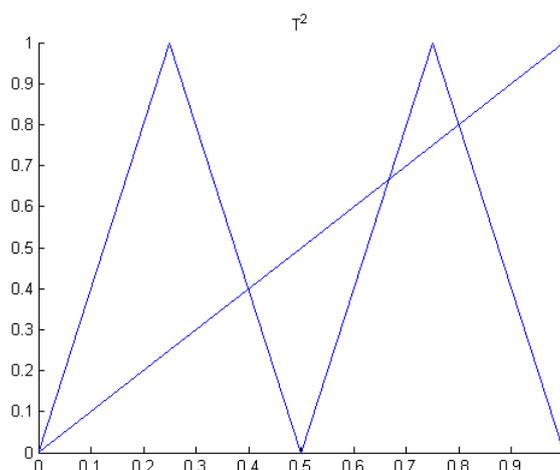


Figure 4: The function T^2 .

Before going on to the dynamics of the successive iterates of T , it is important to establish a notion of equivalence for dynamical systems defined by iterating functions.

Definition 3.1. [1] The functions $f : X \rightarrow X$ and $g : Y \rightarrow Y$ are said to be **topologically conjugate** if there exists a homeomorphism $h : X \rightarrow Y$ such that $g \circ h = h \circ f$. The homeomorphism h is called a topological conjugacy between f and g .

Remark 3.2. (1) In some cases, h can be a diffeomorphism, in which case we have smooth conjugacy.
 (2) Mappings which are topologically conjugate are completely equivalent in terms of their dynamics.

Example 3.3. The dynamics of the functions $f(x) = 2x$ and $g(x) = 3x$ appear qualitatively the same. The function $h : \mathbb{R} \rightarrow \mathbb{R}$, defined by $h(x) = x^{\log_2 3}$ is a homeomorphism that satisfies $g \circ h = h \circ f$ and hence the two functions are topologically conjugate.

Remark 3.4. (1) From the example above, a topological conjugacy between two functions f and g naturally maps orbits of f to orbits of g .
 (2) An important observation is that there is a one-to-one correspondence between periodic orbits of two conjugate maps.

4. Iterates of the Tent map

Higher-order iterates of the tent map, which are involved in the study of the asymptotic dynamics, are piecewise-linear maps and are easy to compute as is shown for T^3 and T^4 .

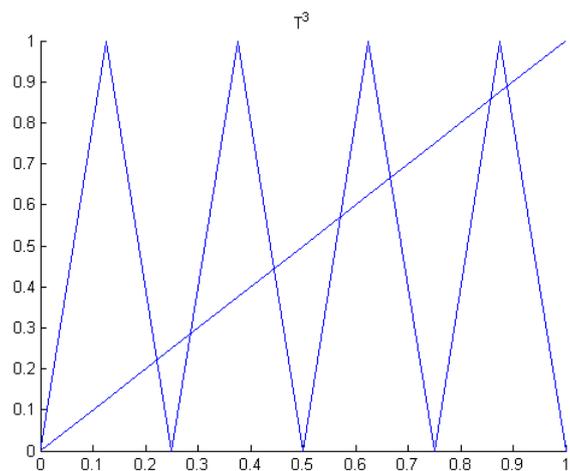


Figure 5: The graph of T^3 .

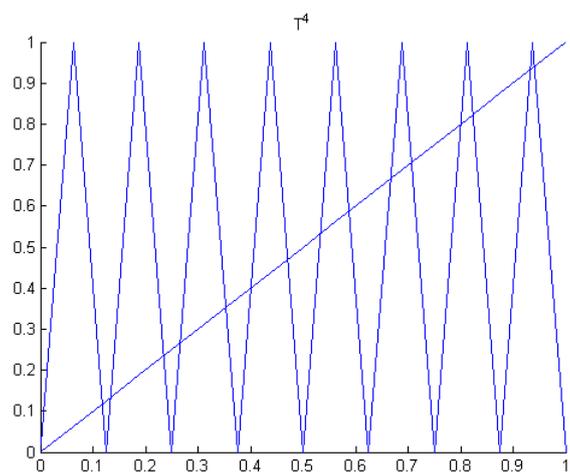


Figure 6: The graph of T^4 .

The pattern for successive iterates is apparent, with the orbits appearing as uniformly distributed and correlated. Further higher iterates are shown below.

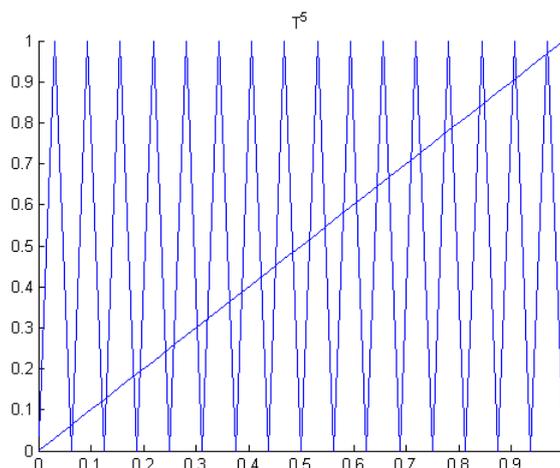


Figure 7: The graph of T^5 .

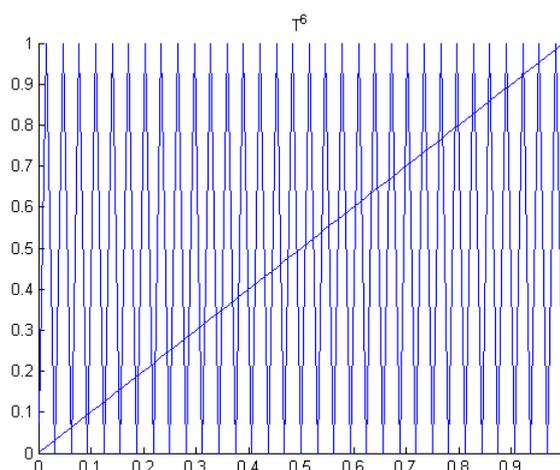


Figure 8: The graph of T^6 .

From the graphs of T^1 right unto T^6 we observe a regular periodicity of the orbits spanning over a long time. We observe a regular duplication of the orbits of a previous iterate in a subsequent one and so forth. In other physical phenomena this behaviour of a dynamical system can span over a very long time, so long that the thought of irregularity is almost impossible.

We also observe from the phase diagrams of the various tents that much of the structure of the T^n tents can be understood from T mapping linearly $\left[\frac{k-1}{2^{n-1}}, \frac{k}{2^{n-1}}\right]$ onto $[0,1]$ which results in a tent over $\left[\frac{k-1}{2^{n-1}}, \frac{k}{2^{n-1}}\right]$ for each $k = 1, 2, \dots, 2^{n-1}$. That is, the graph of the restriction of T^2 to each of the two components $\left[\frac{k-1}{2^{n-1}}, \frac{k}{2^{n-1}}\right]$ reproduces the graph of T on $[0, 1]$. This explains the two-tent structure of T^2 and so on and so forth in that order. Similarly, the relation

$$\forall x \in \left[\frac{k-1}{2^{n-1}}, \frac{k}{2^{n-1}}\right], T^n(x) = T(T^{n-1}(x)) = T^{n-1}(T(x))$$

indicates that the graph of T^n consists of two copies of T^{n-1} .

As n gets larger and larger, the intervals $\left[\frac{k-1}{2^{n-1}}, \frac{k}{2^{n-1}}\right]$ partition $[0, 1]$ into smaller and smaller intervals each of which contains a replication of the orbits in the preceding iterate or tent (over a smaller interval).

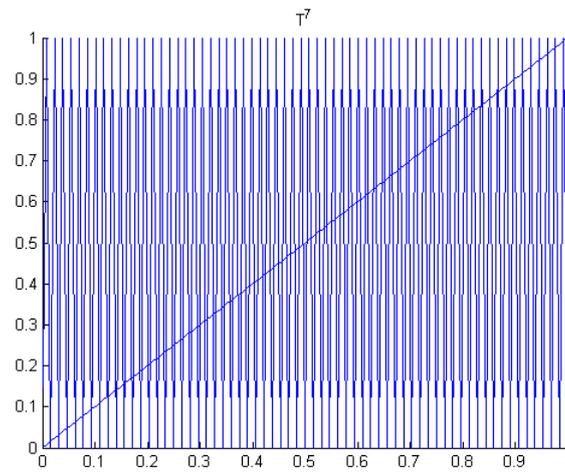


Figure 9: The graph of T^7 .

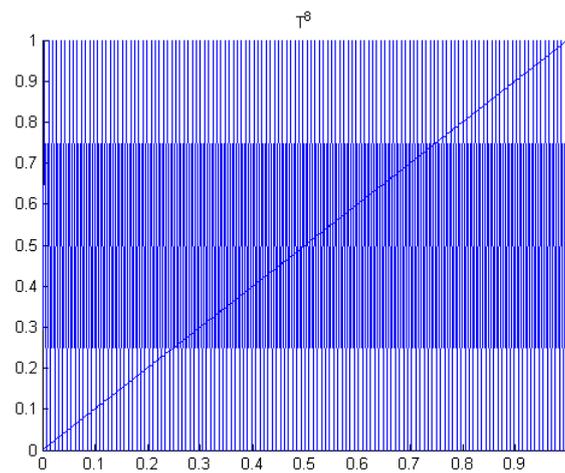


Figure 10: The graph of T^8 .

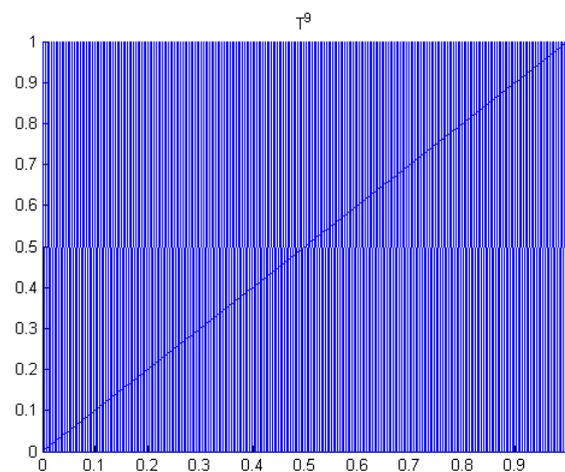


Figure 11: The graph of T^9 .

Definition 4.1. [8] A map $f : [0, 1] \rightarrow [0, 1]$ is said to be topologically mixing if for each pair of open sets $U, V \subset [0, 1]$, $n > p \implies f^n(U) \cap V \neq \emptyset$ for $p > 0$

From T^7, T^8 and T^9 , we observe from the phase diagrams that, smaller and smaller intervals are getting spread out or mixed by T^n over the whole interval $[0, 1]$, the phase space as increases. We can ascertain the eventual spreading and mixing of orbits which were once in regular periodicity as the iteration continues. The graph of T^{10} is shown below.

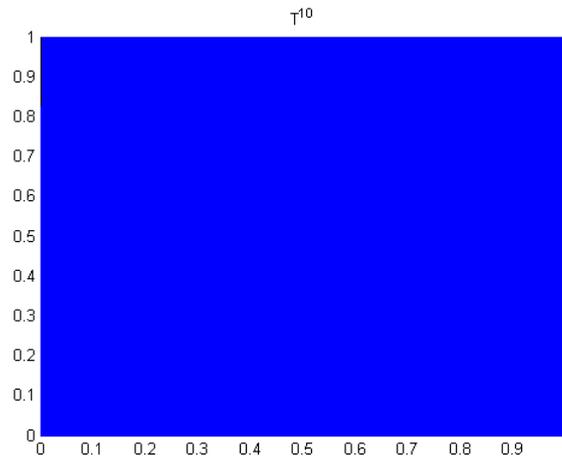


Figure 12: The graph of T^{10} .

The graph of T^{10} shows a very different kind of behavior - non-periodicity or perhaps a different kind of periodicity that is obviously not the same as the regular periodicity exhibited earlier in the lower iterates.

Definition 4.2. [2] A discrete dynamical system $f : [0, 1] \rightarrow [0, 1]$ is Devaney chaotic on an infinite subset $A \subseteq [0, 1]$ if:

- (1) There exists some point $x_0 \in A$ such that the orbit of x_0 is dense in A .
- (2) The set of all periodic orbits is dense in A

From the phase diagram of T^{10} we see a total spreading out of the orbits into each other. There is an indication of (regular) periodic behavior densely distributed in the phase space. The extreme (infinitesimal) closeness of the orbits as observed in T^8 and T^9 and obviously as the iteration continues shows this dense distribution. Topological transitivity also follows in that any given interval U we could find always an interval in the form $\left[\frac{k-1}{2^{n-1}}, \frac{k}{2^{n-1}}\right]$ inside U for sufficiently large n . T^n maps U onto the whole phase space $[0, 1]$. Having showed the main components of chaos by Definition 4.2, we conclude that the tent function has infinitely many chaotic orbits. Further proof of chaos in the tent map is shown by the asymptotic non-periodicity of the orbits as observed in the phase diagrams above. By topological conjugacy, chaos in the tent map can be used to establish chaos in the logistic map by Proposition below.

Proposition 4.3. [10] Suppose $f : I \rightarrow I$ and $g : J \rightarrow J$ are conjugate via h where both I and J are closed intervals in \mathbb{R} of finite length. If f is chaotic on I , then g is chaotic on J .

PROOF. Let U be an open subinterval of J and consider $h^{-1}(U) \subset I$. Since periodic points of f are dense in I , there is a periodic point $x \in h^{-1}(U)$ for f . Say x has period n . Then

$$g^n(h(x)) = h(f^n(x)) = h(x),$$

by the conjugacy equation. This gives a periodic point $h(x)$ for g in U and shows that periodic points of g are dense in J . If U and V are open subintervals of J , then $h^{-1}(U)$ and $h^{-1}(V)$ are open intervals in I . By

transitivity of f , there exists $x_1 \in h^{-1}(U)$ such that $f^m(x_1) \in h^{-1}(V)$ for some m . But then $h(x_1) \in U$ and we have $g^m(h(x_1)) = h(f^m(x_1)) \in V$, so g is transitive also.

For sensitivity, suppose that f has sensitivity constant β . Let $I = [\alpha_0, \alpha_1]$. We may assume that $\beta < \alpha_1 - \alpha_0$. For any $x \in [\alpha_0, \alpha_1 - \alpha_0]$, consider the function $|h(x + \beta) - h(x)|$. This is a continuous function on $[\alpha_0, \alpha_1 - \beta]$ that is positive. Hence it has a minimum value β' . It follows that h takes intervals of length β in I to intervals of length at least β' in J . Then it is easy to check that β' is a sensitivity constant for g . This completes the proof.

References

- [1] C. Adams and R. Franzosa, *Introduction to topology: Pure and applied*, Pearson Prentice Hall, 2008.
- [2] W. Basener, *Topology and its applications*, John Wiley & sons Inc. Hoboken, New Jersey, 2006.
- [3] M. Brin and G. Stuck, *Introduction to dynamical systems*, Cambridge University Press, The Pitt Building, Trumpington Street, Cambridge CB2 1RP, 2002.
- [4] Jan de Vries, *Elements of topological dynamics*, Springer Science and Business Media, 1993.
- [5] R. L. Devaney, *An introduction to chaotic dynamical systems*, second ed., Boulder, Co.: Westview Press, 2003.
- [6] Neven Elezovi, Vesna Āupanovi, and Darko Āubrini, *Box dimension of trajectories of some discrete dynamical systems*, *Chaos, Solitons & Fractals* **34** (2007), no. 2, 244 – 252.
- [7] B. Feger and T. Spier, *Geometrical picture of reaction in enzyme kinetics*, *Journal of Medical Physics* (1989), 186–193.
- [8] R. Gilmore and M. Lefranc, *The topology of chaos*, John Wiley & Sons, Inc, The Pitt Building, Trumpington Street, Cambridge CB2 1RP, 2002.
- [9] W. Gottschalk and G. A. Hedlund, *Topological dynamics*, vol. 36, Amer. Math. Soc. Colloquium Publications, The Pitt Building, Trumpington Street, Cambridge CB2 1RP, 1955.
- [10] M. Hirsch, S. Smale, and R. Devaney, *Differential equations, dynamical systems and an introduction to chaos*, second edition ed., vol. 60, Academic Press, 525 B Street, Suite 1900, San Diego, California 92101-4495, USA., 2004.
- [11] P. Holmes, *Poincare, celestial mechanics, dynamical-systems theory and 'chaos'*, *Physics Report* **193** (1990), no. 3, 137–163.
- [12] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Amer. Math. Soc. Colloquium Publications, Cambridge, 1995.
- [13] Kellert and Stephen H., *In the wake of chaos: Unpredictable order in dynamical systems.*, University of Chicago Press, 1993.
- [14] Stphane Mallat, *Geometrical grouplets*, *Applied and Computational Harmonic Analysis* **26** (2009), no. 2, 161 – 180.
- [15] An Hoang Nguyen and Simon J. Fraser, *Geometrical picture of reaction in enzyme kinetics*, *The Journal of Chemical Physics* **91** (1989), no. 1.
- [16] H. Ohtsuki, C. Hauert, E. Lieberman, and M. A. Nowak, *A simple rule for the evolution of cooperation on graphs and social networks*, *Natur* **441** (2006), 502–505.
- [17] K. Piela, R. Delfos, G. Ooms, J. Westerweel, and R.V.A. Oliemans, *On the phase inversion process in an oilwater pipe flow*, *International Journal of Multiphase Flow* **34** (2008), no. 7, 665 – 677.
- [18] R. H. Price, C. Markasis, and J. L. Friedman, *Iteration stability for simple newtonian stellar systems*, *Journal of Mathematical Physics* (2009).
- [19] J. Ren and B. Zhang, *An improvement on a non-deterministic one-ton substitution scheme in outsourcing association rule mining work* (60502032).
- [20] P. Tabuada, A. D. Ames, A. Julius, and G. J., *Approximate reduction of dynamical systems*, *Dynamical Systems* (2007), 1–11.
- [21] T. N. Wells, *Natural products as starting points for future anti-malarial therapies: going back to our roots?*, *Malaria Journal* **10** (2011), no. 1.
- [22] I. B. Yaacov, *Topometric spaces and perturbations of metric structures. logic and analysis*, 1(3-4), *Logic and Analysis* **1** (2008), 3–4.